A seismometer for Itō semimartingales*

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Abstract

We propose a novel statistical method for detecting explosive behavior in Itō semimartingales with unbounded drift and volatility. We show that the high-frequency asymptotic behavior of the localized autocovariance reveals both the intensity and the source of the explosion. Building on this insight, we develop a testing procedure that classifies explosion regimes and remains valid under irregular sampling and measurement noise. In particular, strongly negative values of the test statistic provide evidence of volatility-driven explosions. Realistic simulations of financial data demonstrate the effectiveness of our methodology. We apply the test to show that financial markets were not stabilized by the market-wide circuit breakers in March 2020, which were activated during the turmoil triggered by COVID-19.

Keywords: Volatility burst; drift burst; local autocovariance; nonparametric statistics; Itō semimartingale.

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1 Introduction

Over the past few decades, the analysis of financial data collected at a high-frequency level has become increasingly important in the econometric and statistical literature. In this framework, it is usually assumed that the data generating process for log-prices can be modeled as an Itō semimartingale in continuous time, see e.g. Jacod and Protter (2012) and Aït-Sahalia and Jacod (2015). This widely used process, fundamental to most modern asset pricing models, decomposes returns into three components (drift, volatility, and jumps) and mostly assumes locally bounded coefficients. In particular, the assumption that drift and volatility coefficients are bounded appears to be broadly consistent with the efficient market hypothesis and with empirical observations of financial time series, where short-term price fluctuations are primarily driven by the volatility component.

However, assuming bounded coefficients is not always realistic. For instance, flash crashes are a relevant example of market inefficiency which has been recently investigated by the high-frequency econometric literature, as a result of an increased attention by financial institutions and investors to them (see, e.g., Easley, de Prado, and O'Hara, 2011; Madhavan, 2012; Andersen and Bondarenko, 2014; Kirilenko, Kyle, Samadi, and Tuzun, 2017; Menkveld and Yueshen, 2019; Bank of England, 2019). In the literature, similar instances of market inefficiency are temporary violations of the no-arbitrage principle (Andersen, Todorov, and Zhou, 2024), gradual jumps (Christensen, Oomen, and Podolskij, 2014), and extreme return persistence (Andersen, Li, Todorov, and Zhou, 2023). These phenomena can be modeled by relaxing the unboundedness coefficients assumption. In particular, volatility may also explode without drift explosion, thus without compromising market efficiency. Turmoil events in financial markets have thus inspired numerous recent contributions in financial econometrics, leading to the development of new test statistics aiming at detecting unboundedness (or, more informally, explosions) in the coefficients, see Christensen, Oomen, and Renò (2022); Andersen, Li, Todorov, and Zhou (2023); Kolokolov (2023); Christensen and Kolokolov (2024); Andersen, Todorov, and Zhou (2024); Shi and Phillips (2024); Zhao, Hong, and Linton (2025); Boswijk, Yu, and Zu (2025); Flora and Renò (2025). Importantly, the literature still lacks a dedicated test for volatility explosions, as existing contributions focus on detecting explosive behavior in the drift or jump components.

We contribute to this literature by introducing a new test for drift and volatility explosion. The test is based on localized autocovariance (LAC) of the return process. Using the autocovariance to detect drift is intuitive, as shown in the recent contributions of Laurent, Renò, and Shi (2024) and Kolokolov, Renò, and Zoi (2024). The autocovariance is also a classical measure of market inefficiency/illiquidity, see e.g. Roll (1984); Lo and MacKinlay (1988); Poterba and Summers (1988), and, more recently, Li and Yang (2025). The key novelty of our approach lies in testing based on the localization of the self-normalized autocovariance of averaged prices around a given time point. We formally show that, when the price process follows an Itō semimartingale with unbounded coefficients, the localized autocovariance can identify the region to which the drift and volatility explosion rates belong, and, in particular, isolate episodes driven by volatility explosions. Despite the presence of jumps, the test statistic does not need to introduce truncation (Mancini, 2009) and can be implemented on tick-by-tick noisy data after suitable pre-averaging of observed prices. When the new LAC test becomes large and negative, it indicates the presence of volatility explosions. When it is large and positive, it may signal either a drift or a volatility explosion. This ambiguity can be resolved by jointly using the LAC test and the recent test of Christensen, Oomen, and Renò (2022) (henceforth, the COR test), which allows us to disentangle drift from volatility explosions in all cases.

We apply the new test to both synthetic and real data. Simulation experiments on realistic high-frequency synthetic data confirm that our methodology effectively detects both drift and volatility explosions in small samples. As an additional robustness check, we conduct our numerical study for a data-generating process that is more complex than that assumed by our theory. In our empirical application, we use the LAC test to demonstrate that the market-wide circuit breakers activated in March 2020 failed to calm the turmoil triggered by the COVID-19 pandemic, at least immediately after the break. In fact, we document a strong volatility explosion upon market reopening in all examined cases, underscoring the importance of a targeted test for volatility explosions.

The remainder of this paper is organized as follows. Section 2 derives the main theoretical results. Section 3 extends the results to noisy tick-by-tick prices. In Section 4, we conduct a simulation study to evaluate the practical performance of our test. Section 5 applies the test to assess the effectiveness of market-wide circuit breakers. Finally, Section 6 concludes. The Supplementary Material contains the mathematical proofs.

2 Methodology

2.1 The model

We work on the standard complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration is $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$. We assume that the logarithmic price follows an Itō semimartingale, which we express in the form of the Grigelionis decomposition (see Jacod and Protter, 2012, Theorem 2.1.2, or Aït-Sahalia and Jacod, 2015, Section 1.4.3), with some suitable restrictions on the coefficients and with explicit formulation of potential explosions in the drift and in the volatility.

Assumption 1 *The one-dimensional log-price process X is driven by the following dynamic:*

$$X_t = X_0 + \int_0^t \mu_u (1-u)^{-\alpha} du + \int_0^t \sigma_u (1-u)^{-\beta} dW_u + (\delta \mathbf{1}_{\{|\delta|\leq 1\}}) \star (\mathbf{p} - \mathbf{q})_t + (\delta \mathbf{1}_{\{|\delta|>1\}}) \star \mathbf{p})_t, \tag{1}$$

for each $t \in [0,1]$, where α and β are real numbers, W_t is a Brownian motion, $\mathbf{p}_t$ is a Poisson random measure with Lévy mesure λ, $\mathbf{q}(dt, dx) = \lambda(dx)dt$ is the compensator of $\mathbf{p}_t$, δ is a predictable jump size, and X_0 is $\mathcal{F}_0$-measurable. Moreover, we assume the following:

(i) The explosion rates satisfy $0 \leq \alpha < 1, 0 \leq \beta < \frac{1}{2}$;

