Fixed-k Inference for Explosive Drift

Yi He* Jia Li† Yifan Li‡ Yuhong Zhu†

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Abstract

We propose a new framework for uniform inference on explosive drifts in high-frequency data, where conventional Gaussian methods can fail because short-window spot statistics are far from Gaussian. Under fixed-window asymptotics, these statistics are coupled with dependent Student-t variables, so their maximum converges to a Fréchet rather than a Gumbel distribution. We establish a new anti-clustering condition for dependent t-processes under overlapping windows and develop a feasible coupling-based test. Empirically, the proposed method achieves better size control and suggests that significant intraday price explosions are much rarer than Gaussian methods imply.

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*School of Mathematical Sciences, Eastern Institute of Technology, Ningbo
†School of Economics, Singapore Management University
‡Alliance Manchester Business School, University of Manchester

1 Introduction

Expected returns are of first-order importance in asset pricing, serving as the central object of inquiry in both academic theories of risk premia and practical investment decisions. In continuous-time models (Merton, 1990; Duffie, 2001), these instantaneous conditional expectations are stochastic and captured by the drift component of the asset price process. High-frequency econometric methods have arguably become the standard toolkit for analyzing such models using intraday data. Yet, a fundamental tension exists between the economic significance of the drift and its role in the statistical analysis. At high frequency, the drift signal is asymptotically dominated by the diffusive component, rendering it statistically negligible in standard high-frequency inference procedures. This theoretical "blindness" to the drift creates a gap between the econometric framework and the economic reality of price discovery, where assets exhibit "explosive" local trends—abrupt yet continuous shifts in price levels precipitated by major news or liquidity shocks (Kyle, 1985; Back, 1992). To bridge this gap, one may entertain a model of explosive drift—a regime where the local signal-to-noise ratio diverges, allowing the drift to become statistically detectable against the background of stochastic volatility.

A natural procedure to detect such phenomena is to compute "spot" t-statistics for the instantaneous expected returns over "short" windows of k observations and test whether their maximum over a certain trading period exceeds a critical value. This inference problem is inherently non-Donsker, as the joint process of spot statistics does not admit a functional central limit theorem. Consequently, the asymptotic distribution of the maximum statistic is derived not via standard weak convergence, but through strong approximations combined with extreme value theory. In the conventional setting where spot statistics are asymptotically Gaussian, the limit of their maximum is Gumbel. These classical results, known as the Smirnov–Bickel–Rosenblatt limit theorems (Smirnov, 1950; Bickel and Rosenblatt, 1973), have been extended extensively in econometrics (Rio, 1994; Ghosal, Sen, van der Vaart, 2000; Lee, Linton and Whang, 2009) and applied to high-frequency jump detection (Fan and

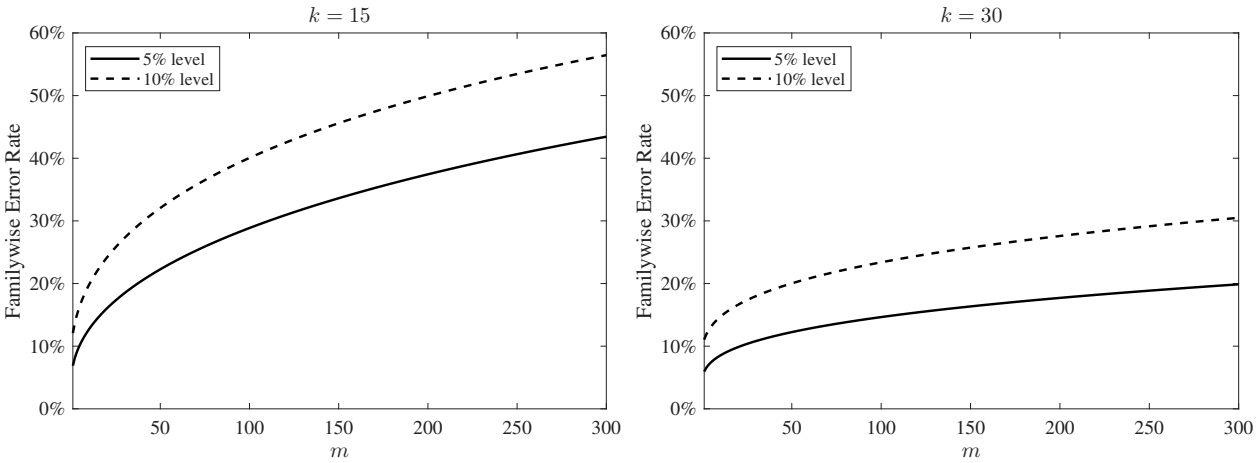


Figure 1: Family-wise error rate for m independent two-sided t-tests with $k-1$ degrees of freedom when the critical value corresponds to the 5% (solid) and 10% (dashed) significance levels of the maximum absolute value of m independent standard Gaussian variables. If the Gaussian approximation were accurate, the curves would be nearly flat and close to the corresponding nominal levels.

Wang, 2008; Lee and Mykland, 2008, 2012; Christensen, Oomen, and Renò, 2022; Bibinger, Hautsch and Ristig, 2026).

However, Gumbel convergence is notoriously slow (Hall, 1979, 1991). To avoid relying on explicit Gumbel asymptotics, a modern strand of literature pioneered by Chernozhukov, Chetverikov, and Kato (2013, 2014a,b) approximates the supremum of a stochastic process via the maximum of a high-dimensional Gaussian vector. This Gaussian coupling approach yields improved finite-sample size control and has been widely adopted in econometrics and statistics (Chernozhukov, Lee and Rosen, 2013; Belloni et al., 2015; Jirak, 2015; Zhang and Wu, 2017; Koike, 2019; Li and Liao, 2020; Chang, Jiang, McElroy, and Shao, 2025), including high-frequency-based functional inference for the stochastic volatility process (Jacod, Li, and Liao, 2021).

These Gaussian coupling approaches crucially rely on the premise that the spot statistics possess sufficiently large degrees of freedom to be approximately normal. This requirement, however, is often unrealistic in our context. As illustrated in Figure 1, even for empirically reasonable window sizes (e.g., $k = 15$ or 30, corresponding to 15- and 30-minute windows for minute-by-minute observations), the Gaussian approximation induces substantial size

distortions when the spot statistics are Student-t distributed.

To address this, we propose a framework based on fixed-k asymptotics, drawing insights from the broad literature on "nonstandard" asymptotic embedding; see, for example, Jansson (2004); Kiefer and Vogelsang (2002, 2005); Sun, Phillips, and Jin (2008); Ibragimov and Müller (2010); Bester, Conley and Hansen (2011); Sun (2014b); Müller and Wang (2017); Lazarus, Lewis, and Stock (2021) for examples in various econometric contexts. By treating the window size k as fixed, we establish high-dimensional Student-t coupling for the collection of spot test statistics.[1] Consequently, the limiting distribution of the uniform test statistic is Fréchet, not Gumbel. This Fréchet limit explicitly captures the tail thickness stemming from studentization, thereby providing superior size control even when k is small.

Our analysis of the uniform test statistic under the fixed-k framework addresses a theoretically challenging high-dimensional non-Donsker inference problem with non-Gaussian dependence. When estimation windows overlap—a standard practice for smooth screening and better finite-sample power—the collection of spot t-statistics forms a dependent process. While Gaussian theory relies on the Smirnov–Bickel–Rosenblatt condition to ensure that extremes do not "cluster," no such tools exist for dependent Student-t processes. Extending the idea of Clarke and Hall (2009) to continuous-time Itô semimartingales, we derive a novel anti-clustering condition for this heavy-tailed setting. We prove that tail independence holds despite the serial correlation. This result allows us to establish the Fréchet limit theorem for the uniform test statistic in a general setting, filling a gap in the literature on high-dimensional inference with non-Gaussian dependence.

We further develop a local power theory for uniform inference in this non-Gaussian setting, complementing the classical Gaussian analysis of Bickel and Rosenblatt (1973). The presence of an explosive drift operates through a fundamentally different channel for uniform inference: whereas in pointwise inference it manifests as an additive location shift in

[1] In the high-frequency econometric setting, the fixed-k approach is first developed by Bollerslev, Li, and Liao (2021) and Bollerslev, Li, and Ren (2024) for spot covariance inference, which is distinct from the present inference problem on the local mean parameter.

the limiting distribution, under uniform inference it instead induces a multiplicative power transformation of the Fréchet limit in our fixed-k framework. Our approach therefore contrasts sharply with the literature on structural change (see, e.g., Andrews (1993); Bai and Perron (1998); Müller (2011); Andersen, Todorov, and Zhou (2025)). While those methods typically rely on cumulative sums and Donsker's invariance principle to detect permanent parameter shifts, our framework targets short-lived, explosive episodes.

Guided by these theoretical results, we recommend a coupling-based approach for constructing critical values, which is robust to the choice of k and the window overlap structure. Empirically, our study sheds new light on intraday price dynamics. By correcting the size distortions inherent in conventional Gaussian-based inference, we demonstrate that statistically significant explosive drifts are far rarer than what would be suggested by conventional procedures.

The remainder of the paper is organized as follows. Section 2 develops the asymptotic theory under the null hypothesis. Section 3 analyzes local power. Section 4 presents Monte Carlo experiments, and Section 5 details the empirical application. Section 6 concludes. Proofs are provided in the Appendix and the Supplementary Material (He, Li, Li, and Zhu, 2026).

2 A Uniform Test for Explosive Drift

In this section, we propose a uniform test for explosive drift and analyze its size properties under the null hypothesis, deferring the power analysis to Section 3. We begin in Section 2.1 by formalizing the model setting and the construction of the test statistics. Section 2.2 establishes the asymptotic size properties in a baseline setting with non-overlapping windows, providing the intuition for the more general framework with overlapping windows presented in Section 2.3. Finally, Section 2.4 elucidates the theoretical connection between our fixed-k methodology and the conventional large-k asymptotics, highlighting the former as a unifying

framework that yields superior finite-sample performance.

2.1 The Setting

We consider a univariate log-price process X on a finite time interval $[0, \mathrm{T}]$, modeled as an Itô semimartingale of the form

$$X_{\mathrm{t}} = X_0 + \int_0^{\mathrm{t}} b_{\mathrm{s}}\, d\mathrm{s} + \int_0^{\mathrm{t}} \sigma_{\mathrm{s}}\, dW_{\mathrm{s}} + J_{\mathrm{t}}, \tag{1}$$

where the drift process b and the stochastic volatility process σ are both assumed to be càdlàg and adapted, and W is a standard Brownian motion. The process J is a finite-activity pure-jump process driven by a homogeneous Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$, with jump times independent of W. Unless stated otherwise, all random elements are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{\mathrm{t}})_{\mathrm{t}\geq 0}, \mathbb{P})$. Throughout, we impose some mild regularity conditions that are stated in the following assumption.

Assumption 1. *There exists a sequence* $(\mathrm{T}_m)_{m\geq 1}$ *of stopping times increasing to infinity and a sequence* $(K_m)_{m\geq 1}$ *of constants such that the following conditions hold for each* $m \geq 1$*:*

(a) $|b_{\mathrm{t}}| + |\sigma_{\mathrm{t}}| + |\sigma_{\mathrm{t}}|^{-1} + F_{\mathrm{t}}(\mathbb{R} \setminus \{0\}) \leq K_m$ *for all* $\mathrm{t} \in [0, \mathrm{T} \wedge \mathrm{T}_m]$*, where* F_{t} *denotes the spot Lévy measure of* J*.*

(b) For some constant $\kappa > 0$*,* $\mathbb{E}\left[|\sigma_{\mathrm{t}\wedge\mathrm{T}_m} - \sigma_{\mathrm{s}\wedge\mathrm{T}_m}|^2\right] \leq K_m|\mathrm{t} - \mathrm{s}|^{2\kappa}$ *for all* $\mathrm{t}, \mathrm{s} \in [0, \mathrm{T}]$*.*

Condition (a) imposes local boundedness on the volatility process σ (and its reciprocal), the drift process, and the jump activity. Condition (b) requires the volatility process to be locally κ-Hölder continuous under the L_2 norm. This regularity condition is satisfied with $\kappa = 1/2$ if the volatility is an Itô semimartingale or a long-memory process driven by fractional Brownian motion, and it also accommodates rough volatility models where $\kappa < 1/2$.

The price process X is observed at discrete times $\{i\Delta_n\}_{i=0}^n$, where the sampling interval $\Delta_n = \mathrm{T}/n$ satisfies $\Delta_n \to 0$ as $n \to \infty$. The corresponding returns are denoted as $\Delta_i^n X \equiv X_{i\Delta_n} - X_{(i-1)\Delta_n}$. We group these returns into possibly overlapping blocks covering $[0, \mathrm{T}]$, each containing a *fixed* number k of returns. Let $\{i_j\}_{j\geq 1}$ denote the strictly increasing sequence of block starting indices with $i_1 = 0$ and satisfying

$$1 \leq i_{j+1} - i_j \leq k, \quad \forall j \geq 1.^2 \tag{2}$$

The total number of blocks required to cover $[0, \mathrm{T}]$ is $m_n = \min\{j \in \mathbb{N} : i_j + k \geq n\}$. For $1 \leq j \leq m_n - 1$, the indices of the k returns in the jth block are collected in $\mathcal{I}_{n,j} \equiv \{i_j + 1, \ldots, i_j + k\}$, spanning the time interval $\mathcal{T}_{n,j} \equiv [i_j\Delta_n, (i_j + k)\Delta_n)$. The final block, which includes the right endpoint T, is given by $\mathcal{T}_{n,m_n} \equiv [\mathrm{T} - k\Delta_n, \mathrm{T}]$ with corresponding index set $\mathcal{I}_{n,m_n} = \{n-k+1, \ldots, n\}$ for the returns. This construction ensures that, for each n, the interval $[0, \mathrm{T}]$ is fully covered by the collection of blocks, that is, $\bigcup_{j=1}^{m_n} \mathcal{T}_{n,j} = [0, \mathrm{T}]$.