(ii) The process μ_t is locally bounded and predictable, and satisfies $\mu_t^2 > 0$ a.s. for all $t \in [0,1]$. Moreover, with $C > 0$, $|u-s|$ small enough and $\Gamma > 2\alpha + \min(1-2\beta, 1/2) - 2$,

$$\mathbb{E}\left[|\mu_s - \mu_u|^2 \middle| \mathcal{F}_{u\wedge s} \right] \leq C|u-s|^{\Gamma}. \tag{2}$$

(iii) The process σ_t is driven by the following dynamics:

$$\sigma_t = \sigma_0 + \int_0^t \xi_u du + \int_0^t \eta_u dW'_u + (\delta^\sigma \mathbf{1}_{\{|\delta^\sigma|\leq 1\}}) \star (\mathbf{p}^\sigma - \mathbf{q}^\sigma)_t + (\delta^\sigma \mathbf{1}_{\{|\delta^\sigma|>1\}}) \star \mathbf{p}^\sigma)_t, \tag{3}$$

where W'_t is a Brownian motion (possibly correlated with W_t), $\mathbf{p}^\sigma_t$ is a Poisson random measure with Lévy mesure λ^σ, $\mathbf{q}^\sigma(dt, dx) = \lambda^\sigma(dx)dt$ is the compensator of the volatility jumps, δ^σ is a predictable jump size, ξ_t is locally bounded, η_t is càdlàg, and σ_0 is $\mathcal{F}_0$-measurable.

(iv) The jumps of X satisfy $\min(|\delta(\omega, t, z)|, 1) \leq \overline{\Gamma}(z)$ *where* $\overline{\Gamma}(z)$ *satisfies* $\int \overline{\Gamma}(z)^r \lambda(dz) < \infty$ *for a given* $0 \leq r < 2$.

(v) The jumps of σ satisfy $\min(|\delta^\sigma(\omega, t, z)|, 1) \leq \overline{\Gamma}^\sigma(z)$ *where* $\overline{\Gamma}^\sigma(z)$ *satisfies* $\int \overline{\Gamma}^\sigma(z)^2 \lambda(dz) < \infty$.

Model 1 is a straightforward modification of the standard model employed in financial econometrics for the efficient price. The specific assumptions $(i) - (v)$ on the coefficients of the Itō semimartingale are mild and their role will be discussed in the specific parts of the proofs in which these assumptions are needed.

The continuous part of X is written as in Laurent, Renò, and Shi (2024). When $\alpha = \beta = 0$, the model is the classical case of a Itō semimartingale with locally bounded coefficients, see Jacod and Protter (2012) for an extensive treatment. The modification allows for drift explosion (at rate α) and for volatility explosion (at rate β). We place both explosions in $t = 1$. Allowing for joint drift and volatility explosion is motivated by the empirical observation that drift explosions are typically accompanied by volatility explosions, which limits the violation of absence of arbitrage occurring when $\alpha - \beta > 1/2$, see Remark 2.6 in Laurent, Renò, and Shi (2024). Our results can be generalized for explosions to occur at a random time between 0 and 1 (see Remark 3.1 in Laurent, Renò, and Shi, 2024). In particular, volatility explosions enrich existing models in the statistical literature by introducing an additional mechanism through which volatility can change abruptly, thereby extending recent approaches based on volatility jumps (see Bibinger, Jirak, and Vetter, 2017).

The discontinuous part is also standard. In particular, assumption (iv) implies that the Blumenthal-Geetor index of price jumps is smaller than r, which we assume to be smaller than 2. This requirement, as discussed below, will dispense our testing methodology with truncation, even when r is arbitrarily close to 2. The treatment of market microstructure noise is more delicate and postponed to Section 3.

2.2 The Localized Auto-Covariance (LAC) test

We introduce a bandwidth sequence $h_n \to 0$ and a kernel function $K : \mathbb{R} \to \mathbb{R}^+$. The kernel function obeys standard conditions (see, e.g., Bandi, 2002; Kristensen, 2010). The last condition requires rapidly vanishing tails for the kernel function and it is the most important for guaranteeing the limit results below.

Assumption 2 *The following conditions hold:*

1. *K is bounded and differentiable, with bounded first derivative $K'(x)$;*

2. *The kernel function satisfies:*

$$\int_{-\infty}^{\infty} K(x)dx = 1; K_2 := \int_{-\infty}^{\infty} K^2(x)dx < \infty;$$
$$\int_{-\infty}^{\infty} |x|^{\kappa}|K(x)|dx < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |x|^{\kappa}|K^2(x)|dx < \infty \quad \text{for} \quad \kappa > -1;$$
$$xK(x) \to 0 \quad \text{and} \quad xK'(x) \to 0 \quad \text{for} \quad x \to \pm\infty.$$

3. *For $B, C > 0$ and every positive sequence $s_n \to \infty$, $\int_{-\infty}^{-s_n} K(x)dx \leq Cs_n^{-B}$ and $\int_{s_n}^{\infty} K(x)dx \leq Cs_n^{-B}$.*

The next assumption is about observation times, which we allow to be uneven (but dense enough around any point in time).

Assumption 3 *We assume to observe a realization of the data generating process* (1) *on the discrete unevenly spaced time grid* $\{0 = t_0 < t_1 < t_2 \cdots < t_n = 1\}$*, and, writing*

$\Delta_{n,i} = t_i - t_{i-1}$ and $\Delta_n = 1/n$, $c_1\Delta_n \le \Delta_{n,i} \le c_2\Delta_n$ uniformly on i and n for two suitable constants $c_1 > 0$ and $c_2 > 0$. Moreover, given $H(t) = \lim_{n\to\infty} H_n(t)$ where $H_n(t) = \frac{1}{\Delta_n}\sum_{t_i \le t}(\Delta_{n,i})^2$, we assume that $H(t)$ exists and is Lebesgue-almost surely differentiable on $(0,1)$ with derivative H' such that, for a suitable constant c, $\left|H'(t_i) - \frac{\Delta_{n,i}}{\Delta_n}\right| \le c\Delta_{n,i}$, for any t_i in which H is differentiable.

We denote returns on the time grid by $\Delta_i^n X = X_{t_i} - X_{t_{i-1}}$. The standardized Local Auto-Covariance (LAC) at lag 1 and time $s \in]0,1]$ is defined as:

$$\mathrm{LAC}_s^n := \frac{1}{h_n}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-s}{h_n}\right)\Delta_i^n X \frac{\Delta_{i-1}^n X}{\sqrt{\Delta_{n,i-1}}}. \tag{4}$$

We standardize only one return in Eq. (4) by $\sqrt{\Delta_{n,i-1}}$ because, as made clear in the proofs, this delivers a second moment for LAC which has order $1/h_n$ under the null.

In order to build the test statistic, we standardize LAC_s^n with a spot variance estimator (see, e.g., Zu and Boswijk, 2014 and Chen, Mykland, and Zhang, 2020):

$$\widehat{\sigma^2}_s^n := \frac{1}{h_n}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-s}{h_n}\right)(\Delta_i^n X)^2. \tag{5}$$

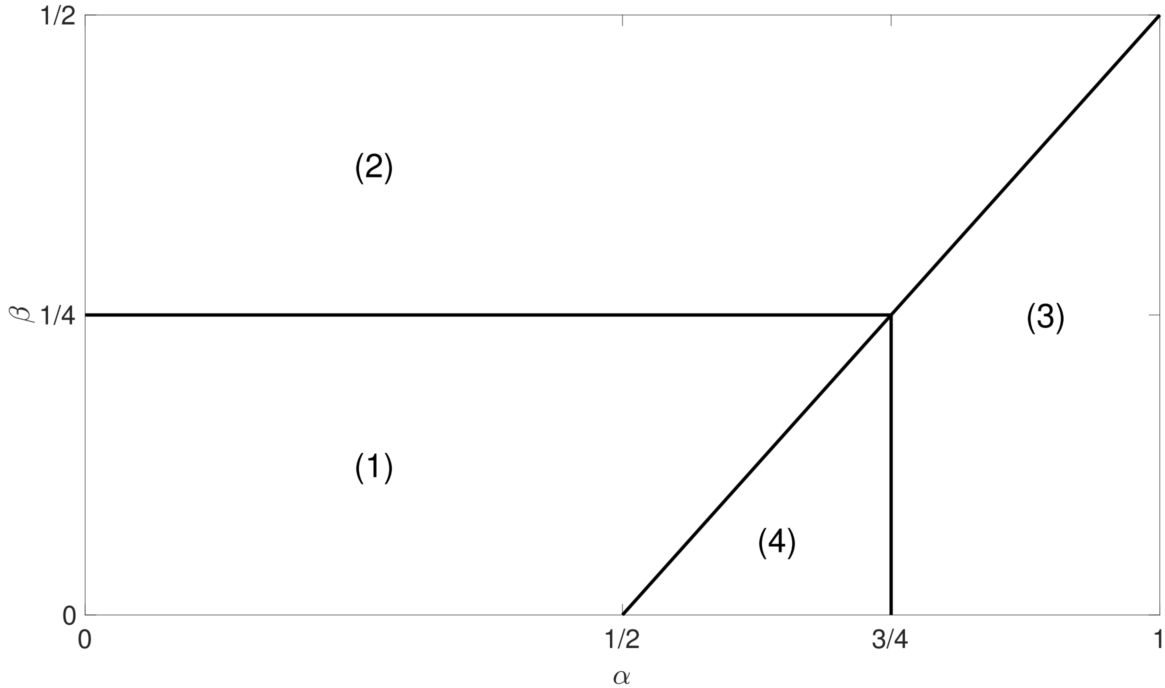
It is well known that $\widehat{\sigma^2}_s^n$ is a consistent estimator, after an appropriate rescaling, of the variance of the numerator when $\alpha = \beta = 0$. We finally define our test statistic as:

$$T_{n,s}^{\mathrm{LAC}} := \sqrt{\frac{h_n}{K_2}}\frac{\mathrm{LAC}_s^n}{\widehat{\sigma^2}_s^n}, \tag{6}$$

where K_2 is defined in point 2 of Assumption 2.

By construction, we expect $T_{n,s}^{\mathrm{LAC}}$ to be normally distributed when $s < 1$. In $s = 1$, its behavior depends on α and β. The following Theorem states the asymptotic limit of the test $T_{n,1}^{\mathrm{LAC}}$ in $s = 1$. The asymptotic theory splits the (α, β) parameter space into four distinct areas, represented in Figure 1. The enumeration in the Theorem corresponds to that in Figure 1.

Figure 1: (α, β)-plane



Note. The different regions in the (α, β) plane which characterize the different zones of Theorem 2.1.

Theorem 2.1 *Let Assumptions 1-2-3 hold, and* $n \to \infty$*,* $h_n n \to \infty$*, and* $h_n n^c \to c'$ *for some* $\frac{1}{2} < c < 1$ *and* $0 < c' < +\infty$*. Also, let* $r < \frac{2}{2-c}$*. The limiting distribution of* $T_{n,1}^{LAC}$ *defined in Eq.* (6) *is the following:*

(1) When $\beta < 1/4$ *and* $\alpha - \beta < 1/2$*:*

$$T_{n,1}^{LAC} \xrightarrow{d} N(0,1).$$

(2) When $\beta > 1/4$ *and* $\alpha - \beta < 1/2$*:*

$$|T_{n,1}^{LAC}| \xrightarrow{\mathbb{P}} \infty.$$

(3) When $\alpha > 3/4$ *and* $\alpha - \beta > 1/2$*:*

$$T_{n,1}^{LAC} \xrightarrow{\mathbb{P}} \infty.$$

(4) When $1/2 < \alpha < 3/4$ *and* $\alpha - \beta > 1/2$*, if* $c > \frac{3/2-2\alpha}{1/2-2\beta}$*:*

$$T_{n,1}^{LAC} \xrightarrow{\mathbb{P}} \infty;$$

if $c \leqslant \frac{3/2-2\alpha}{1/2-2\beta}$*:*

$$T_{n,1}^{LAC} \xrightarrow{d} N(0,1).$$

Proof. See Supplementary Material. □

Asymptotic theory identifies four zones in the (α, β) plane (see Figure 1).

- In zone (1), which contains the no-explosion point $\alpha = \beta = 0$, the test converges to a standard normal. In this zone, neither α nor β are big enough to make $T_{n,1}^{\text{LAC}}$ explode.

- In zone (2), volatility explosion dominates drift explosion and $T_{n,1}^{\text{LAC}}$ diverges with a positive or negative sign. At a first glance, this may sound counterintuitive, since as volatility explodes the denominator of $T_{n,1}^{\text{LAC}}$ (which is a spot volatility estimator) diverges. However, in this zone, where the rate of divergence is driven by the martingale component of the price process (1), the numerator diverges at a faster rate. The test thus explodes with the random sign of the numerator, which generates the oscillating behavior of the statistics in volatility explosion zones (hence the name "seismometer"), see empirical results in Section 5.

- In zone (3) the explosion of the drift dominates the variance, resulting in an asymptotic explosion of the test with a positive sign.

- In zone (4), the explosion of the drift is stronger than in zone (1), but not as strong as in zone (3); at the same time, the volatility explosion is weak. This gives rise to two distinct asymptotic behaviors of the test, depending on the rate at which the bandwidth h_n tends to zero. If h_n decreases slowly enough, the test statistic follows the distribution described in zone (1); otherwise, the test statistic diverges.