For each time block $\mathcal{T}_{n,j}$, whose length shrinks to zero asymptotically, we compute the "spot" t-statistic as[3]

$$\widehat{t}_{n,j} \equiv \frac{\sqrt{k}\bar{X}_{n,j}}{S_{n,j}}, \tag{3}$$

where

$$\bar{X}_{n,j} \equiv \frac{1}{k}\sum_{i\in\mathcal{I}_{n,j}} \frac{\Delta_i^n X}{\sqrt{\Delta_n}}, \quad S_{n,j}^2 \equiv \frac{1}{k-1}\sum_{i\in\mathcal{I}_{n,j}} \left(\frac{\Delta_i^n X}{\sqrt{\Delta_n}} - \bar{X}_{n,j}\right)^2.$$

Since the drift process governs the "local mean" of the returns, employing the spot t-statistic to detect explosive drift appears rather natural. In addition, given the well-known local Gaussianity of the Itô semimartingale model (see, e.g., Bollerslev, Li, and Liao (2021)), the original testing problem may be locally coupled with a Gaussian location experiment, which further motivates the use of the spot t-statistic, as the t-test is finite-sample optimal

[2]When the left inequality in (2) holds with $i_j = j - 1$ for all j, we obtain maximally overlapping rolling windows. When the right inequality in (2) holds with $i_j = (j-1)k$ for all j, we obtain non-overlapping windows (except for the final block when n is not divisible by k, which does not affect our results).

[3]The following notational convention is adopted in this paper: we use t to denote a generic time point, and reserve the notation t for the t-statistic and its limiting counterpart.

in such experiments.

Our main goal is to analyze the *joint* asymptotic behavior of the entire collection of spot t-statistics and, based on this analysis, develop a uniform test. This problem presents several theoretical challenges. First, under the fixed-k framework, the number of spot t-statistics, m_n, diverges at the same rate as the sample size (i.e., $m_n \asymp n$), resulting in a high-dimensional, non-Donsker inference problem. Second, as we show below, the marginal limiting distribution of each spot t-statistic is Student-t rather than Gaussian; this places our setting outside the scope of the extensive literature on high-dimensional Gaussian coupling developed by Chernozhukov, Chetverikov, and Kato (2013, 2014a,b) and others. Third, when estimation windows overlap, the spot statistics are mutually dependent, with their limiting behavior described by a high-dimensional, serially correlated Student-t process. This structure is qualitatively distinct from prior work, which focuses almost exclusively on Gaussian limits and their associated anti-concentration properties. To overcome the lack of anti-concentration inequalities for this dependent Student-t setting, we adopt an alternative strategy inspired by extreme value theory to establish the distributional regularity required for inference.

We abstract from market microstructure noise in the present paper in order to focus on the fixed-k inference problem under the baseline semimartingale model. This restriction is mainly for expositional simplicity rather than a fundamental limitation of the approach. In principle, the framework can be extended to noisy high-frequency observations by replacing raw returns with suitable noise-robust local statistics, such as pre-averaged returns (Mykland and Zhang, 2016; Jacod, Li, Mykland, Podolskij, and Vetter, 2009). The key requirement for our analysis is a coupling step: just as the noiseless local returns can be coupled with Gaussian variables, one may show that the corresponding pre-averaged returns admit an analogous Gaussian coupling under standard high-frequency asymptotics. Once such a coupling is established, the resulting studentized local mean statistics should again behave like fixed-k t-type variables, and the associated uniform statistic can then be analyzed using the

same arguments developed in this paper for the pointwise and sup statistics.

To streamline the exposition, we first present results in Section 2.2 for the baseline case of non-overlapping estimation windows, where the spot t-statistics are asymptotically independent. In this setting, we derive the limiting Student-t distributions and detail the construction of the uniform test. Section 2.3 extends the theory to the general case of overlapping windows, introducing mutual dependence among the high-dimensional spot statistics. We emphasize that this analysis is nonstandard because we adopt the fixed-k framework; this choice avoids the secondary approximation of sending $k \to \infty$ (necessary for Gaussian limits) and yields improved finite-sample performance. Finally, Section 2.4 demonstrates that the fixed-k theory recovers the standard large-k asymptotics as $k \to \infty$. This result generalizes the classical convergence of Student-t variables to Gaussian variables, formally justifying the fixed-k approach as a more robust, unifying framework that entails no loss of generality.

2.2 Testing With Non-Overlapping Windows

In this subsection, we consider the baseline case with non-overlapping estimation windows. Under this setting, each window contains exactly k return observations, and the total number of windows is $m_n = n/k$. For expositional simplicity, we assume that n/k is an integer, noting that this restriction can be easily relaxed and is immaterial in the preferred overlapping-window setting discussed later in Section 2.3. For each time point $\mathrm{t} \geq 0$, we define the mapping

$$j_n(\mathrm{t}) = \{j \in \mathbb{N} : \mathrm{t} \in \mathcal{T}_{n,j}\} \tag{4}$$

to identify the block(s) containing time t. Because the windows are non-overlapping in the present setting, each time point $\mathrm{t} \in [0, \mathrm{T}]$ belongs to exactly one window, and hence the set $j_n(\mathrm{t})$ contains a single element.

As we are interested in detecting both positive and negative explosive drifts, we base inference on the absolute value of the t-statistic, referred to as the z-statistic. Accordingly,

the spot z-statistic at any given time t is defined as

$$\widehat{z}_{n,\mathrm{t}} \equiv \left|\widehat{t}_{n,j}\right|, \quad j \in j_n(\mathrm{t}). \tag{5}$$

To characterize the asymptotic behavior of $\widehat{z}_{n,\mathrm{t}}$, we define

$$\bar{W}_{n,j} \equiv \frac{1}{k}\sum_{i\in\mathcal{I}_{n,j}} \frac{\Delta_i^n W}{\sqrt{\Delta_n}}, \quad V_{n,j}^2 \equiv \frac{1}{k-1}\sum_{i\in\mathcal{I}_{n,j}} \left(\frac{\Delta_i^n W}{\sqrt{\Delta_n}} - \bar{W}_{n,j}\right)^2,$$

and introduce the corresponding *coupling* spot t-statistic,

$$t_{n,j} \equiv \frac{\sqrt{k}\,\bar{W}_{n,j}}{V_{n,j}} \;\sim\; \mathscr{T}_{k-1}, \tag{6}$$

where "$\sim$" denotes equality in distribution and $\mathscr{T}_\nu$ denotes a generic Student-t random variable with ν degrees of freedom.[4] Analogously, the coupling variable associated with the z-statistic is defined as

$$z_{n,\mathrm{t}} \equiv |t_{n,j}| \;\sim\; |\mathscr{T}_{k-1}|. \tag{7}$$

The following theorem establishes the pointwise coupling result at any fixed time point.

Theorem 1. *Let $k \geq 2$. For the process X in* (1) *satisfying Assumption 1, we have $\widehat{z}_{n,\mathrm{t}} = z_{n,\mathrm{t}} + o_p(1)$ for any fixed* t.

Since $z_{n,\mathrm{t}}$ is pivotal, this coupling result immediately implies the following pointwise weak convergence, where $\xrightarrow{d}$ denotes convergence in distribution.

Corollary 1. *Under the conditions of Theorem 1, $\widehat{z}_{n,\mathrm{t}} \xrightarrow{d} |\mathscr{T}_{k-1}|$.*

For our main purpose of conducting uniform inference, we consider the sup-z statistic over

[4]The t-distribution follows from the Gaussianity of the Brownian increments in the price process and the classical finite-sample distribution of t-statistics in the Gaussian location model.

the interval $[0, \mathrm{T}]$, which is equivalent to the maximum absolute value of the spot t-statistics:

$$\sup_{\mathrm{t}\in[0,\mathrm{T}]} \widehat{z}_{n,\mathrm{t}} = \max_{1\leq j\leq m_n} |\widehat{t}_{n,j}|. \tag{8}$$

This maximum statistic can be approximated by the maximum of the independent Student-t variables $\{|t_{n,j}|\}_{j=1}^{m_n}$, provided that condition (b) in Assumption 1 is strengthened as follows.

Assumption 2. *There exists some constant $\kappa > 0$ such that for each $p \geq 2$, and any stopping time T_m from Assumption 1, there exist constants $K_{m,p}$ such that*

$$\mathbb{E}[|\sigma_{\mathrm{t}\wedge\mathrm{T}_m} - \sigma_{\mathrm{s}\wedge\mathrm{T}_m}|^p] \leq K_{m,p}|\mathrm{t}-\mathrm{s}|^{\kappa p}, \ \ \forall \mathrm{t}, \mathrm{s} \in [0, \mathrm{T}].$$

This condition requires the volatility process to be κ-Hölder continuous under all L_p-norms.[5] It is weaker than prior work on the uniform inference for the stochastic volatility process (see, e.g., Assumption 2 in Bollerslev, Li, and Liao (2021)), which imposes a stronger requirement by placing the supremum over t and s inside the expectation operator.

Theorem 2. *Let $k \geq 6$ be fixed. Suppose that the process X follows* (1) *and satisfies Assumptions 1 and 2 with $\kappa > 2/(k-1)$. Then:*

(a) The normalized sup-z statistic converges in distribution to a Fréchet limit:

$$a_{m_n,k-1}^{-1}\left(\sup_{\mathrm{t}\in[0,\mathrm{T}]} \widehat{z}_{n,\mathrm{t}} - q_{m_n,k-1}\right) \xrightarrow{d} \xi_{k-1},$$

where ξ_ν denotes a Fréchet random variable with shape parameter ν, having the distribution function

$$\mathbb{P}\left(\xi_\nu \leq x\right) = \exp\left(-\left(1+\frac{x}{\nu}\right)^{-\nu}\right), \quad x > -\nu. \tag{9}$$

[5]This continuity condition, as well as that in Assumption 1, can be relaxed to a weaker version such that the Hölder continuity holds only within each disjoint sub-interval of the form $[\mathrm{T}_l, \mathrm{T}_{l+1})$, so as to accommodate the presence of volatility jumps at the boundary times, possibly corresponding to pre-scheduled events. Since only finitely many local windows contain these times, and each contains at most one for large n, the associated absolute t-statistics do not affect the asymptotic behavior of the maximum over a large number of windows. When there are infinitely many volatility jumps at points $i\Delta_n$, it is possible to extend the insight of Ibragimov and Müller (2010) to show that the t-test remains valid for control size, possibly with conservativeness.

Closed-form expressions for the normalizing constants $q_{m_n,k-1}$ and $a_{m_n,k-1}$ are provided in Appendix A.

(b) For any significance level $\alpha \in (0,1)$,

$$\mathbb{P}\left(\sup_{\mathrm{t}\in[0,\mathrm{T}]} \widehat{z}_{n,\mathrm{t}} > z^*_{n,1-\alpha}\right) \to \alpha,$$

where $z^*_{n,1-\alpha}$ denotes the $(1-\alpha)$-quantile of $\max_{1\le j\le m_n}|t^*_j|$, with $\{t^*_j\}$ being a sequence of i.i.d. Student-t variables with $k-1$ degrees of freedom.[6]

Theorem 2 justifies two alternative approaches for computing critical values for the sup-z statistic: the asymptotic Fréchet approximation derived in part (a) and the coupling-based simulation method detailed in part (b). We recommend the coupling-based approach, as it avoids the additional layer of extreme-value asymptotics (associated with $m_n \to \infty$) and is consequently robust across both small and large values of m_n, sparing the user from having to judge whether m_n is large enough for the extreme-value approximation to be accurate. This intuition is corroborated by the simulation results in Section 4, which demonstrate the superior finite-sample size control of the coupling-based critical values.

Although we recommend the coupling-based approach for practical inference, we stress that part (a) of the theorem serves a critical theoretical role in the background. By establishing the distributional convergence of the sup-z statistic and its coupling counterpart, it ensures the requisite regularity of their distribution functions *even* when m_n is large. This argument plays a role analogous to the anti-concentration inequalities used in high-dimensional Gaussian coupling theory (see, e.g., Chernozhukov, Chetverikov, and Kato (2013, 2014a,b)), circumventing the lack of such tools for non-Gaussian settings. This theoretical foundation proves particularly valuable when extending the analysis to the dependent case of overlapping estimation windows, to which we now turn.

[6] In the non-overlapping setting, the coupling-based critical value $z^*_{n,1-\alpha}$ admits the closed-form expression $z^*_{n,1-\alpha} = Q_{k-1}\big(\frac{1}{2} + \frac{1}{2}(1-\alpha)^{1/m_n}\big)$, where Q_{k-1} denotes the quantile function of the Student-t distribution with $k-1$ degrees of freedom.

2.3 Testing With Overlapping Windows

We now turn to the general case of possibly overlapping windows. In this setting, each time point t may be covered by multiple estimation windows; that is, the set $j_n(\mathrm{t})$ defined in (4) may contain multiple elements. We therefore generalize the z-statistic as the following "local maximum" statistic:[7]

$$\widehat{z}_{n,\mathrm{t}} = \max_{j \in j_n(\mathrm{t})} \left| \widehat{t}_{n,j} \right|.$$

In the fixed-k setting, this maximization involves only a bounded number of spot t-statistics. Consequently, the asymptotic behavior of $\widehat{z}_{n,\mathrm{t}}$ follows as a direct extension of Theorem 1, as stated below.

Theorem 3. *Under the conditions of Theorem 1, for any fixed time* t *and significance level* $\alpha \in (0, 1)$*,*

$$\mathbb{P}\left(\widehat{z}_{n,\mathrm{t}} > c_{n,\alpha}\right) \to \alpha,$$

where $c_{n,\alpha}$ *denotes the* $(1-\alpha)$*-quantile of* $z_{n,\mathrm{t}} = \max_{j \in j_n(\mathrm{t})} |t_{n,j}|$*.*

Theorem 3 establishes that the quantile of the coupling variable $z_{n,\mathrm{t}} = \max_{j \in j_n(\mathrm{t})} |t_{n,j}|$ provides a valid critical value for the pointwise test. Even in this pointwise setting, the constituent variables $t_{n,j}$ are mutually dependent, as they load on common Brownian increments by construction. The distribution of $z_{n,\mathrm{t}}$ is thus nonstandard, though its quantiles can be calculated via simulation.

For uniform inference, we again consider the sup-z statistic, $\sup_{\mathrm{t} \in [0,\mathrm{T}]} \widehat{z}_{n,\mathrm{t}}$, defined in (8). As noted previously, the primary complication relative to the non-overlapping baseline is the serial dependence among the coupling Student-t variables $t_{n,j}$, which stems from their involvement of the same Brownian shocks and precludes the application of classical extreme value theory for *i.i.d.* variables. We instead draw upon the literature on extreme value theory for dependent variables; see, e.g., Leadbetter, Lindgren, and Rootzén (1983), Drees (2000),

[7] When the windows are non-overlapping, $j_n(\mathrm{t})$ is a singleton for all t, and the definition of $\widehat{z}_{n,\mathrm{t}}$ reduces to (5).

and Drees and Rootzén (2010).