Remark 2.1 *The occurrence $T_{n,1}^{LAC} \longrightarrow -\infty$ can be generated only by a volatility explosion. Instead, $T_{n,1}^{LAC} \longrightarrow +\infty$ can be generated either by a volatility explosion or a drift explosion.*

Remark 2.2 *The condition $c > 1/2$ implies $h_n n^{1/2} \to 0$. This is necessary to prevent the volatility of volatility η_t from appearing in the asymptotic variance of LAC_1^n (see e.g. the discussion in Jacod and Rosenbaum, 2013).*

Remark 2.3 *The rate of convergence of h_n to zero depends on the value of r, which determines the "vibrancy" of the jump process, as fixed by Assumption 1, point (iv). If $r \leq \frac{4}{3}$ (e.g. for a compound Poisson process), the condition $r < \frac{2}{2-c}$ is satisfied for any value of c in the interval $(1/2, 1)$. For more vibrant jumps with $r > \frac{4}{3}$, we need a more restrictive $c > \frac{1}{2}$, that is h_n must converge to zero sufficiently fast to prevent jumps from contributing to the asymptotic variance of LAC_1^n. For the spot variance estimator, this result is established in Theorem 13.3.3 in Jacod and Protter (2012). In our case, the proof of Theorem 2.1 generalizes this result to spot autocovariance.*

Remark 2.4 *An important theoretical property of the LAC test is that it does not require truncation (Mancini, 2009) even when jumps are assumed to be present in the data generating process. This is due to the fact that we localize around $t = 1$ at a sufficiently fast rate (the condition $h_n n^c \to c'$ with $c > 1/2$). Dispensing with truncation is theoretically important: truncation would indeed reduce the power of the test under the alternative of drift explosion, since large returns due to drift explosion would be truncated too (see e.g. Andersen, Li, Todorov, and Zhou, 2023 and their theoretical analysis of truncated returns).*

It is also important practically, since there is no consensus in the financial econometric literature on how to choose the truncation level in small samples. As discussed in Remark 2.3, dispensing with truncation requires jumps in price not to be too "vibrant". This requirement is however practically harmless, since r can be arbitrarily close to 2 as the parameter c gets closer to 1 (consequently, a smaller bandwidth h_n is required). The irrelevance of jumps for self-normalized statistics also holds for the COR test (Mancini, 2023).

Remark 2.5 *The zone splitting in our Figure 1 provides a more refined partitioning for the behavior of the test statistic across different regions, compared to the zone partitioning used for the estimator in Figure 1 of Laurent, Renò, and Shi (2024). This is a consequence of the localization and the standardization by the spot variance estimator in the construction of $T^{LAC}_{n,1}$. The most important difference with respect to their result is in our zone (2), where our proposed test statistic explodes, as we discuss below, while the non-localized autocovariance (the realized drift estimator) satisfies a standard central limit theorem.*

2.3 A joint test of drift and volatility explosion

The COR test is defined as:

$$T^{\mathrm{COR}}_{n,s} := \sqrt{\frac{h_n}{K_2}} \frac{\frac{1}{h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1}-s}{h_n}\right) \Delta_i^n X}{\sqrt{\widehat{\sigma^2}^n_s}}. \tag{7}$$

The limiting theory of $T^{\mathrm{COR}}_{n,1}$ splits the parameter space into two areas only, since the test diverges when $\alpha - \beta > 1/2$ (that is in regions (3) + (4) of Figure 1), and converges to a standard normal distribution when $\alpha - \beta < 1/2$ (that is in region (1) + (2) of Figure 1). The sign of $T^{\mathrm{COR}}_{n,1}$ when this test diverges is informative about the direction of the drift burst (negative or positive).

T^{COR}	T^{LAC}	(α, β) region	type of explosion
$\mathcal{N}(0,1)$	$\mathcal{N}(0,1)$	(1)	no explosion
$\mathcal{N}(0,1)$	$+\infty$	(2)	volatility explosion
$\mathcal{N}(0,1)$	$-\infty$	(2)	volatility explosion
$\pm\infty$	$\mathcal{N}(0,1)$	(4)	drift explosion
$\pm\infty$	$+\infty$	(3) + (4)	drift explosion
$\pm\infty$	$-\infty$	impossible	-

Table 1: Localization of the region in the (α, β) space based on the asymptotic theory of the COR and the LAC test. See Figure 1 for the definition of the various regions.

Thus, the joint behavior of $T_{n,1}^{\text{COR}}$ and $T_{n,1}^{\text{LAC}}$ provides information about the nature of the explosion when $T_{n,1}^{\text{LAC}} \to +\infty$. Table 1 summarizes all the six possibilities implied by the asymptotic theory (two possible outcomes from $T_{n,1}^{\text{COR}}$, that is standard normal and $\pm\infty$, and three possible outcomes from $T_{n,1}^{\text{LAC}}$, that is standard normal, $+\infty$ and $-\infty$). Using this table, we can identify the region where the point (α, β) lies using the asymptotic behavior of the two statistics. In particular, the case $T_{n,1}^{\text{LAC}} \to -\infty$ is possible only in zone (2).

Figure 2 provides an illustration of volatility explosion detections, the main novelty of the testing procedure. We compute the LAC test every second on 12 March 2020, a day initiated by a market wide circuit breaker (more details about data and implementation in Section 5). A volatility explosion is detected by the LAC test before and after the circuit breaker, and in the middle of the day. Indeed, only volatility explosions can generate strongly negative values of the LAC test.

We end this section by providing the joint distibution of $T_{n,1}^{\text{COR}}$ and $T_{n,1}^{\text{LAC}}$ under the hypothesis of no explosion (or limited explosion) of the drift and the volatility, showing that the two tests are asymptotically uncorrelated under the null. Below I_2 denotes the identity matrix of dimension 2.