It is instructive to contrast our analysis with prior work relying on asymptotic Gaussianity, where the validity of the associated Gumbel extreme-value approximation hinges on the Smirnov–Bickel–Rosenblatt condition. This condition ensures that extreme values do not "cluster"—effectively implying asymptotic tail independence despite the construction-induced serial dependence—thereby preserving the anti-concentration properties of the independent case. In the fixed-k framework, however, the tail behavior is fundamentally different: it is no longer governed by the Gaussian numerator but is instead dominated by the heavy-tailed denominator, $V_{n,j}$.

A key finding of our analysis is that, perhaps surprisingly, an analogous anti-clustering condition holds even in this setting. Given its independent theoretical interest, we formalize this result in Lemma 1 below. To our knowledge, this represents the first derivation of such a property in the context of high-dimensional inference with non-Gaussian dependence.

Lemma 1 (Anti-Clustering)**.** *The Student-t variables $t_{n,j}$ are tail independent uniformly in the sense that, for any positive sequence $u_n \to \infty$,*

$$\max_{1 \le j < j' \le m_n} \mathbb{P}\left(|t_{n,j'}| > u_n \mid |t_{n,j}| > u_n\right) \to 0.$$

Using this anti-clustering property, the following theorem generalizes the results of Theorem 2 to the case of arbitrarily overlapping windows.

Theorem 4. *Let $k \ge 6$ be fixed. Suppose that the process X follows* (1) *and satisfies Assumptions 1 and 2 with $\kappa > 2/(k-1)$. Then:*

(a) The sup-z statistic converges in distribution to the Fréchet limit from Theorem 2:

$$a_{m_n,k-1}^{-1}\left(\sup_{\mathrm{t}\in[0,\mathrm{T}]} \widehat{z}_{n,\mathrm{t}} - q_{m_n,k-1}\right) \xrightarrow{d} \xi_{k-1},$$

for the same normalizing constants $a_{m_n,k-1}$ and $q_{m_n,k-1}$ in Theorem 2 (also see Appendix

A).

(b) For any significance level $\alpha \in (0,1)$*,*

$$\mathbb{P}\left(\sup_{\mathrm{t}\in[0,\mathrm{T}]} \widehat{z}_{n,\mathrm{t}} > z^*_{n,1-\alpha}\right) \to \alpha, \tag{10}$$

where $z^*_{n,1-\alpha}$ *denotes the* $(1-\alpha)$*-quantile of the variable* $\max_{1\le j\le m_n} |t_{n,j}|$.

As in the non-overlapping setting detailed in Section 2.2, we recommend employing the coupling-based critical value $z^*_{n,1-\alpha}$ described in part (b), as it better captures the finite-sample behavior of the sup-z test statistic. In the overlapping window context, this approach not only avoids the extreme-value approximation but also accounts for the finite-sample dependence of the Student-t coupling variables, without relying on the tail independence property that holds "only" asymptotically. Also, part (a) of the theorem establishes a type of distributional regularity for the sup-z statistic, which is needed for justifying the asymptotic validity of the coupling-based inference, even though the inference procedure does not explicitly exploit the Fréchet limiting distribution.

2.4 Fixed-k Versus Large-k Approximations

We have established the asymptotic behavior of the pointwise and uniform test statistics under the fixed-k asymptotic framework. In contrast to the conventional large-k framework, the spot t-statistic follows a Student-t distribution and the sup-z statistic follows a Fréchet distribution, rather than the Gaussian and Gumbel distributions. We advocate the fixed-k approach for constructing critical values because it avoids the additional approximation error associated with imposing $k \to \infty$. Moreover, because our theory does not require k to be large, the practitioner may use a relatively small window size to better isolate the economic event of interest, without concern that sharper event alignment comes at the cost of distorted statistical inference. In this section, we show that the fixed-k framework is theoretically more

general than the large-k framework, in the sense that the latter is recovered as a limit of the former. Hence, there is no loss of generality in adopting the fixed-k approach.

The discussion for the pointwise test based on the spot t-statistic is straightforward, essentially relying on the well-known fact that the Student-t distribution converges to the normal distribution as the degrees of freedom grow to infinity. This implies that the critical values derived from the fixed-k theory will automatically recover those implied by the large-k theory when k is indeed "large," while the gap can be non-trivial when k is relatively small. For completeness and ease of comparison with the sup-test below, we provide in Corollary 2 a formal statement for the convergence of the distribution of the spot t-statistic towards the large-k Gaussian limit.

Corollary 2. *Consider the process X from (1) satisfying Assumption 1. For every $\varepsilon > 0$, there exists a constant $\underline{k} = \underline{k}(\varepsilon)$ such that, for any fixed $k \geq \underline{k}$ and all sufficiently large n,*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\big(\widehat{t}_{n,j_n^*(\mathrm{t})} \leq x\big) - \Phi(x) \right| < \varepsilon,$$

where Φ denotes the standard normal distribution function and $j_n^*(\mathrm{t})$ is any element of $j_n(\mathrm{t})$ at any fixed time t. If, in addition, Assumption 2 holds with $\kappa > 3/(k-1)$ and there are no jumps with $J \equiv 0$, then the approximation is uniform in the sense that

$$\sup_{x \in \mathbb{R},\, 1 \leq j \leq m_n} \left| \mathbb{P}\big(\widehat{t}_{n,j} \leq x\big) - \Phi(x) \right| < \varepsilon,$$

possibly requiring a larger $\underline{k} = \underline{k}(\varepsilon)$.

We next establish an analogous result for the uniform test by exploiting the fact that the Gumbel distribution arises as the large-k limit of the Fréchet distribution in Theorem 4. Corollary 3 below shows that the distribution of the normalized sup-z statistic converges to the Gumbel distribution as $k \to \infty$. This provides a formal sense in which fixed-k inference entails no loss of generality even when k is large, since the implied limiting distributions

coincide. Put differently, the user may simply adopt the former without having to decide whether k is "large enough" in practice.

Corollary 3. *Under the same setting as in Theorem 4, for every $\varepsilon > 0$, there exists a constant $\underline{k} = \underline{k}(\varepsilon)$ such that, for any fixed $k \geq \underline{k}$ and all sufficiently large n,*

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(a_{m_n,k-1}^{-1}\left(\sup_{\mathrm{t}\in[0,\mathrm{T}]}\widehat{z}_{n,\mathrm{t}}-q_{m_n,k-1}\right)\leq x\right)-\mathbb{P}(\xi\leq x)\right|<\varepsilon,$$

where ξ denotes the Gumbel limit with distribution function

$$\mathbb{P}(\xi\leq x)=\lim_{k\to\infty}\mathbb{P}(\xi_{k-1}\leq x)=\exp(-\exp(-x))\,.^{8}$$

3 Power Theory Against Local Alternatives

In this section, we analyze the power of the proposed uniform test against a sequence of local alternatives exhibiting explosive drift. Specifically, we consider a sequence of semimartingales, $X^{(n)}$, defined by

$$X_{\mathrm{t}}^{(n)}=X_0+\int_0^{\mathrm{t}}b_{\mathrm{s}}^{(n)}\,d\mathrm{s}+\int_0^{\mathrm{t}}\sigma_{\mathrm{s}}\,dW_{\mathrm{s}}+J_{\mathrm{t}},\tag{11}$$

where

$$b_{\mathrm{t}}^{(n)}=b_{\mathrm{t}}+\Delta_n^{-1/2}\varpi_{\mathrm{t}}\sigma_{\mathrm{t}},\tag{12}$$

captures the explosive drift through the $\Delta_n^{-1/2}\varpi_{\mathrm{t}}\sigma_{\mathrm{t}}$ component, with ϖ governing the signal-to-noise ratio. Under the null hypothesis $\varpi\equiv 0$, the process $X^{(n)}$ reduces to (1). Under the alternative $\varpi\neq 0$, the perturbation is "local" in the sense that the drift increment is of the

[8] To further substitute $a_{m_n,k-1}$ and $q_{m_n,k-1}$ with their limits from Gaussian coupling theory, namely $q_m=\lim_{k\to\infty}q_{m,k-1}$ and $a_m=\lim_{k\to\infty}a_{m,k-1}$, one must balance k and m to trade off between the coupling error and the Gaussian approximation error; see, e.g., Theorem 3 of Christensen, Oomen, and Renò (2022). Our adaptive choice of $a_{m_n,k-1}$ and $q_{m_n,k-1}$ circumvents this trade-off and yields robust approximations for both small and large k.

same order, $\sqrt{\Delta_n}$, as the Brownian component, yielding a non-degenerate asymptotic power function.

To obtain a tractable characterization of the local power, we assume that ϖ is a deterministic function on $[0, \mathrm{T}]$ (or, more generally, stochastic but independent of the underlying driving processes), satisfying the following assumption.

Assumption 3. *The function ϖ admits the decomposition $\varpi = \varpi^c + J^{\varpi}$, where ϖ^c is its continuous component and the step function J^{ϖ} is its jump component, with J^{ϖ} having only finitely many discontinuities on $[0, \mathrm{T}]$. Moreover, there exist constants $\kappa > 0$ and $K > 0$ such that, for all $\mathrm{t}, \mathrm{s} \in [0, \mathrm{T}]$, $|\varpi^c_{\mathrm{t}} - \varpi^c_{\mathrm{s}}| \leq K\,|\mathrm{t} - \mathrm{s}|^{\kappa}$.*

To characterize the local power of the uniform test, we first examine the behavior of the spot t-statistic. Under the alternative $\varpi \neq 0$, the $\widehat{t}_{n,j}$ statistics are no longer asymptotically pivotal. Instead, they are coupled with the noncentral Student-t variables

$$t^a_{n,j} = t_{n,j} + \frac{\sqrt{k}\varpi_{n,j}}{V_{n,j}},$$

where $t_{n,j}$ is the centered Student-t variable from (6) and $\varpi_{n,j}$ can be taken as any value in $\{\varpi_{\mathrm{t}} : \mathrm{t} \in \mathcal{T}_{n,j}\}$. Consequently, $|t^a_{n,j}|$ follows the distribution $\widetilde{F}_{k-1,\mu_{n,j}}$ with noncentrality parameter $\mu_{n,j} = \sqrt{k}\,\varpi_{n,j}$; here, $\widetilde{F}_{\nu,\mu}$ denotes the distribution function of the absolute value of a noncentral t-variable with ν degrees of freedom and noncentrality parameter μ, defined formally as $\widetilde{F}_{\nu,\mu}(x) = \mathbb{P}(|Z + \mu|/\sqrt{\chi^2_\nu/\nu} \leq x)$, where Z is a standard normal variable independent of χ^2_ν.

Each spot constituent contributes to the uniform test's power through changes in tail probabilities relative to the null. To characterize this effect, we introduce a scaling function $\beta_{k-1} : \mathbb{R} \to [1, \infty)$ corresponding to the limit of the ratio of tail probabilities:

$$\beta_{k-1}(\mu) = \lim_{x \to \infty} \frac{1 - \widetilde{F}_{k-1,\mu}(x)}{1 - \widetilde{F}_{k-1,0}(x)}.$$

The function $\beta_{k-1}(\mu)$ is symmetric, strictly increasing in $|\mu|$, and strictly increasing in k; its analytical expression is provided in Lemma 15 in Appendix C.3.

While the function $\beta_{k-1}(\cdot)$ captures the tail behavior of a single spot statistic, the power of the uniform test depends on the aggregate behavior across all constituents. To obtain a well-defined asymptotic power function, we impose the following regularity condition on the ϖ function, requiring the existence of the limit

$$\frac{1}{m_n}\sum_{j=1}^{m_n}\beta_{k-1}\left(\sqrt{k}\,|\varpi_{n,j}|\right)\to\bar{\beta}_{k-1}, \tag{13}$$

for some constant $\bar{\beta}_{k-1}$.

The limit $\bar{\beta}_{k-1}$ precisely characterizes how the signal-to-noise ratio process ϖ influences local power, serving as the key parameter indexing the asymptotic distribution of the sup-z statistic under the local alternative. We further let $\xi_{k-1}(\bar{\beta})$ denote a generalized Fréchet random variable with distribution function

$$\mathbb{P}\left(\xi_{k-1}(\bar{\beta})\le x\right)=\left[\mathbb{P}\left(\xi_{k-1}\le x\right)\right]^{\bar{\beta}},\quad x\in\mathbb{R}, \tag{14}$$

where ξ_{k-1} is the standard Fréchet limit from Theorem 4.

Equipped with these definitions, we are now ready to present the asymptotic behavior of the sup-z statistic and the uniform test under the local alternative.

Theorem 5. *Let $k\ge 6$ be fixed. Suppose that the process X follows* (11)*, satisfies Assumptions 1–3 with $\kappa>2/(k-1)$, and that the condition* (13) *holds. Then:*

(a) The normalized sup-z statistic converges in distribution to $\xi_{k-1}(\bar{\beta}_{k-1})$:

$$a_{m_n,k-1}^{-1}\left(\sup_{\mathrm{t}\in[0,\mathrm{T}]}\widehat{z}_{n,\mathrm{t}}-q_{m_n,k-1}\right)\xrightarrow{d}\xi_{k-1}(\bar{\beta}_{k-1}),$$

for the same normalizing constants $a_{m_n,k-1}$ and $q_{m_n,k-1}$ in Theorem 4.

(b) For any significance level $\alpha \in (0,1)$, the local power of the coupling-based test satisfies

$$\mathbb{P}\left(\sup_{\mathrm{t}\in[0,\mathrm{T}]} \widehat{z}_{n,\mathrm{t}} > z^*_{n,1-\alpha}\right) \to 1-(1-\alpha)^{\bar{\beta}_{k-1}},$$

where $z^*_{n,1-\alpha}$ denotes the coupling-based critical value from Theorem 4.

The key insight revealed by the theorem is the analytical form of the asymptotic power function, $1-(1-\alpha)^{\bar{\beta}_{k-1}}$, where the parameter $\bar{\beta}_{k-1}$ encapsulates the aggregate impact of the explosive drift on the tail behavior of the spot statistics. The condition (13) governing this parameter is mild; for standard estimation schemes—including both non-overlapping and fully overlapping windows—$\bar{\beta}_{k-1}$ is well-defined as the integral average

$$\bar{\beta}_{k-1} = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \beta_{k-1}\left(\sqrt{k}|\varpi_{\mathrm{t}}|\right)\, d\mathrm{t}.$$

Since $\beta_{k-1}(\cdot)$ is strictly increasing, the presence of a signal ensures $\bar{\beta}_{k-1} > 1$. Consequently, the logarithm of the Type II error probability, $\bar{\beta}_{k-1}\log(1-\alpha)$, decreases linearly as the aggregate signal strength $\bar{\beta}_{k-1}$ increases.[9]

This characterization offers a distinct heavy-tailed counterpart to the classical Gaussian power theory of Bickel and Rosenblatt (1973). As the signal manifests through a multiplicative power transformation of the limiting extreme value distribution, detection in finite samples is driven by the amplification of tail probabilities rather than by a simple shift in central tendency. In addition, the precise link between the signal-to-noise ratio process ϖ and the tail exponent $\bar{\beta}_{k-1}$ differs between Gaussian and Student-t approximations. The tail behavior under the Gaussian approximation is driven by mean statistics in a large-k framework, whereas the tail behavior under our Student-t approximation is driven by spot volatility estimation in a fixed-k framework. Our analysis establishes a new benchmark for power analysis using heavy-tailed statistics.

[9]This linear relationship implies that the semi-elasticity of the Type II error probability with respect to the signal strength $\bar{\beta}_{k-1}$ is constant and equal to $\ln(1-\alpha)$. In other words, a unit increase in $\bar{\beta}_{k-1}$ always yields the same proportional reduction in the failure-to-reject probability, regardless of the initial signal level.

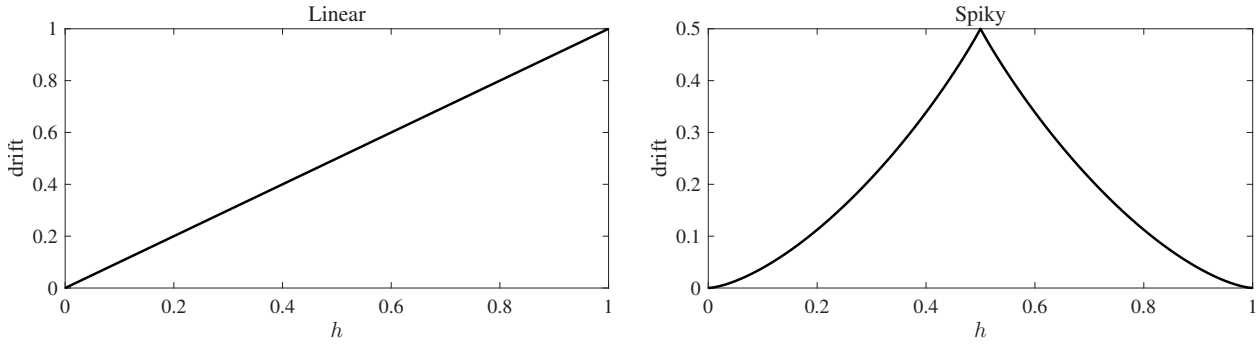


Figure 2: Explosive drift patterns in the simulation design. The cumulative signal $\int_0^h \omega(u)\,du$ is plotted as a function of h for the linear (left) or spiky (right) drift patterns.

4 Simulation Study

In this section, we conduct Monte Carlo experiments to examine the finite-sample performance of the proposed uniform t-test. With the unit time interval normalized to represent one trading day of $n = 390$ minutes, we simulate the semimartingale $X^{(n)}$ on $[0, 1]$ according to (11) and (12). The signal-to-noise ratio process is specified as

$$\varpi_{\mathrm{t}} = \begin{cases} c \cdot \omega\left(\dfrac{\mathrm{t} - \mathrm{t}^*}{\delta_0}\right), & \mathrm{t} \in (\mathrm{t}^*,\, \mathrm{t}^* + \delta_0), \\ 0, & \text{otherwise}, \end{cases}$$

where the signal persists for a fixed duration of $\delta_0 = 1$ hour (or $\delta_0 = 60/390$ day) and starts at a random time $t^* \sim \mathrm{Unif}(0,\, 1 - \delta_0)$. We consider two specifications of the function $\omega(\cdot)$: $\omega(u) \equiv 1$ and $\omega(u) = 2\,\mathrm{sign}(1 - 2u)\,F\big(1 - \big|1 - 2u\big|\big)$ with the arcsine distribution function $F(x) = 2/\pi \arcsin(\sqrt{x})$ for $0 \le x \le 1$. Figure 2 plots the cumulative signal $\int_0^h \omega(u)\,du$ over the one-hour event window, illustrating the resulting linear pattern under the constant design and the "spiky" pattern under the second design.

We perform 10,000 Monte Carlo replications to examine test size under the null hypothesis $c = 0$ and to evaluate power for alternatives with $c \in [-2,\, 2]$. Throughout, the sampling interval is fixed at $\Delta_n = 1$ minute (or $\Delta_n = 1/390$ day). We consider three choices of the window length, $k \in \{10,\, 15,\, 30\}$, corresponding to window sizes $k\Delta_n$ of 10, 15, and 30

minutes, respectively. In addition, we study two estimation schemes: (a) non-overlapping windows, for which $m_n = 1/(k\Delta_n) \in \{39, 26, 13\}$, and (b) rolling windows, where the window start is given by $i_j = j\Delta_n$ and $m_n = n - k + 1$.

Following Bollerslev and Todorov (2011), we simulate the spot variance process as

$$\sigma_{\mathrm{t}}^2 = h_{\mathrm{t}}^{(1)} + h_{\mathrm{t}}^{(2)},$$

where two stochastic volatility factors, $h^{(1)}$ and $h^{(2)}$, evolve according to

$$\begin{aligned} dh_{\mathrm{t}}^{(1)} &= 0.0128\left(0.4068 - h_{\mathrm{t}}^{(1)}\right)d\mathrm{t} + 0.0954\sqrt{h_{\mathrm{t}}^{(1)}}\left(\rho\, dW_{\mathrm{t}} + \sqrt{1-\rho^2}\, dB_{1,\mathrm{t}}\right), \\ dh_{\mathrm{t}}^{(2)} &= 0.6930\left(0.4068 - h_{\mathrm{t}}^{(2)}\right)d\mathrm{t} + 0.7023\sqrt{h_{\mathrm{t}}^{(2)}}\left(\rho\, dW_{\mathrm{t}} + \sqrt{1-\rho^2}\, dB_{2,\mathrm{t}}\right), \end{aligned}$$

where B_1 and B_2 are independent standard Brownian motions that are also independent of W. While Bollerslev and Todorov (2011) set $\rho = 0$, we choose $\rho = -0.7$ to capture the leverage effect, following Bollerslev, Li, and Liao (2021). The first volatility factor, $h^{(1)}$, is highly persistent, whereas the second factor, $h^{(2)}$, is fast mean-reverting. The process is initialized at $\sigma_0^2 = 1$ by setting $h_0^{(1)} = h_0^{(2)} = 0.5$.

We generate the jump component J as a compound Poisson process that is independent of W, B_1, and B_2, with intensity $\lambda = 3$. Jump sizes are specified as $S \cdot R$, where the sign variable S takes values $+1$ and -1 with equal probability 0.5. Conditional on $S = +1$, the jump magnitude R follows a Pareto distribution with mean 0.08 and shape parameter 5. Conditional on $S = -1$, R follows a Pareto distribution with the same mean but a heavier tail, characterized by a shape parameter of 3.

Table 1 reports the finite-sample size of our uniform t-test for three choices of critical values at significance levels $\alpha \in \{10\%, 5\%, 1\%\}$: the recommended coupling-based critical value $z^*_{n,1-\alpha}$, the closed-form Fréchet approximation (with finite-sample corrections given in Appendix A), and the conventional Gumbel approximation corresponding to $k \to \infty$. The results show that the Gumbel approximation performs well only when k is large and

Table 1: Size of the Uniform Test for Alternative Critical Values

	$k = 10$			$k = 15$			$k = 30$		
Significance Level	10%	5%	1%	10%	5%	1%	10%	5%	1%
Non-Overlapping									
Coupling	9.6	4.6	0.9	9.7	4.8	0.9	9.6	4.8	0.8
Fréchet	11.1	4.7	0.7	11.0	4.7	0.6	9.3	3.6	0.2
Gumbel	39.4	27.1	12.0	23.0	13.6	4.0	10.8	5.0	0.7
Overlapping									
Coupling	9.5	4.5	0.8	9.0	4.3	0.8	9.6	4.7	0.8
Fréchet	11.0	5.3	0.9	10.6	5.1	0.9	10.5	5.3	0.9
Gumbel	61.4	51.0	31.1	32.1	23.3	10.0	8.9	5.4	1.6

exhibits notable size distortions when k is small, whereas both the coupling-based and Fréchet approximations deliver accurate size control across all values of k. Overall, we recommend the coupling approach, as it provides robust finite-sample performance and adapts naturally to a wide range of empirical settings without requiring ad hoc corrections.

Figure 3 depicts the finite-sample power curves for 5%-level coupling-based tests over different window sizes $k \in \{10, 15, 30\}$. The top row reports results for the linear explosive-drift alternative with constant $\omega(\cdot)$, where the finite-sample power curves for both non-overlapping and overlapping windows are nearly indistinguishable. The bottom row presents results for the spiky explosive-drift alternative, illustrating that the test implemented with overlapping windows attains noticeably higher finite-sample power than its non-overlapping counterpart, which may be attributed to the former's ability to screen for signals in a more granular manner. Taken together, the comparison between the linear and spiky designs highlights the advantage of the overlapping window design in detecting nonlinear explosive-drift alternatives, which we recommend for practical applications.

5 Empirical Application

We apply the proposed uniform test to detect intraday explosive drift among S&P 500 constituent stocks listed at the beginning of 2023. For this analysis, we collect minute-

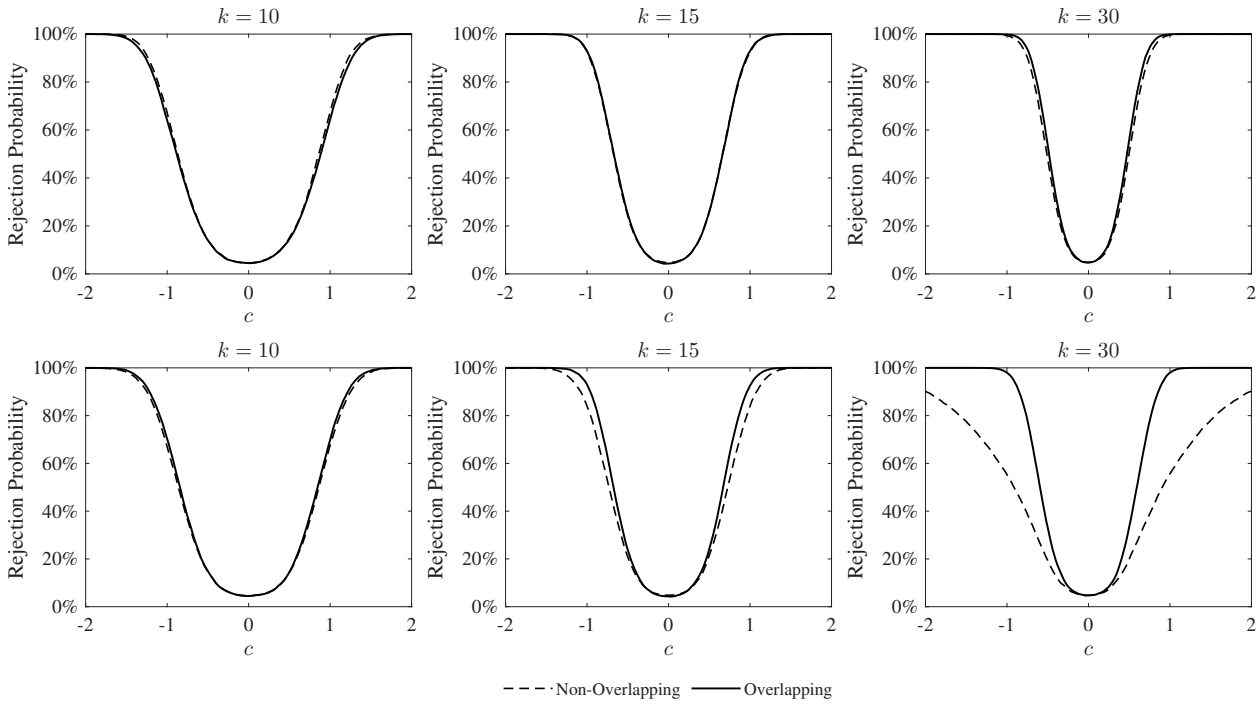


Figure 3: Finite-sample power curves of the uniform test using non-overlapping (dashed) and overlapping (solid) estimation windows with window size $k \in \{10, 15, 30\}$. The significance level is fixed at $\alpha = 5\%$. The top row corresponds to the linear explosive-drift alternative with constant $\omega(\cdot)$, while the bottom row reports results for the spiky explosive-drift alternative with $\omega(u) = 2\ \mathrm{sign}(1-2u)\, F\big(1 - \big|1-2u\big|\big)$ and the arcsine distribution function F.

by-minute return data for each stock over the full calendar year of 2023.[10] Motivated by the superior finite-sample performance of the overlapping window design in the simulation study, we employ this approach with 1-minute rolling steps, while noting that results based on non-overlapping windows are qualitatively similar and thus omitted for brevity.

For each stock and each trading day, we conduct the uniform test for intraday explosive drift at the 5% significance level, using critical values obtained from the recommended coupling approach. As demonstrated in the simulation study, the coupling-based method exhibits excellent size control, whereas the conventional extreme-value approach based on Gumbel asymptotics can suffer from substantial over-rejection and thus false detections of explosive drift. To highlight this contrast in the empirical setting, we also report results obtained using Gumbel-based critical values. We consider two window sizes, $k \in \{15, 30\}$

[10]The data are sourced from PiTrading.