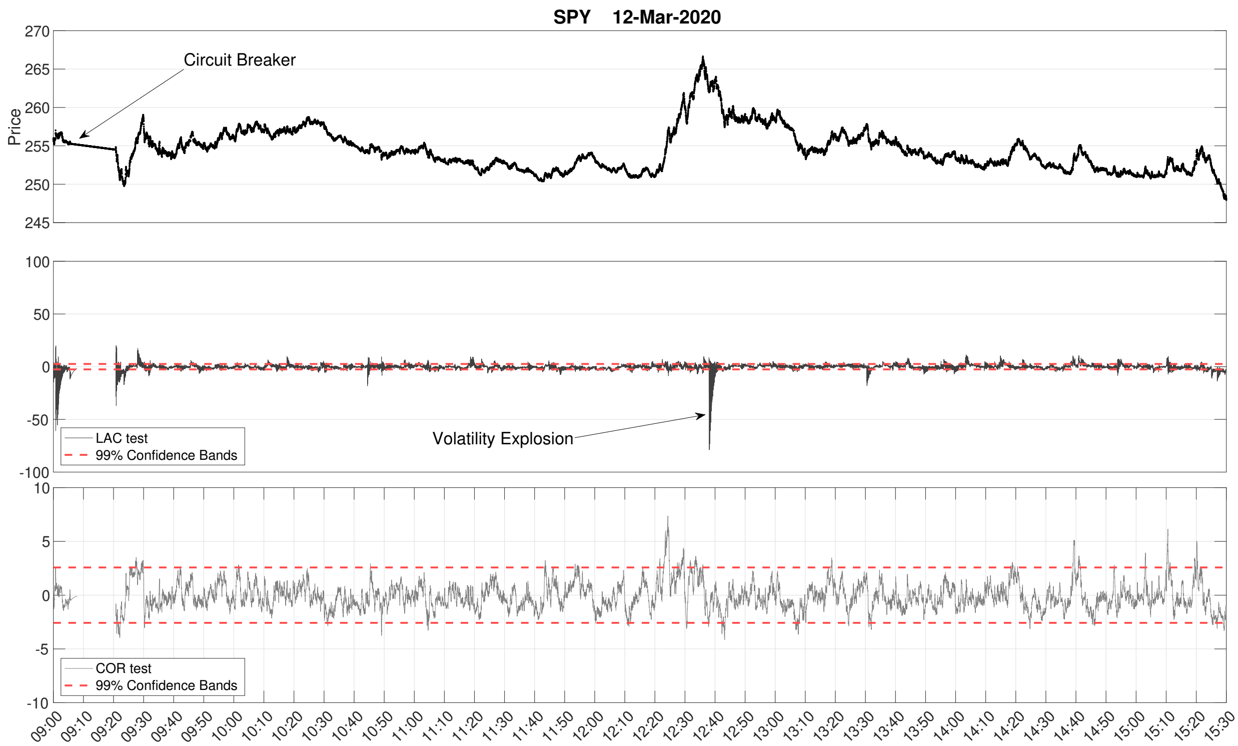


Figure 2: LAC and COR test computed every second on March 12, 2020.

Corollary 2.1 *Under the same assumptions of Theorem 2.1, when* $\beta < 1/4$ *and* $\alpha - \beta < 1/2$*:*

$$\begin{pmatrix} T_{n,1}^{LAC} \\ \\ T_{n,1}^{COR} \end{pmatrix} \xrightarrow{d} \mathcal{N}\big(\mathbf{0},\, I_2\big), \tag{8}$$

Proof. See Supplementary Material. □

3 Extension to noisy prices

In this section we assume that the observed price process is a contaminated version of the true price X:

$$Y_{t_i} = X_{t_i} + \epsilon_{t_i}, \tag{9}$$

where ϵ is the noise, an iid zero-mean process, independent from X, with finite variance ω and finite fourth moment. Our theoretical analysis deals with iid noise for simplicity, but our results easily extend to the case of q-dependent noise (for a finite q).

To implement the test on noisy prices, we carry out three steps:

1. Block partitioning: Split the price series into non-overlapping blocks of length L_n.

2. Noise-reduction averaging: Within each block, further divide the data into sub-blocks of length k_n and average the price in each sub-block to attenuate microstructure noise.

3. Autocovariance calculation: Compute the autocovariance of averaged prices over the original blocks of length L_n, making sure that each block is non-overlapping.

Figure 3 illustrates how we aggregate the data to compute the noise-robust test. Specifically, we proceed as follows. We choose a first window of data k_n, and for each $i \in \{k_n - 1, n - k_n + 1\}$ we define the average backward and forward-looking prices, respectively, as $\overline{Y}^-_{i,n}$ and $\overline{Y}^+_{i,n}$:

$$\overline{Y}^-_{i,n} := k_n^{-1} \sum_{j=0}^{k_n-1} Y_{t_{i-j}}, \qquad \overline{Y}^+_{i,n} := k_n^{-1} \sum_{j=0}^{k_n-1} Y_{t_{i+j}}.$$

Now we define our pre-averaged version of $T^{\text{LAC}}_{n,s}$:

$$\overline{T^{\text{LAC}}_{n,s}} := \sqrt{\frac{h_n}{K_2}} \frac{\overline{\text{LAC}}^n_s}{(\tilde{\sigma}^n_s)^2}, \tag{10}$$

where

$$\overline{\text{LAC}}^n_s = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n} - s}{h_n}\right) \left(\overline{Y}^-_{(i+1)L_n,n} - \overline{Y}^+_{iL_n,n}\right) \frac{\left(\overline{Y}^-_{iL_n,n} - \overline{Y}^+_{(i-1)L_n,n}\right)}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}}, \tag{11}$$

and

$$(\tilde{\sigma}^n_s)^2 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n} - s}{h_n}\right) \left(\overline{Y}^-_{(i+1)L_n,n} - \overline{Y}^+_{iL_n,n}\right)^2. \tag{12}$$

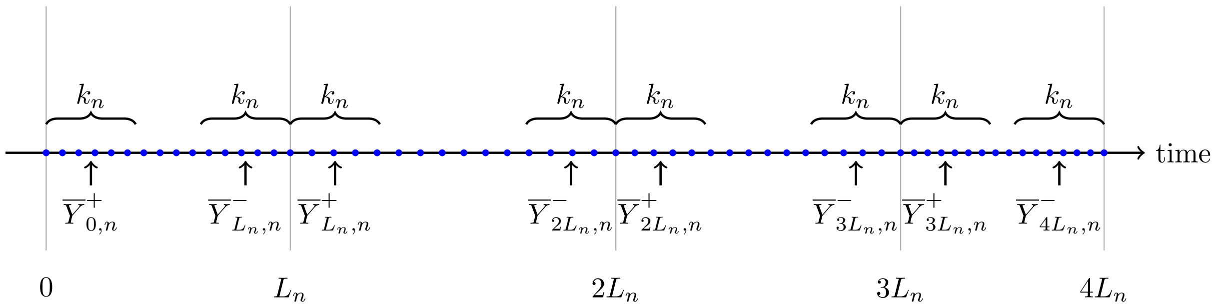


Figure 3: Illustration of our pre-averaging scheme to implement the test on tick-by-tick data. The data are divided in blocks with L_n data point. The illustration shows the first four blocks. Then, we use k_n data points to average prices, as illustrated. The autocovariance is then computed (localizing with a uniform kernel) as $(\overline{Y}^{-}_{L_n,n} - \overline{Y}^{+}_{0,n})(\overline{Y}^{-}_{2L_n,n} - \overline{Y}^{+}_{L_n,n}) + (\overline{Y}^{-}_{3L_n,n} - \overline{Y}^{+}_{2L_n,n})(\overline{Y}^{-}_{4L_n,n} - \overline{Y}^{+}_{3L_n,n}) + \dots$

We prove that $(\tilde{\sigma}^n_s)^2$ is the appropriate standardization for the variance of the dominating term of (11) in Theorem 3.1 below. This approach allows to replicate the structure of our earlier results without noise. Thus the noise-robust version of the LAC test in Eq. (10) detects bursts in drift and volatility even in the presence of noise contamination in the data. In the theoretical treatment in this section, we also assume that jumps are absent. Jumps and microstructure noise are considered jointly in the simulation experiments in Section 4, where we further allow the noise to be heteroskedastic and dependent.