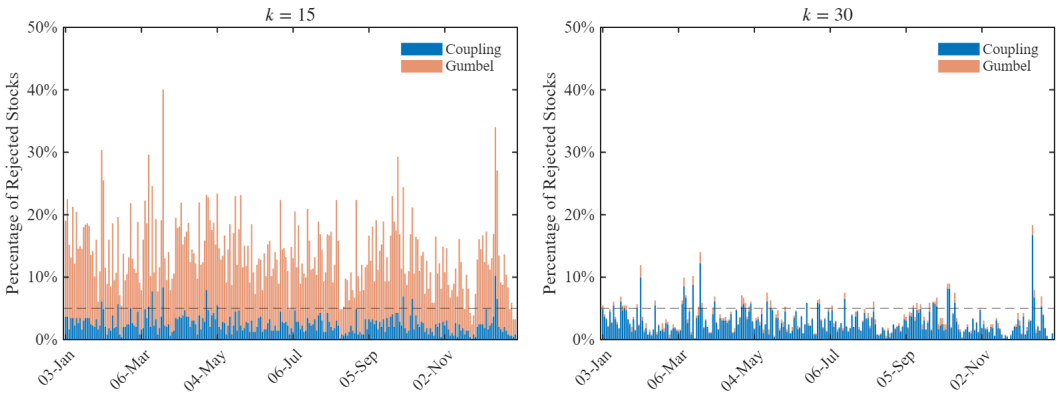


Figure 4: Daily Percentage of S&P 500 Stocks Rejecting the Null of No Explosive Drift in 2023. The dashed line indicates the 5% significance level.

corresponding to 15 and 30 minutes, respectively, to examine the sensitivity of results to this key tuning parameter. For ease of exposition, we summarize the findings by reporting daily rejection rates averaged across all S&P 500 constituent firms.

Figure 4 plots the daily rejection rates of the coupling- and Gumbel-based tests for $k = 15$ and $k = 30$, and yields several important empirical insights. First, the rejection rates based on the coupling approach are relatively stable with respect to the choice of window size k. Although the larger window leads to modestly higher rejection frequencies, which is consistent with the use of additional information, the overall magnitude and time-series patterns remain similar, indicating that the coupling-based test is not overly sensitive to the choice of k.

Second, the behavior of the Gumbel-based test differs markedly across window sizes. When $k = 30$, its rejection rates are broadly similar to those obtained under the coupling approach. In contrast, when $k = 15$, the Gumbel-based test rejects far more frequently, often by a wide margin. An empirical researcher relying solely on Gumbel-based inference—common in applications grounded in classical extreme value theory—may therefore reach drastically different conclusions about the prevalence of intraday explosive drift depending on the choice of window length. In such settings, one might be tempted to attribute the lower rejection rates at larger k to a "power effect," arguing that short-lived explosive-

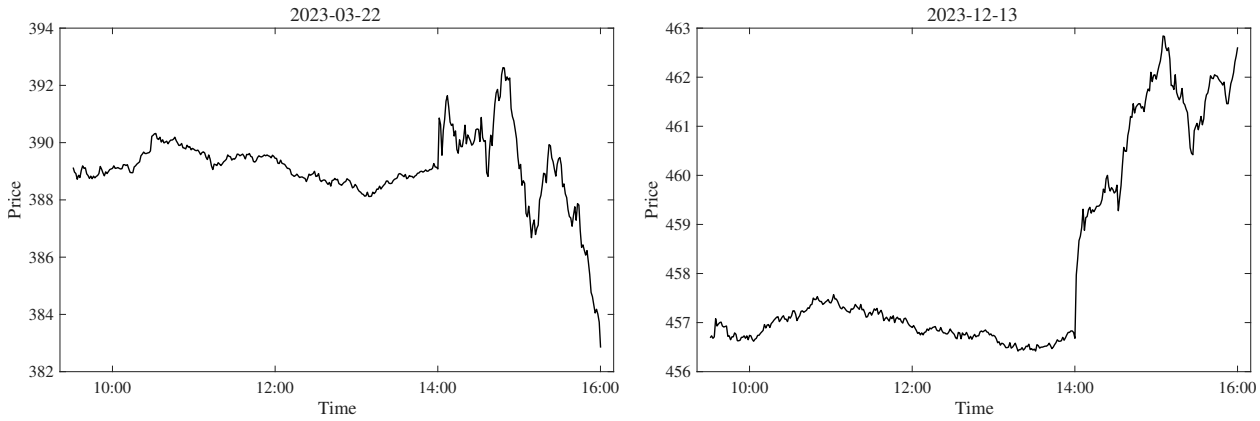


Figure 5: Price paths of the S&P 500 market portfolio on two FOMC announcement dates.

drift episodes are "averaged out" over longer estimation windows. While this interpretation contains some element of truth, it can be misleading in light of our earlier simulation evidence, which shows that Gumbel-based critical values may suffer from severe over-rejection even under the null hypothesis when k is moderate as considered here.

These findings underscore the superior robustness of the proposed coupling-based test. In contrast, the Gumbel-based approach is highly sensitive to the tuning parameter k and may produce misleading empirical conclusions due to size distortions. Importantly, these results are based on real data covering a large cross section of S&P 500 stocks over an extended sample period, and therefore provide a comprehensive empirical assessment rather than isolated case evidence. Consistent with the simulation results, inference based on the coupling-based test—which offers more reliable size control—yields substantially weaker evidence for widespread intraday explosive drift than what would be suggested by conventional extreme-value theory and Gumbel-based critical values.

Our main empirical findings discussed above indicate that episodes of intraday explosive drift are generally infrequent during normal trading days. This finding is consistent with the idea that such extreme local price dynamics are most likely to arise during periods of unusually intense trading activity, such as major news announcements. To illustrate how explosive drift manifests in practice, we next present two concrete case studies.

Specifically, we consider March 22 and December 13, which are both Federal Open Market

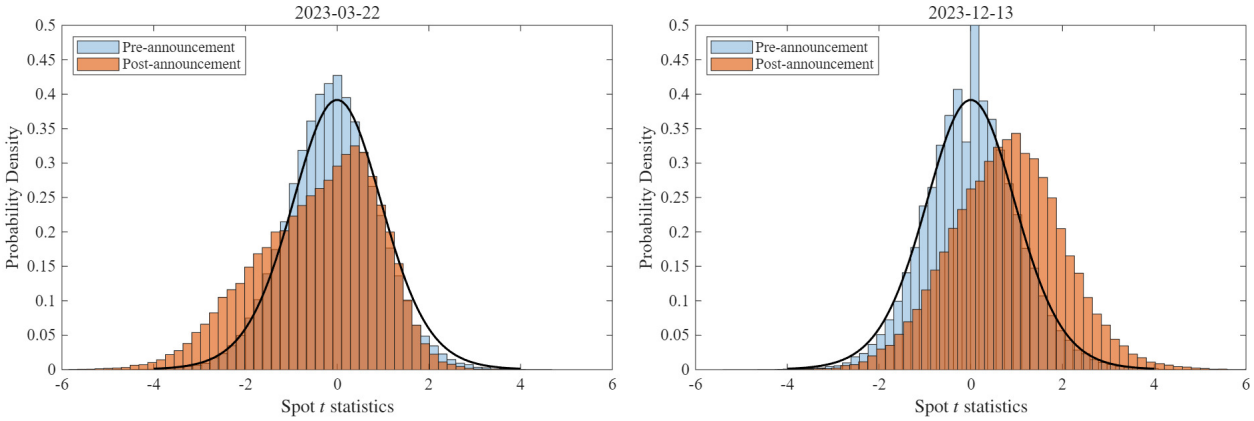


Figure 6: Empirical distributions of firm-level spot t-statistics on two FOMC announcement dates over pre- and post-announcement periods. The rolling window size is fixed at $k = 15$. The solid curve shows the density function of the coupling t_{14} distribution under the null hypothesis.

Committee (FOMC) announcement dates, as clear exceptions to the typical pattern. On these days, the daily rejection rates as shown in Figure 4 are 8.33% and 10.18% for $k = 15$ (and even larger for $k = 30$), respectively, well exceeding the nominal 5% level of the test. Figure 5 shows the intraday price paths of the market portfolio on these two days, revealing pronounced post-announcement drifts in opposite directions.

To illustrate more precisely how signals for explosive drift are revealed by the test statistics, we collect the spot t-statistics across all stocks computed over $k = 15$-minute rolling windows.[11] They are then separated into pre- and post-announcement groups. Figure 6 plots the empirical distributions for the two groups. As expected, before the announcement, the empirical distribution of the spot t-statistic closely aligns with the coupling t_{14} distribution implied by the null hypothesis, serving as a placebo check. In contrast, the empirical distributions of the post-announcement spot t-statistics are markedly distinct from the null distribution, strongly suggesting the occurrence of explosive drift across the stock market after the announcement.

Overall, the empirical evidence indicates that the proposed uniform test yields empirically sensible and economically interpretable results when applied to large panels of high-

[11] Results for $k = 30$ are qualitatively similar and therefore omitted for brevity.

frequency equity data. Once the substantial over-rejection inherent in conventional extreme-value–based methods is accounted for, intraday explosive drift appears to be a relatively rare phenomenon under normal trading conditions. At the same time, the case studies show that explosive drift remains economically important, as it arises sharply and systematically around unexpected news shocks—such as those associated with FOMC announcements—when trading activity and information flows are unusually intense. Taken together, these findings confirm both the reliability of the proposed inference procedure and its ability to distinguish economically meaningful episodes of explosive price dynamics from spurious statistical artifacts.

6 Conclusion

We propose a new framework for uniform inference on explosive drifts in high-frequency data. By treating the window size k as fixed, our approach departs from the conventional large-k paradigm to accommodate the empirically relevant setting of short estimation windows, which is essential for detecting short-lived explosive drift episodes. In such settings, conventional large-k asymptotics often fail because spot statistics are far from Gaussian. In contrast, our fixed-k theory is robust: it remains valid regardless of whether the window size is small or large, whereas the validity of standard Gaussian approximations hinges critically on k being sufficiently large.

Theoretically, our work addresses a challenging high-dimensional non-Donsker inference problem characterized by dependent, non-Gaussian coupling. Unlike prior literature that relies on Gaussian approximations and associated anti-concentration inequalities, we tackle the dependence inherent in overlapping windows by establishing a novel anti-clustering condition for the heavy-tailed Student-t process. This allows us to derive a Fréchet limiting distribution for the test statistic, in sharp contrast to the Gumbel limit typically found in Gaussian settings. We anticipate that this insight into handling dependent coupling variables is of

general interest for other non-Gaussian high-dimensional inference problems where standard Gaussian-type tools are inapplicable (see, e.g., Kiefer and Vogelsang (2002), Ibragimov and Müller (2010)).

Based on this theoretical foundation, we recommend the use of coupling-based critical values. Our analysis and simulations show that this method offers superior finite-sample performance, mimicking the behavior of the supremum statistic more accurately than asymptotic approximations. Crucially, the coupling-based approach is robust with respect to the key dimensions of the research design, including the window size k, the block number m_n, and the choice between overlapping versus non-overlapping windows.

Empirically, our study sheds new light on the nature of intraday price dynamics. By correcting the substantial size distortions inherent in conventional Gaussian- and Gumbel-based inference, we demonstrate that statistically significant explosive drifts are far rarer than previously suggested. This result aligns with the economic intuition that such extreme local price movements are exceptional phenomena, typically precipitated by major information shocks. Indeed, our case studies of FOMC announcements confirm that while rare, these events are economically meaningful and are cleanly detected by our procedure.

Appendix A Closed-Form Fréchet Approximation

This appendix collects the closed-form expressions underlying the Fréchet approximation. First, we report the standardization parameters appearing in Theorems 2–5. Second, we present the closed-form critical value based on the Fréchet approximation that is used in the simulation study, incorporating a finite-sample correction.

Appendix A.1 Standardization Parameters

For simplicity, we write m in place of m_n for the number of windows. We define the function ψ_k by

$$\psi_k(x) \equiv (k-1)\big[x^{2/(k-1)} - 1\big], \qquad x > 0.$$

The scale parameter is

$$a_{m,k-1} = \frac{q_{m,k-1}}{k-1} + \frac{1}{q^{(0)}_{m,k-1}}, \qquad q^{(0)}_{m,k-1} = \sqrt{\psi_k(2c_k m)}, \tag{15}$$

where $q_{m,k-1}$ is the location parameter to be defined later on and the constant c_k is given by

$$c_k = f_{k-1}(0) = \frac{1}{\sqrt{\pi(k-1)}} \frac{\Gamma(k/2)}{\Gamma((k-1)/2)}.$$

Here f_{k-1} denotes the density of the Student-t distribution with $k-1$ degrees of freedom, and Γ denotes the gamma function. The location parameter is

$$\begin{aligned} q_{m,k-1} = \; & \exp\left(-\frac{\log(k-1)}{2(k-1)}\right) q^{(0)}_{m,k-1} \\ & - \frac{1}{2}\left(1 + \frac{\left(q^{(0)}_{m,k-1}\right)^2}{k-1}\right)^{-1/2} \frac{\psi_k\left(q^{(0)}_{m,k-1}\right) - \psi_k(2c_k)}{q^{(0)}_{m,k-1}}. \end{aligned} \tag{16}$$

Both $a_{m,k-1}$ and $q_{m,k-1}$ admit leading terms that satisfy power laws. For the location parameter,

$$\lim_{m\to\infty} \frac{q_{m,k-1}}{m^{1/(k-1)}} = \exp\left(\frac{k-2}{2(k-1)} \log(k-1)\right) (2c_k)^{1/(k-1)}, \tag{17}$$

and for the scale parameter,

$$\lim_{m\to\infty} \frac{a_{m,k-1}}{q_{m,k-1}} = \frac{1}{k-1}. \tag{18}$$

In addition, $a_{m,k-1}$ and $q_{m,k-1}$ include second-order correction terms. These ensure consistency with their Gaussian counterparts (see the comment at the end of Section 2.4). Taking

the limit as $k \to \infty$ yields

$$q_m \equiv \lim_{k\to\infty} q_{m,k-1} = \frac{1}{a_m} - \frac{a_m}{2}\log\Big(\pi\log(m) - \frac{\pi}{2}\log(2/\pi)\Big)\,,$$
$$a_m \equiv \lim_{k\to\infty} a_{m,k-1} = \frac{1}{\sqrt{2\log(m) - \log(2/\pi)}}.$$

Appendix A.2 Critical Value for the Fréchet Approximation

For any given level $\alpha \in (0,1)$, we employ the following closed-form critical value, for the sup-z statistics to correct for the clustering effects in finite samples:

$$q_{m_n,k-1} + a_{m_n,k-1}\, Q_{\xi_{k-1}}\big((1-\alpha)^{1/\theta_n}\big)\,.$$

Here $Q_{\xi_{k-1}}$ denotes the quantile function of ξ_{k-1}, and $\theta_n \to 1$ is a finite-sample correction factor. The following specification performs well in our simulations:

$$\theta_n = \frac{1}{m_n - 1}\sum_{j=1}^{m_n-1} \theta_n(|\mathcal{I}_{n,j} \cap \mathcal{I}_{n,j+1}|)\,, \qquad \theta_n(\ell) = \frac{1}{\sqrt{2\ell}} + \left(1 - \frac{1}{\sqrt{2\ell}}\right)\frac{1}{1 + \ell/(\sqrt{2}\log n)}.$$

Here $|\mathcal{I}|$ denotes the cardinality of a set $\mathcal{I}$. We define $\theta_n(0) = \lim_{\ell\downarrow 0}\theta_n(\ell) = 1$. When the windows are non-overlapping, then $\theta_n \equiv 1$ and the critical value here coincides with the coupling-based critical value $z^*_{n,1-\alpha}$; see footnote 6 on page 12.

Appendix B Proofs of Pointwise Inference

In this appendix we prove Theorem 3 for pointwise inference. We work under the null hypothesis with the semimartingale X in (1). Since the theorem allows for a general covering mechanism, it implies Theorem 1 for non-overlapping windows as a special case.

Under Assumption 1, with probability approaching one, there is no jump in the interval $[\mathrm{t} - k\Delta_n,\, \mathrm{t} + k\Delta_n]$ containing all windows $\{\mathcal{T}_{n,j}\}_{j\in j_n(\mathrm{t})}$. Recall that $j_n(\mathrm{t}) = \{j : \mathrm{t} \in \mathcal{T}_{n,j}\}$

denotes the set of indices of blocks covering time t. It therefore suffices to consider the case in which $X_{\mathrm{s}} = X_{\mathrm{s}}^c$ for all $\mathrm{s} \in \bigcup_{j \in j_n(\mathrm{t})} \mathcal{T}_{n,j}$. We do not distinguish between X and X^c unless needed.

Before presenting the main steps of the argument, we state two intermediate lemmas. The first is a coupling result for the collection of t-statistics constructed from all blocks covering time t in the overlapping case.

Lemma 2. *Let* $\mathrm{t} \in [0, \mathrm{T}]$ *be fixed. Then*

$$\max_{j \in j_n(\mathrm{t})} \left| \widehat{t}_{n,j} - t_{n,j} \right| \xrightarrow{\mathbb{P}} 0.$$

The second lemma establishes an anti-concentration property for the supremum of absolute t-statistics over finitely many windows.

Lemma 3. *Let* $\{t_j^*\}_{j=1,2,\ldots}$ *be a (possibly dependent) sequence of Student-t random variables with* $\nu > 0$ *degrees of freedom. There exists a constant* $K > 0$*, depending only on* ν*, such that for any fixed* $r \geq 1$ *and all* $\delta \geq 0$*,*

$$\sup_{x \in \mathbb{R}} \mathbb{P}\left(x - \delta < \max_{1 \leq j \leq r} |t_j^*| \leq x + \delta \right) \leq rK\delta.$$

The proof of Lemma 2 follows similar arguments as in Bollerslev, Li, and Liao (2021) and Bollerslev, Li, and Ren (2024), with modifications to accommodate multiple windows and the t-statistic. The proof of Lemma 3 is elementary, relying only on the uniform boundedness of the Student-t density. Technical details are provided in the supplement (He, Li, Li, and Zhu, 2026). Now we are ready to prove the theorem.

Proof of Theorem 3. It follows directly from Lemma 2 that

$$\widehat{z}_{n,\mathrm{t}} = z_{n,\mathrm{t}} + o_{\mathbb{P}}(1).$$

Now take any small $\delta > 0$.

$$\mathbb{P}\left(\widehat{z}_{n,\mathrm{t}} > c_{n,\alpha}\right) \leq \mathbb{P}\left(\widehat{z}_{n,\mathrm{t}} > c_{n,\alpha}, |\widehat{z}_{n,\mathrm{t}} - z_{n,\mathrm{t}}| \leq \delta\right) + \mathbb{P}\left(|\widehat{z}_{n,\mathrm{t}} - z_{n,\mathrm{t}}| > \delta\right)$$

Therefore, for all large n,

$$\begin{aligned}\mathbb{P}\left(\widehat{z}_{n,\mathrm{t}} > c_{n,\alpha}\right) \leq & \mathbb{P}\left(z_{n,\mathrm{t}} + \delta > c_{n,\alpha}\right) + \delta \\ \leq & \mathbb{P}\left(z_{n,\mathrm{t}} > c_{n,\alpha}\right) + kK\delta + \delta = \alpha + kK\delta + \delta,\end{aligned}$$

with some constant $K > 0$ not depending on n by Lemma 3, as each time t can be covered by at most k windows. Similarly, for all large n,

$$\begin{aligned}\mathbb{P}\left(\widehat{z}_{n,\mathrm{t}} > c_{n,\alpha}\right) \geq & \mathbb{P}\left(z_{n,\mathrm{t}} - \delta > c_{n,\alpha}\right) - \delta \\ \geq & \mathbb{P}\left(z_{n,\mathrm{t}} > c_{n,\alpha}\right) - kK\delta - \delta = \alpha - kK\delta - \delta.\end{aligned}$$

Since δ is arbitrary, we conclude that $\mathbb{P}\left(\widehat{z}_{n,\mathrm{t}} > c_{n,\alpha}\right) \to \alpha$. □

Appendix C Proofs of Uniform Inference

In this appendix we outline the main steps in the proof of Theorem 5 for simultaneous inference. The results for non-overlapping windows arise as special cases of those for overlapping windows with $i_j = (j-1)k$. The null hypothesis is also a special case of the general model (11). Consequently, Theorems 2 and 4 follow directly as corollaries.

By a standard localization argument, we can and will strengthen Assumptions 1 and 2 to hold with $T_1 = \infty$ without loss of generality; see Section 4.4.1 of Jacod and Protter (2012). Throughout, K denotes a generic positive constant that may change from line to line, and K_p highlights its dependence on a parameter p. For brevity, we write $X^{(n)}$ as X when no confusion arises, and denote by $X^c = X - J$ the continuous component, excluding jumps.

We divide the proofs into four sections. The first three introduce key lemmas. The first bounds the coupling error (Section Appendix C.1). The second establishes an extreme value result for possibly non-stationary coupled statistics (Section Appendix C.2). The third extends the anti-clustering property in Lemma 1 to the coupled t-statistics under the general model (11) (Section Appendix C.3). These results may rely on additional intermediate lemmas, whose proofs are provided in the supplement (He, Li, Li, and Zhu, 2026). We then use these lemmas to establish Theorem 5 in Section Appendix C.4.

Appendix C.1 A Lemma for Uniform Coupling

Lemma 4. *Let $k \geq 6$ be fixed. Let $X = X^{(n)}$ follow the model (11) and satisfy Assumptions 1–3 with $\kappa > 2/(k-1)$. Then, for all $0 < \iota < \min\{\kappa - 2/(k-1),\, 1/2 - 2/(k-1),\, 1/(k-1)\}$*

$$a_{m_n,k-1}^{-1} \sup_{\mathrm{t}\in[0,\mathrm{T}]} \left|\widehat{z}_{n,\mathrm{t}} - z_{n,\mathrm{t}}^{a}\right| = o_{\mathbb{P}}\left(\Delta_n^{\iota}\right), \quad z_{n,\mathrm{t}}^{a} = \max_{j\in j_n(\mathrm{t})} |t_{n,j}^{a}|. \tag{19}$$

If there are no jumps, that is, if $J \equiv 0$ and $J^{\varpi} \equiv 0$, the result holds for all $0 < \iota < \min\{\kappa,\, 1/2\} - 2/(k-1)$.

To establish this lemma, we work with the general sequence of semimartingales $X = X^{(n)}$ defined in (11). For brevity, we write $X^{(n)}$ simply as X whenever no confusion arises, and denote by $X^c = X - J$ its continuous component, excluding the jump process J. We will invoke several auxiliary lemmas. Throughout this section, we assume that the conditions of Lemma 4 hold unless stated otherwise. For convenience, we introduce the process μ and its continuous component μ^c,

$$\mu_{\mathrm{t}} = \varpi_{\mathrm{t}}\sigma_{\mathrm{t}}, \qquad \mu_{\mathrm{t}}^{c} = \varpi_{\mathrm{t}}^{c}\sigma_{\mathrm{t}}, \qquad \mathrm{t} \in [0, \mathrm{T}],$$

and, when convenient, work with μ rather than ϖ.

We begin with some preliminary lemmas. The first shows that the continuous component μ^c satisfies a smoothness condition analogous to that imposed on σ.

Lemma 5. *Under Assumptions 1-3, there exists some constant $\kappa > 0$ such that for each $p \geq 2$, and any stopping time T_m from Assumption 1, there exist constants $K_{m,p}$ such that*

$$\mathbb{E}[\left|\mu^c_{\mathrm{t}\wedge\mathrm{T}_m} - \mu^c_{\mathrm{s}\wedge\mathrm{T}_m}\right|^p] \leq K_{m,p}|\mathrm{t}-\mathrm{s}|^{\kappa p}, \quad \forall \mathrm{t},\mathrm{s} \in [0,\mathrm{T}].$$

The second lemma provides estimates for the returns of the continuous part X^c of X, defined by $\Delta^n_i X^c \equiv X^c_{i\Delta_n} - X^c_{(i-1)\Delta_n}$.

Lemma 6. *For all $i \in \mathcal{I}_{n,j}$ and $\tau(n,j) \equiv \inf \mathcal{T}_{n,j}$, define*

$$\delta^{(r)}_{n,i,j} \equiv \left(\frac{\Delta^n_i X^c}{\sqrt{\Delta_n}} - \frac{1}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n} \mu_{\mathrm{s}} d\mathrm{s} - \frac{1}{\sqrt{\Delta_n}}\int_{(i-1)\Delta_n}^{i\Delta_n} b_{\mathrm{s}} d\mathrm{s}\right)^r - \left(\sigma_{\tau(n,j)}\frac{\Delta^n_i W}{\sqrt{\Delta_n}}\right)^r.$$

Then for all $0 < \epsilon < \kappa$ and $r \in \{1,2\}$,

$$\max_{1\leq j\leq m_n} \max_{i\in\mathcal{I}_{n,j}} \left|\delta^{(r)}_{n,i,j}\right| = o_{\mathbb{P}}(\Delta^{\epsilon}_n).$$

The next lemma is a well-known bound for the standard normal survival distribution.

Lemma 7. *Let $Z \sim N(0,1)$ be a standard normal variable. Then, for all $x > 0$*

$$\mathbb{P}(|Z| > x) \leq x^{-1}\sqrt{2/\pi}\exp\left(-x^2/2\right).$$

Using this lemma one can obtain the following bound for the maximum of Gaussian variables. The bound is not sharp but sufficient for our proofs.

Lemma 8. *Let $Z_{n,1},\ldots,Z_{n,n}$ be i.i.d. standard normal variables for every integer n. For any sequence $n \to \infty$,*

$$\max_{1\leq i\leq n} |Z_i| = o_{\mathbb{P}}(\log n).$$

From the previous lemmas, we can deduce the coupling results for both the mean and variance statistics. We present these results in two separate lemmas: one for windows without

jumps, and the other for windows that contain jumps. For the ease of exposition, we shall define the following set of indices to identify windows without jumps:

$$\mathcal{J}_n = \{1 \leq j \leq m_n : \Delta J_{\mathrm{t}} = 0, \Delta J^{\varpi}_{\mathrm{t}} = 0 \ \forall \mathrm{t} \in (\tau(n,j), \tau(n,j) + k\Delta_n]\}\,. \tag{20}$$

The corresponding set of window indices with jumps is thus defined as the complement of $\mathcal{J}_n$, i.e., $\mathcal{J}_n^{\complement} = \{1 \leq j \leq m_n, j \notin \mathcal{J}_n\}$. The following lemma establishes the coupling results on windows without jumps.

Lemma 9. *For any* $0 < \epsilon < \min\{1/2, \kappa\}$,

$$\max_{j \in \mathcal{J}_n} \max_{i \in \mathcal{I}_{n,j}} \left| \frac{1}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} \mu_{\mathrm{s}} d\mathrm{s} - \mu_{\tau(n,j)} \right| = o_{\mathbb{P}}(\Delta_n^{\epsilon}).$$

Lemma 10. *For any* $0 < \epsilon < \min\{1/2, \kappa\}$,

(i) $\max_{j \in \mathcal{J}_n} \left| \bar{X}_{n,j} - \mu_{\tau(n,j)} - \sigma_{\tau(n,j)} \bar{W}_{n,j} \right| = o_{\mathbb{P}}(\Delta_n^{\epsilon})$;

(ii) $\max_{j \in \mathcal{J}_n} \left| S^2_{n,j} - \sigma^2_{\tau(n,j)} V^2_{n,j} \right| = o_{\mathbb{P}}(\Delta_n^{\epsilon})$.

Lemma 11. *Recall the index set of the windows containing no jumps* $\mathcal{J}_n$ *from* (20). *Then, for all* $0 < \iota < \min\{\kappa, 1/2\} - 2/(k-1)$,

$$a^{-1}_{m_n, k-1} \max_{j \in \mathcal{J}_n} \left| \widehat{t}_{n,j} - t^a_{n,j} \right| = o_{\mathbb{P}}(\Delta_n^{\iota})$$

We now study the windows with jumps indexed by the set $\mathcal{J}_n^{\complement}$.

Lemma 12. *It holds as* $n \to \infty$ *that* $\max_{j \in \mathcal{J}_n^{\complement}} \left| t^a_{n,j} \right| = O_{\mathbb{P}}(1)$ *and* $\max_{j \in \mathcal{J}_n^{\complement}} \left| \widehat{t}_{n,j} \right| = O_{\mathbb{P}}(1)$.