Theorem 3.1 *Let Assumptions 1-2-3 hold with no jumps, and* $n \to \infty$*,* $k_n \to \infty$*,* $L_n \to \infty$*,* $h_n(n/L_n) \to \infty$*,* $h_n(n/L_n)^{1/2} \to 0$ *such that* $k_n/n \to 0$*,* $L_n/n \to 0$*,* $L_n/\sqrt{n} \to \infty$ $L_n/k_n \to \infty$*,* $\frac{k_n L_n}{n} \to \infty$*,* $h_n k_n \to \infty$*, and* $\frac{k_n}{h_n n} \to 0$*. Then* $\overline{T^{LAC}_{n,1}}$ *follows the asymptotic distribution of* $T^{LAC}_{n,1}$ *reported in Theorem 2.1.*

Proof. See Supplementary Material. □

Remark 3.1 *Theorem 3.1 holds for* $\frac{k_n L_n}{n} \to \infty$*,* $h_n k_n \to \infty$ *and* $\frac{k_n}{h_n n} \to 0$*; these conditions are more general than imposing a pre-averaging rate of* $k_n \asymp n^{1/2}$*, which is the standard* *choice in literature. Indeed, for every choice of*

$$k_n \asymp n^a, \quad L_n \asymp n^b, \quad h_n \asymp n^{-\gamma},$$

where $1/2 < b < 1$, $0 < a < b$, $\gamma > 0$ *and* $a + b > 1$, *together with* $\frac{1-b}{2} < \gamma < \min\{1 - b, a, 1 - a\}$, *the conditions of Theorem 3.1 are satisfied. For instance, we could choose* $k_n \asymp n^{0.6}$, $L_n \asymp n^{0.9}$, $h_n \asymp n^{-0.07}$. *In fact, since we are interested in testing (and not in estimating integrated variance), we do not need to balance the stochastic orders of the pre-averaged price and noise components (see, e.g., Jacod, Li, Mykland, Podolskij, and Vetter, 2009).*

Remark 3.2 *If in Theorem 3.1 we furthermore assume* $h_n(n/L_n) \to \infty$ *and* $h_n(n/L_n)^c \to c'$ *for some* $\frac{1}{2} < c < 1$ *and* $0 < c' < +\infty$, *the bandwidth conditions are still satisfied with* $\gamma = c(1 - b)$ *in place of the weaker requirement* $\frac{1-b}{2} < \gamma$, *with* γ *defined in Remark 3.1.*

4 Simulations

We conduct a Monte Carlo study in which we apply our test to simulated financial prices. The Section has two main purposes: (i) check whether the asymptotic results established in the previous sections allow to approximate the behaviour of the test statistics in small samples; (ii) check robustness to the joint presence of jumps and dependent and heteroskedastic market microstructure noise. To benchmark the behaviour of the LAC test we use the COR test, with the latter being designed to test for drift explosions only.

To simulate realistic financial returns we simulate log-prices over a high-frequency grid over 6.5 hours of trading (one trading day in the U.S.) using a jump-diffusion model with stochastic volatility and stochastic drift:

$$dX_t = a(1-\alpha)(1-t)^{-\alpha}\mathbf{1}_{\{t>\theta\}}dt + \mu_t dt + \gamma_t\sqrt{V_t}dW_{1,t} + dJ_{1,t}, \tag{13}$$

$$d\mu_t = -\kappa\mu_t dt + \xi\sqrt{2\kappa}dW_{\mu,t}, \tag{14}$$

$$V_t = e^{\ell_t} + b(1-2\beta)\int_\theta^t (1-s)^{-2\beta}ds \cdot \mathbf{1}_{\{t>\theta\}} \tag{15}$$

$$d\ell_t = (\alpha_V - \beta_V \log \ell_t)dt + \eta dW_{2,t} + dJ_{2,t}, \tag{16}$$

where $W_{1,t}$ and $W_{2,t}$ are correlated Brownian motions (with correlation parameter ρ), $J_{1,t}$ and $J_{2,t}$ are correlated jump processes, $W_{\mu,t}$ is independent of $W_{1,t}$ and $W_{2,t}$, and γ_t is an adjustment for the intraday effect. The parameterization in Eq. (14) implies that the unconditional distribution of the drift process μ_t is $\mathcal{N}(\theta, \xi^2)$, and its persistence is driven by the parameter κ. Parameters are for daily unites (one day = 1). We set $\kappa = 0.1$ and $\xi = 10$ as in Kolokolov, Renò, and Zoi (2024). We set the other parameters of the model as in Table IV of Andersen, Benzoni, and Lund (2002), corresponding to the column $SV_1J, \rho \neq 0$. The intraday effect takes the following form:

$$\gamma_t = \frac{1}{1033}(0.1271t^2 - 0.1260t + 0.1239), \tag{17}$$

as estimated by Caporin, Kolokolov, and Renò (2017) on S&P500 index data.

For the drift and volatility explosion, we start them at $\theta = 0.99$ (that is, 3.9 minutes before the end of the day) and we use $a = 0.025$ and $b = 60\sqrt{e^{\alpha_V/\beta_V}}$. These choices are calibrated to have a cumulated excess drift (from explosion only) of $\approx 0.5\%$ when $\alpha = 0.5$, and $\approx 4\%$ when $\alpha = 0.95$; and a cumulative excess daily volatility of $\approx 10\%$ when $\beta = 0.3$ and $\approx 40\%$ when $\beta = 0.45$. These high values are not uncommon in financial markets.

We consider settings without jumps, with a single jump in price only and with a single price/volatility co-jump. Jumps occur at a random time in the interval $[\theta, 1]$, that is near to the explosion point. Jumps in the log-price are $\mathcal{N}(0,(1\%)^2)$. Jumps in the log-volatility are exponentially distributed with mean e^{α_V/β_V}, which is the exponential of the long-run mean of the logarithmic volatility ℓ_t.

We finally add market microstructure noise. The log-price we use in simulation is $Y_t = X_t + \varepsilon_t$, with ε_t distributed normally with zero mean and standard deviation $\frac{1}{2}\gamma_\varepsilon\sqrt{V_t/n}$, where $n = 2,340,000$ is the number of sampled prices in a day (corresponding to ten prices per second on average). Simulated market microstructure noise then departs from the iid assumption in the theory allowing for heteroskedasticity, intraday effects and serial dependence. When $\gamma_\varepsilon = 1$, the magnitude of the noise is consistent with the findings in Christensen, Oomen, and Podolskij (2014).