Finally, we are ready to prove Lemma 4.

Proof of Lemma 4. First consider ι such that $0 < \iota < \min\{\kappa, 1/2\} - 2/(k-1)$. The triangle

inequality gives that

$$a_{m_n,k-1}^{-1} \sup_{\mathrm{t}\in[0,\mathrm{T}]} \left|\widehat{z}_{n,\mathrm{t}} - z_{n,\mathrm{t}}^{a}\right| \leq a_{m_n,k-1}^{-1} \max_{1\leq j\leq m_n} \left|\widehat{t}_{n,j} - t_{n,j}^{a}\right|$$

Furthermore,

$$\begin{aligned} & a_{m_n,k-1}^{-1} \max_{1\leq j\leq m_n} \left|\widehat{t}_{n,j} - t_{n,j}^{a}\right| \\ \leq & a_{m_n,k-1}^{-1} \max_{j\in\mathcal{J}_n} \left|\widehat{t}_{n,j} - t_{n,j}^{a}\right| + a_{m_n,k-1}^{-1} \max_{j\in\mathcal{J}_n^{\complement}} \left|\widehat{t}_{n,j}\right| + a_{m_n,k-1}^{-1} \max_{j\in\mathcal{J}_n^{\complement}} \left|t_{n,j}^{a}\right|. \end{aligned}$$

By Lemma 11, the first term is of order $o_{\mathbb{P}}(\Delta_n^{\iota})$. If there is no jumps, that is, $\mathcal{J}_n^{\complement} = \emptyset$, then the last two terms vanish and the lemma holds.

When there are jumps (of finite activities), use Lemma 12 and the power law of $a_{m_n,k-1}$ according to (17) and (18) to obtain that

$$a_{m_n,k-1}^{-1} \max_{j\in\mathcal{J}_n^{\complement}} \left|\widehat{t}_{n,j}\right| = O_{\mathbb{P}}(\Delta_n^{1/(k-1)}) \cdot O_{\mathbb{P}}(1) = o_{\mathbb{P}}(\Delta_n^{\iota}),$$

and similarly

$$a_{m_n,k-1}^{-1} \max_{j\in\mathcal{J}_n^{\complement}} \left|t_{n,j}^{a}\right| = O(\Delta_n^{1/(k-1)}) \cdot O_{\mathbb{P}}(1) = o_{\mathbb{P}}(\Delta_n^{\iota})$$

for ι above but satisfying the additional constraint $\iota < 1/(k-1)$. □

Appendix C.2 A Lemma for Nonstationary Extremes

A technical complication arises in deriving the asymptotic distribution of the coupled maximum statistics with overlapping windows. Existing extreme value theory for time-dependent variables is developed for sequences, whereas our setting involves arrays indexed by the discretization level (that is, the sample size n). To address this issue, the following lemma extends the framework of Hüsler (1983, 1986) for anti-clustering nonstationary sequences to an array setting. This extension allows for local alternatives indexed by n. The arguments

are largely analogous, except that sequential convergence is replaced by convergence along all sequences indexed by n. We therefore omit the details and refer the reader to the supplement (He, Li, Li, and Zhu, 2026).

Lemma 13. *Let $\{B_{n,j} : 1 \le j \le m_n\}$ be an array of events indexed by n, where $m_n \to \infty$ as $n \to \infty$. Assume that $m_n \mathbb{P}\big(B_{n,j}^{\complement}\big) = m_n\big(1 - \mathbb{P}(B_{n,j})\big)$ is bounded away from zero and infinity uniformly in n and j, where $B_{n,j}^{\complement}$ denotes the complement of $B_{n,j}$. Suppose that the following conditions hold:*

(a) There exists a positive integer k such that $B_{n,j}$ and $B_{n,j'}$ are independent whenever $|j - j'| \ge k$.

(b) $\max_{1 \le j < j' \le m_n} \mathbb{P}\big(B_{n,j'}^{\complement} \,\big|\, B_{n,j}^{\complement}\big) \to 0$.

Then, as $n \to \infty$,

$$\mathbb{P}\left(\bigcap_{j=1}^{m_n} B_{n,j}\right) - \prod_{j=1}^{m_n} \mathbb{P}(B_{n,j}) \to 0.$$

This lemma requires sufficient anti-clustering conditions in the proportional framework. Under these conditions, the joint probability of $(k-1)$-dependent events is asymptotically equivalent to that under independence. In particular, condition (a) holds trivially for the tail events $B_{n,j} = \big\{|t_{n,j}^a| \le u_n\big\}$, $1 \le j \le m_n$. With an appropriately chosen threshold $u_n \to \infty$, we obtain

$$\mathbb{P}\left(\bigcap_{j=1}^{m_n} B_{n,j}\right) = \mathbb{P}\left(\max_{1 \le j \le m_n} |t_{n,j}^a| \le u_n\right) = \prod_{j=1}^{m_n} \mathbb{P}\big(|t_{n,j}^a| \le u_n\big) + o(1),$$

where condition (b) is verified in the next section.

Appendix C.3 A Lemma for Tail Independence

In this section, we prove a stronger result than Lemma 1 on the tail dependence of coupled t-statistics. We work under the general model (11), which nests both the null and local alternatives.

Lemma 14. *The noncentral Student-t variables $t^a_{n,j}$ are uniformly tail independent. That is, for any positive sequence $u_n \to \infty$,*

$$\max_{1\le j<j'\le m_n} \mathbb{P}\left(\left|t^a_{n,j'}\right| > u_n \,\middle|\, \left|t^a_{n,j}\right| > u_n\right) \to 0.$$

To establish this result, we require several auxiliary lemmas. Recall the noncentral t distribution function $\widetilde{F}_{\nu,\mu}$ defined in Section 3. For brevity, write $\widetilde{F}_\nu \equiv \widetilde{F}_{\nu,0}$, the half-$t$ distribution function under the null hypothesis $\varpi \equiv 0$. We first establish the following uniform refinement of Breiman's theorem for noncentral t distribution; see Proposition 7.5 of Resnick (2007).

Lemma 15. *Let G denote the distribution function of $\sqrt{\nu}V_\nu^{-1}$. Uniformly for μ in any compact set of $\mathbb{R}$, as $u \to \infty$*

$$\frac{1-\widetilde{F}_{\nu,\mu}(u)}{1-G(u)} \to \mathbb{E}|Z+\mu|^\nu$$

and consequently

$$\frac{1-\widetilde{F}_{\nu,\mu}(u)}{1-\widetilde{F}_\nu(u)} \to \frac{\mathbb{E}|Z+\mu|^\nu}{\mathbb{E}|Z|^\nu} \equiv \beta_\nu(\mu).$$

The next lemma gives bounds for the Mills ratio of Student-t distributions.

Lemma 16. *Let*

$$f_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\Gamma(\nu/2)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}, \quad F_\nu(x) = \int_{-\infty}^x f_\nu(u)du, \quad x \in \mathbb{R},$$

denote the probability density function and the cumulative distribution function of the Student-t distribution with ν degrees of freedom, respectively. Then, for all $x > 0$

$$1 - \frac{\nu}{(\nu+2)x^2} < \frac{1-F_\nu(x)}{f_\nu(x)} \cdot \frac{x}{1+x^2/\nu} < 1.$$

Now we are ready to prove Lemma 14.

Proof of Lemma 14. The results are trivial uniformly for $j \neq j'$ such that $\mathcal{I}_{n,j} \cap \mathcal{I}_{n,j'} = \emptyset$,

for which $t_{n,j}$ and $t_{n,j'}$ satisfy

$$\mathbb{P}\left(|t^a_{n,j}| > u_n \mid |t^a_{n,j'}| > u_n\right) = \mathbb{P}\left(|t^a_{n,j}| > u_n\right) \to 0.$$

We only need to consider the cases $\mathcal{I}_{n,j} \cap \mathcal{I}_{n,j'} \neq \emptyset$. All results will hold uniformly over j, j' and we do not repeat this argument for simplicity. Let $\delta_n \equiv \log u_n/u_n \to 0$ such that $u_n\delta_n = \log u_n \to \infty$. Then we have

$$\begin{aligned}
&\mathbb{P}\left(|t^a_{n,j}| > u_n, |t^a_{n,j'}| > u_n\right) \\
\leq &\mathbb{P}\left(\frac{1}{V_{n,j}} > \frac{1}{\delta_n}, \frac{1}{V_{n,j'}} > \frac{1}{\delta_n}\right) \\
&+ \mathbb{P}\left(\sqrt{k}\left|\bar{W}_{n,j} + \varpi_{n,j}\right| > u_n\delta_n, \text{ or } \sqrt{k}\left|\bar{W}_{n,j'} + \varpi_{n,j'}\right| > u_n\delta_n\right) \\
\leq &\mathbb{P}\left(V_{n,j} < \delta_n, V_{n,j'} < \delta_n\right) + 2\mathbb{P}\left(\sqrt{k}\left|\bar{W}_{n,j}\right| > \log u_n/2\right),
\end{aligned} \tag{21}$$

where the last step used the uniform boundedness of ϖ:

$$\max_{1\leq j\leq m_n} |\varpi_{n,j}| \leq \sup_{\mathrm{t}\in[0,\mathrm{T}]} |\varpi_{\mathrm{t}}| < \log u_n/(2\sqrt{k}),$$

for all large n, as $\log u_n \to \infty$.

For all large $u_n > 1$, from Lemmas 15 and 16, we know that

$$\mathbb{P}\left(|t^a_{n,j}| > u_n\right) \asymp u_n^{-(k-1)}, \quad u_n \to \infty, \tag{22}$$

uniformly over $1 \leq j \leq m_n$. Looking at the second term in (21), Lemma 7 gives that

$$\mathbb{P}\left(\sqrt{k}\left|\bar{W}_{n,j}\right| > \log u_n/2\right) \leq \frac{1}{\log u_n/2}\sqrt{\frac{2}{\pi}}\exp\left(-\frac{(\log u_n/2)^2}{2}\right).$$

Therefore, combining the last two displayed equations, as $u_n \to \infty$

$$\frac{\mathbb{P}\left(\sqrt{k}\left|\bar{W}_{n,j}\right| > u_n\delta_n\right)}{\mathbb{P}\left(|t^a_{n,j}| > u_n\right)} = O\left(\frac{1}{\log u_n}\exp\left(\frac{k-1}{2}\log u_n - \frac{(\log u_n)^2}{8}\right)\right) \to 0. \tag{23}$$

To complete the proof, we claim that

$$\mathbb{P}\left(V_{n,j} < \delta_n, V_{n,j'} < \delta_n\right) = O(\delta_n^k) \tag{24}$$

and then it follows (22) that

$$\mathbb{P}\left(V_{n,j} < \delta_n, V_{n,j'} < \delta_n\right) = O\left(\frac{(\log u_n)^k}{u_n^k}\right) = o(u_n^{-(k-1)}) = o(\mathbb{P}(|t^a_{n,j}| > u_n)).$$

To this end, denote by $\ell = i_{j'} - i_j$ for some $1 \le \ell \le k-1$, so that ℓ is the number of sampling intervals between the starting time of the windows. We will establish (24) separately in two difference cases: (1) $\ell \ge 2$ and (2) $\ell = 1$.

First consider the case $\ell \ge 2$. Using the well-known fact that the sample mean minimizes the sum squared errors, for all $j < j'$, with probability 1

$$(k-1)V^2_{n,j'} = \sum_{i\in\mathcal{I}_{j'}}\left(\Delta^n_i W - \bar{W}_{n,j}\right)^2 \ge \sum_{i\in\mathcal{I}_{j'}\setminus\mathcal{I}_{n,j}}\left(\Delta^n_i W - \bar{W}_{n,j}\right)^2$$

where the last term is bounded below by

$$\min_{x\in\mathbb{R}}\sum_{i\in\mathcal{I}_{j'}\setminus\mathcal{I}_{n,j}}\left(\Delta^n_i W - x\right)^2 = \sum_{i\in\mathcal{I}_{j'}\setminus\mathcal{I}_{n,j}}\left(\Delta^n_i W - \ell^{-1}\sum_{i\in\mathcal{I}_{j'}\setminus\mathcal{I}_{n,j}}\Delta^n_i W\right)^2 \equiv \Delta_{n,j',j}.$$

Hence, uniformly for $j < j'$ with $\ell \geq 2$,

$$\begin{aligned}\mathbb{P}\left(V_{n,j} < \delta_n, V_{n,j'} < \delta_n\right) =& \mathbb{P}\left((k-1)V_{n,j}^2 < (k-1)\delta_n^2, (k-1)V_{n,j'}^2 < (k-1)\delta_n^2\right) \\ \leq& \mathbb{P}\left((k-1)V_{n,j}^2 < (k-1)\delta_n^2, \Delta_{n,j',j} < (k-1)\delta_n^2\right) \\ =& \mathbb{P}\left((k-1)V_{n,j}^2 < (k-1)\delta_n^2\right) \cdot \mathbb{P}\left(\Delta_{n,j',j} < (k-1)\delta_n^2\right)\end{aligned}$$

where in the last step we noted that $\Delta_{n,j',j}$ is independent of $(k-1)V_{n,j}$ because they use the returns on the underlying Brownian motion in different time intervals. Now using the well-known fact that $(k-1)V_{n,j}^2 = W_{n,j}' C_k W_{n,j} \sim \chi_{k-1}^2$ with the demeaning matrix C_k of rank $k-1$ and $W_{n,j} = \Delta_n^{-1/2}(\Delta_i^n W)'_{i \in \mathcal{I}_{n,j}}$ denotes the k-by-1 standard normal vector for all returns on $\mathcal{I}_{n,j}$, and similarly $\Delta_{n,j',j} \sim \chi_{\ell-1}^2$. We can conclude that, as $\delta_n \downarrow 0$

$$\begin{aligned}\mathbb{P}\left(V_{n,j} < \delta_n, V_{n,j'} < \delta_n\right) =& O\left(F_{\chi_{k-1}^2}((k-1)\delta_n^2) \cdot F_{\chi_{\ell-1}^2}\left((k-1)\delta_n^2\right)\right) \\ =& O\left(\left((k-1)\delta_n^2\right)^{(k-1)/2} \cdot \left((k-1)\delta_n^2\right)^{(\ell-1)/2}\right) \\ =& O(\delta_n^{k+\ell-2}) = O(\delta_n^k),\end{aligned}$$

where $F_{\chi_\nu^2}(\cdot)$ denotes the χ_ν^2 distribution function and the last step holds for $\ell \geq 2$.