We simulate $1,000$ different paths for high-frequency prices, with n observation times sampled randomly (and unevenly) in the time interval $[0,1]$ using a uniform distribution. We consider combinations of $\alpha = (0, 0.25, 0.5, 0.75, 0.95)$ and $\beta = (0, 0.15, 0.3, 0.45)$. We compute the LAC and COR tests at $t = 1$. We choose $h_n = 60$ seconds as a bandwidth for the numerator and $h_n = 300$ for the denominator to better exploit persistence in volatility when estimating the denominator (as in Christensen, Oomen, and Renò, 2022), in line with the empirical application. Regarding robustness to market microstructure noise, we use $L_n = 100$ and $k_n = 50$ for the LAC test, and $k_n = 50$ for the COR test.

We first analyze the behavior under the null ($\alpha = \beta = 0$) for three different levels of market microstructure noise: $\gamma_\varepsilon = 0$ (no noise), 1 (realistic noise) and 10 (large noise). The distribution of the LAC test (original version, and the version robust to market microstructure noise) and the COR test are displayed in Figure 4. In the no-noise case, the three tests are all close to a Gaussian distribution. In the realistic case, as expected, the original LAC test is strongly negatively biased because of the simulated negative autocovariance of

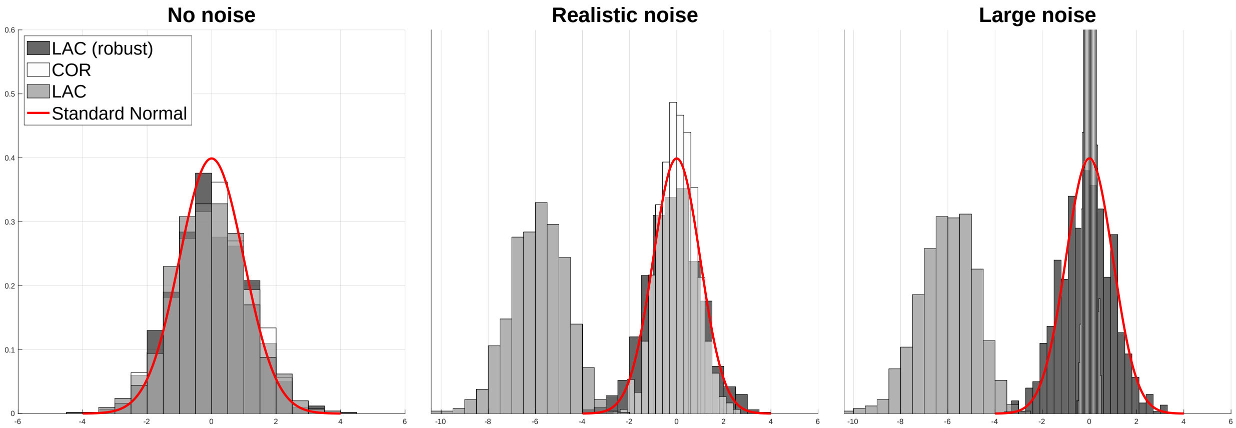


Figure 4: Distribution of the COR and the LAC tests (robust for noise and not) under the null $\alpha = \beta = 0$ for different levels of market microstucture noise: $\gamma_\varepsilon = 0$ (no noise), $\gamma_\varepsilon = 1$ (realistic noise), $\gamma_\varepsilon = 10$ (large noise).

noise differences. The robust LAC test in Eq. (10) is instead close to the standard normal, with the COR test behaving similarly. The robust LAC is unaffected by large noise, while the COR test is severely undersized in this setting.[1] Overall, this experiment shows that the robust LAC test is quite reliable under the null, even in the presence of very large noise with complex realistic features.

The distributions of the two tests, for the various combinations of α and β and in the absence of jumps, are represented in Figure 5. Results are fully consistent with the asymptotic theory illustrated in Section 2.2. When there are no bursts ($\alpha = \beta = 0$), the distribution of both tests is close to a standard normal. As α increases, the COR test diverges positively, with faster divergence for smaller β. Also the LAC test diverges, with faster divergence for smaller β as well. Compared to the COR test, the LAC test appears to be more powerful for large drift explosions and less powerful for moderate explosions. As β increases, the distribution of the LAC test becomes increasingly fatter, potentially originating both negative and positive values (the seismometer-like behavior), as implied

[1]For the COR test, it is advised to use a much smaller k_n, see Christensen, Oomen, and Renò (2022). In our experiments, we keep $k_n = 50$ for comparison with the LAC test.

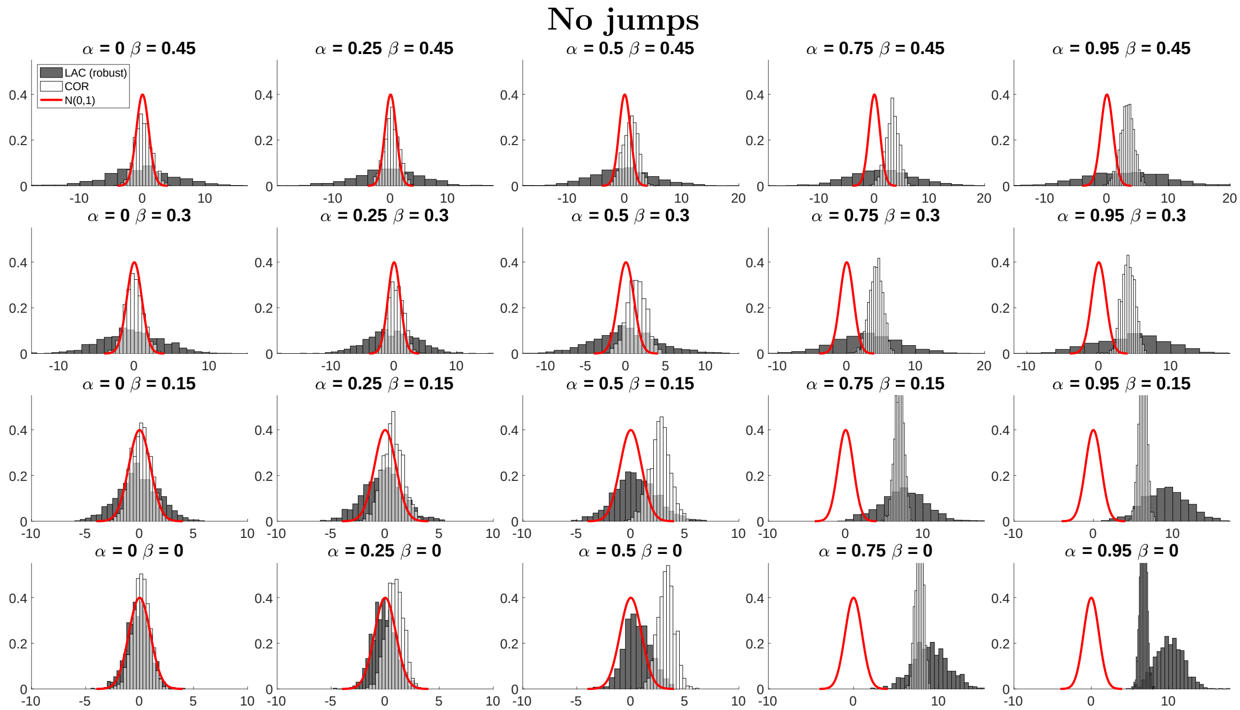


Figure 5: Distribution of the COR and the LAC tests (robust for noise) for different levels of the explosion rates α and β. In these simulations, there are no jumps.

by the theory.