It remains to show (24) for $\ell = 1$. First, we bound the joint probability on the left-hand-side by

$$\begin{aligned}\mathbb{P}\left(V_{n,j} < \delta_n, V_{n,j'} < \delta_n\right) =& \mathbb{P}\left((k-1)V_{n,j}^2 < (k-1)\delta_n^2, (k-1)V_{n,j'}^2 < (k-1)\delta_n^2\right) \\ \leq& \mathbb{P}\left((k-1)V_{n,j}^2 + (k-1)V_{n,j'}^2 < 2(k-1)\delta_n^2\right),\end{aligned}$$

where the summation term in the last term, namely,

$$(k-1)V_{n,j}^2 + (k-1)V_{n,j'}^2 = W_{n,j}' C_k W_{n,j} + W_{n,j'}' C_k W_{n,j'}$$

with the demeaning matrix C_k of rank $k-1$ and $W_{n,j} = \Delta_n^{-1/2}(\Delta_i^n W)'_{i \in \mathcal{I}_{n,j}}$ denotes the

k-by-1 standard normal vector for all returns on $\mathcal{I}_{n,j}$, and $W_{n,j'}$ similarly. Now rewrite this sum of quadratic form into a single quadratic form to get

$$(k-1)V_{n,j}^2 + (k-1)V_{n,j'}^2 = W'_{n,j,j'} A_{k+1} W_{n,j,j'}$$

where $W_{n,j,j'} = \Delta_n^{-1/2}(\Delta_i^n W)'_{i\in\mathcal{I}_{n,j}\cup\mathcal{I}_{j'}}$ denotes the $(k+1)$-by-1 standard normal vector for all returns on $\mathcal{I}_{n,j} \cup \mathcal{I}_{n,j'}$, and the weight matrix $A_{k+1} \in \mathbb{R}^{(k+1)\times(k+1)}$ is given by

$$A_{k+1} := A_{k+1,1} + A_{k+1,2} := \begin{bmatrix} C_k & \mathbf{0}_k \\ \mathbf{0}'_k & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0}'_k \\ \mathbf{0}_k & C_k \end{bmatrix}$$

Observe that $A_{k+1,1}$ has eigenvectors, say, $v_1, \ldots, v_{k-1}$ associated with eigenvalue of 1, orthogonal to the standard basis vector $\mathbf{e}_{k+1} = (0, \ldots, 0, 1)' \in \mathbb{R}^{k+1}$ that has 1 on the last entry but zero elsewhere. Let $\lambda_l(A)$ denotes the l-th largest eigenvalue of matrix A. While the top left $k \times k$ block of $A_{k+1,2}$ is positive semi-definite and observe that $e'_{k+1} A_{k+1} v_l = e'_{k+1} v_l = 0$ for all $1 \le l \le k-1$, the min-max theorem of eigenvalues gives

$$\begin{aligned} \lambda_k(A_{k+1}) &\ge \min\{x' A_{k+1} x : x \in \operatorname{span}(v_1, \ldots, v_{k-1}, \mathbf{e}_{k+1})\} \\ &\ge \min\{\lambda_{k-1}(C_k), 1 - 1/k\} = 1 - 1/k > 0, \end{aligned}$$

where $1 - 1/k$ is the last entry of $A_{k+1,2}$. Using the spectral decomposition of A_{k+1} and the well-known decomposition of quadratic forms of Gaussian variables as linear combination of independent chi-squared variables $\chi^2_{1,j,j',l} \sim \chi^2_1$,

$$W'_{n,j,j'} A_{k+1} W_{n,j,j'} = \sum_{l=1}^{k+1} \lambda_l(A_{k+1}) \chi^2_{1,j,j',l} \ge \sum_{l=1}^{k} \lambda_l(A_{k+1}) \chi^2_{1,j,j',l} \ge (1 - 1/k) Q_{j,j'},$$

where first inequality is due to $\lambda_{k+1}(A_{k+1}) \geq 0$, and in the last step

$$Q_{j,j'} \equiv \sum_{l=1}^{k} \chi^2_{1,j,j',l} \sim \chi^2_k.$$

Hence, we can conclude that, for all $j < j'$ with $\ell = 1$, as $\delta_n \downarrow 0$

$$\begin{aligned}
\mathbb{P}\left(V_{n,j} < \delta_n, V_{n,j'} < \delta_n\right) \leq & \mathbb{P}\left(W'_{n,j,j'} A_{k+1} W_{n,j,j'} < 2(k-1)\delta_n^2\right) \\
\leq & \mathbb{P}\left((1-1/k) Q_{j,j'} < 2(k-1)\delta_n^2\right) \\
= & \mathbb{P}\left(Q_{j,j'} < 2k\delta_n^2\right) = F_{\chi^2_k}\left(2k\delta_n^2\right) = O(\delta_n^k),
\end{aligned}$$

This completes the proof of (24). □

Appendix C.4 Proof of Theorem 5

Recall from the proof of Lemma 4 that $a^{-1}_{m_n,k-1}\left|\widehat{t}_{n,j} - t^a_{n,j}\right| \xrightarrow{\mathbb{P}} 0$. By the identity $\sup_{\mathrm{t}\in[0,\mathrm{T}]} \widehat{z}_{n,\mathrm{t}} = \max_{1\leq j\leq m_n}\left|\widehat{t}_{n,j}\right|$ and the Slutsky's theorem, it suffices to show that

$$a^{-1}_{m_n,k-1}\left(\max_{1\leq j\leq m_n}|t^a_{n,j}| - q_{m_n,k-1}\right) \xrightarrow{d} \xi_{k-1}(\bar{\beta}_{k-1}). \tag{25}$$

First, we verify the so-called von Mises' condition for the centered Student-t distribution.

Lemma 17 (von Mises' condition)**.** *The Student-t distribution function F_ν with $\nu > 0$ degrees of freedom satisfies the von Mises' condition, namely,*

$$\lim_{x\to\infty} \frac{(1-F_\nu(x))f'_\nu(x)}{(f_\nu(x))^2} = -\frac{1}{\nu} - 1, \quad f_\nu = F'_\nu.$$

Now let $\widetilde{F}_\nu$ denote the half-t distribution function with ν degrees of freedom and $\widetilde{f}_\nu = \widetilde{F}'_\nu$, such that

$$1 - \widetilde{F}_\nu(x) = 2(1-F_\nu(x)), \quad \widetilde{f}_\nu = 2f_\nu(x).$$

Then it is obvious that $\widetilde{F}_\nu$ also satisfies the von Mises condition:

$$\frac{(1-\widetilde{F}_\nu(x))\widetilde{f}'_\nu(x)}{(\widetilde{f}_\nu(x))^2} = \frac{(1-F_\nu(x))f'_\nu(x)}{(f_\nu(x))^2} \to -\frac{1}{\nu} - 1, \quad x \to \infty.$$

With this von Mises condition, we can obtain the following well-known consequence in extreme value analysis.

Lemma 18. *Let $\widetilde{Q}_\nu(x) = \widetilde{F}_\nu^{-1}$ denote quantile function of the half-t distribution with degrees of freedom $\nu > 0$. Then, for any sequence $m_n \to \infty$ as $n \to \infty$*

$$\frac{\widetilde{Q}_\nu(1-z/m_n) - q_{m_n,\nu}}{a_{m_n,\nu}} \to \nu(z^{-1/\nu} - 1), \quad z > 0.$$

for our choice of sequences $q_{m,\nu}$ and $a_{m,\nu}$ given in (15) *and* (16).

Proof of Theorem 5, Part (a). From the discussions at the beginning of this section, it suffices to establish (25). That is, for each $x > -(k-1)$,

$$\mathbb{P}\left(\max_{1\le j\le m_n} |t^a_{n,j}| \le a_{m_n,k-1}x + q_{m_n,k-1}\right) \to \mathbb{P}\left(\xi_{k-1}(\bar{\beta}_{k-1}) \le x\right).$$

This is equivalent to for each $z > 0$ and $x = (k-1)(z^{-1/(k-1)} - 1) > -(k-1)$,

$$\begin{aligned}&\mathbb{P}\left(\max_{1\le j\le m_n} |t^a_{n,j}| \le a_{m_n,k-1}(k-1)(z^{-1/(k-1)} - 1) + q_{m_n,k-1}\right)\\ &\to \mathbb{P}\left(\xi_{k-1}(\bar{\beta}_{k-1}) \le (k-1)(z^{-1/(k-1)} - 1)\right) = \exp(-\bar{\beta}_{k-1}z).\end{aligned}$$

By Lemma 18, it suffices to show that

$$\mathbb{P}\left(\max_{1\le j\le m_n} |t^a_{n,j}| \le \widetilde{Q}_{k-1}(1-z/m_n)\right) \to \exp(-\bar{\beta}_{k-1}z),$$

where $\widetilde{Q}_{k-1} = \widetilde{F}_{k-1}^{-1}$ denotes the quantile function of the half-t distribution with $k-1$ degrees

of freedom. This is equivalent to

$$\mathbb{P}\left(\bigcap_{j=1}^{m_n} B_{n,j}\right) \to \exp(-\bar{\beta}_{k-1}z), \quad B_{n,j} \equiv \left\{|t^a_{n,j}| \leq \widetilde{Q}_{k-1}(1-z/m_n)\right\}. \tag{26}$$

By Lemma 15, uniformly over $1 \leq j \leq m_n$,

$$\begin{aligned} m_n \mathbb{P} B^{\complement}_{n,j} =& m_n \left(1 - \widetilde{F}_{k-1,\sqrt{k}\varpi_{n,j}}\left(\widetilde{Q}_{k-1}(1-z/m_n)\right)\right) \\ =& z \cdot \frac{1 - \widetilde{F}_{k-1,\sqrt{k}\varpi_{n,j}}\left(\widetilde{Q}_{k-1}(1-z/m_n)\right)}{1 - \widetilde{F}_{k-1}\left(\widetilde{Q}_{k-1}(1-z/m_n)\right)} = z \cdot \beta_{k-1}\left(\sqrt{k}\varpi_{n,j}\right) + o(1), \end{aligned}$$

where

$$\beta_{k-1}\left(\sqrt{k}\varpi_{n,j}\right) \in \left[\inf_{\mathrm{t}\in[0,\mathrm{T}]} \beta_{k-1}\left(\sqrt{k}\varpi_{\mathrm{t}}\right), \sup_{\mathrm{t}\in[0,\mathrm{T}]} \beta_{k-1}\left(\sqrt{k}\varpi_{\mathrm{t}}\right)\right]$$

are bounded away from 0 and infinity uniformly. Moreover, it is obvious $B_{n,j}$ and $B_{n,j'}$ are independent for $|j - j'| \geq k$. Then using Lemma 13 to these events whose condition (b) is satisfied according to Lemma 14, via (26), it remains to show that

$$\prod_{j=1}^{m_n} \mathbb{P}\left(B_{n,j}\right) \to \exp(-\bar{\beta}_{k-1}z), \quad \text{or} \quad \sum_{j=1}^{m_n} -\log \mathbb{P}\left(B_{n,j}\right) \to \bar{\beta}_{k-1}z.$$

Recall from above that $\mathbb{P}\left(B^{\complement}_{n,j}\right) = 1 - \mathbb{P}\left(B_{n,j}\right) = O(1/m_n) \to 0$ uniformly over $1 \leq j \leq m_n$. Using the well-known approximation $-\log(1-x) = x + o(x^2)$, as $x \to 0$, the last convergence is equivalent to

$$\sum_{j=1}^{m_n} \mathbb{P}\left(B^{\complement}_{n,j}\right) \to \bar{\beta}_{k-1}z.$$

This holds because from above we already know

$$\sum_{j=1}^{m_n} \mathbb{P}\left(B^{\complement}_{n,j}\right) = \frac{1}{m_n}\sum_{j=1}^{m_n} m_n \mathbb{P}\left(B^{\complement}_{n,j}\right) = \frac{1}{m_n}\sum_{j=1}^{m_n} \beta_{k-1}(\sqrt{k}\varpi_{n,j}) \cdot z + o(1) \to \bar{\beta}_{k-1} \cdot z,$$

where the last step holds by assumption. □

Proof of Theorem 5, Part (b). By the convergence in distribution established in Part (a) of Theorem 4 (that follows from Part (a) of Theorem 5), and using the equivalence between convergence in distribution and convergence of quantile functions, we have

$$a_{m_n,k-1}^{-1}\left(z_{n,1-\alpha}^{*}-q_{m_n,k-1}\right)\to q_{1-\alpha}(\xi_{k-1}),$$

where $q_{1-\alpha}(\xi_{k-1})$ denotes the $(1-\alpha)$ quantile of the limiting null distribution ξ_{k-1}, and satisfies

$$\left(\frac{q_{1-\alpha}(\xi_{k-1})}{k-1}+1\right)^{-(k-1)}=-\log(1-\alpha). \tag{27}$$

Together with the continuity of the limiting distribution in Part (a) of Theorem 5,

$$\mathbb{P}\left(\sup_{\mathrm{t}\in[0,\mathrm{T}]}\widehat{z}_{n,\mathrm{t}}>z_{n,1-\alpha}^{*}\right)=\mathbb{P}\left(\frac{\sup_{\mathrm{t}\in[0,\mathrm{T}]}\widehat{z}_{n,\mathrm{t}}-q_{m_n,k-1}}{a_{m_n,k-1}}>\frac{z_{n,1-\alpha}^{*}-q_{m_n,k-1}}{a_{m_n,k-1}}\right)$$

which approaches

$$\begin{aligned}\mathbb{P}\left(\xi_{k-1}(\bar{\beta}_{k-1})>q_{1-\alpha}(\xi_{k-1})\right)=&1-\exp\left(-\bar{\beta}_{k-1}\left(\frac{q_{1-\alpha}(\xi_{k-1})}{k-1}+1\right)^{-(k-1)}\right)\\=&1-\exp\left(\bar{\beta}_{k-1}\log(1-\alpha)\right)=1-(1-\alpha)^{\bar{\beta}_{k-1}},\end{aligned}$$

where the penultimate equality follows from (27). □

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