Table 2 reports the percentage of rejections at 5% significance level for the LAC and the COR test. The table also shows the case for simulated trajectories with jumps in price only and price/volatility co-jumps respectively. In this setting, when $\alpha = \beta = 0$ and in the case in Panel A (the same used in applications below) the LAC test is slightly oversized in the absence of jumps, and slightly undersized in the presence of jumps. This is due to the variance of the spot volatility estimator at the denominator. Indeed, the presence of jumps near explosion points induces a small positive bias in the spot volatility estimator (a finite-sample artifact according to theory) which, in turn, mechanically reduces the LAC test statistic. Increasing α, we can see that in our simulated setting LAC is powerful for drift explosions as α increases above the predicted value of 3/4, and less powerful for volatility explosion (when $\alpha = 0$ and $\beta > 0$). Increasing β, we see that LAC is also powerful for volatility explosions, the novelty with respect to the COR test. The impact of jumps amounts to a slight reduction of power. This is in line with the theoretical argument that localized autocovariances are automatically robust to jumps.

In Panel B of Table 2 we explore the case in which we use the same bandwidth for the numerator and the denominator. If we do so, the LAC test is more correctly sized, but power decreases. Panel B thus illustrates the tradeoff on the selection of the ratio between the two bandwidths. Given the results, we will use the setting in Panel A on empirical applications.

Overall, the results of these simulated experiments suggest that the LAC behaviour in small samples reflects reliably the asymptotic predictions. This holds for models richer than those employed in the theory, which include the joint presence of jumps and heteroskedastic and dependent noise.

5 Do circuit breakers calm financial markets?

Financial markets experienced unprecedented volatility in 2020 due to the onset of the COVID-19 pandemic. In particular, Market Wide Circuit Breakers were activated 4 times in 2020 (9, 12, 16 and 18 of March), stopping the entire market in the US for 15 minutes (for institutional details, see e.g. Li and Yao, 2021). This event happened only once before March 2020 and never after 2020 (to date). Circuit breakers are mechanisms to halt trading during extreme market movements, aim to prevent panic selling and restore investor confidence. An open research question is whether they are able to provide the "calming" effect they are designed for, see e.g. Brugler, Linton, Noss, and Pedace, 2018; Sifat and Mohamad, 2020. Empirical evidence on their efficacy is indeed mixed, see e.g.

Panel A: h_n=60 seconds (numerator), 300 seconds (denominator)

LAC test

	Case 1: no jumps					Case 2: Jumps in price only					Case 3: price/volatility co-jumps				
	α					α					α				
β	0.00	0.25	0.50	0.75	0.95	0.00	0.25	0.50	0.75	0.95	0.00	0.25	0.50	0.75	0.95
0.450	69.4	71.2	70.0	73.9	79.2	63.2	63.6	64.5	70.6	74.3	63.4	61.2	64.9	69.9	74.2
0.300	60.8	64.3	60.3	73.3	80.5	53.2	51.5	47.1	62.7	71.6	48.6	46.7	47.2	60.4	71.7
0.150	31.5	29.0	31.5	96.4	99.7	15.5	16.4	15.5	82.9	92.8	14.8	13.4	17.1	80.4	90.8
0.000	6.9	6.1	16.7	100.0	100.0	3.4	3.6	6.0	85.5	92.9	3.8	4.5	8.5	83.6	94.0

COR test

	Case 1: no jumps					Case 2: Jumps in price only					Case 3: price/volatility co-jumps				
	α					α					α				
β	0.00	0.25	0.50	0.75	0.95	0.00	0.25	0.50	0.75	0.95	0.00	0.25	0.50	0.75	0.95
0.450	14.3	14.8	25.9	85.3	87.9	12.6	14.1	22.4	80.3	77.7	12.8	12.8	22.5	78.1	81.5
0.300	10.0	11.3	29.7	98.6	99.0	8.4	9.7	24.5	91.8	93.0	8.3	8.6	25.9	91.2	90.7
0.150	3.4	9.6	77.7	100.0	100.0	3.2	7.6	54.8	96.4	95.6	3.5	5.5	49.7	95.7	95.0
0.000	2.0	9.2	96.1	100.0	100.0	1.3	4.4	67.7	95.8	95.1	1.4	4.2	60.7	95.6	96.1

Panel B: h_n=60 seconds (numerator), 60 seconds (denominator)

LAC test

	Case 1: no jumps					Case 2: Jumps in price only					Case 3: price/volatility co-jumps				
	α					α					α				
β	0.00	0.25	0.50	0.75	0.95	0.00	0.25	0.50	0.75	0.95	0.00	0.25	0.50	0.75	0.95
0.450	23.6	25.8	24.3	29.8	32.8	21.1	22.2	24.0	26.9	28.9	20.2	21.7	22.7	27.9	29.7
0.300	19.0	18.0	17.5	26.9	33.7	14.1	15.1	17.8	23.3	32.4	14.6	14.6	15.4	24.7	32.3
0.150	8.7	9.2	9.7	80.5	80.2	8.5	7.1	9.0	73.7	74.1	7.3	4.9	9.1	66.0	70.6
0.000	5.3	5.9	9.5	98.0	95.5	4.5	4.7	6.1	88.6	89.4	4.2	5.1	7.8	83.6	82.7

COR test

	Case 1: no jumps					Case 2: Jumps in price only					Case 3: price/volatility co-jumps				
	α					α					α				
β	0.00	0.25	0.50	0.75	0.95	0.00	0.25	0.50	0.75	0.95	0.00	0.25	0.50	0.75	0.95
0.450	1.3	1.2	3.3	44.7	43.3	0.9	1.1	4.5	44.3	43.2	1.2	1.5	4.1	47.4	40.8
0.300	0.9	1.5	7.5	85.9	78.6	0.8	1.8	8.1	82.9	76.3	1.8	0.8	7.7	80.7	77.0
0.150	1.2	3.7	61.6	100.0	100.0	1.5	4.1	54.1	97.6	97.2	0.7	3.8	48.2	96.9	97.6
0.000	1.3	5.7	93.5	100.0	100.0	1.5	7.0	85.8	97.2	97.1	1.6	4.3	74.8	97.3	97.2

Table 2: Percentage of rejections at 5% under several values of the explosion parameters α and β, for three different models (no jumps, jumps in price only, price/volatility co-jumps). In Panel A we report results with different bandwidths for numerator and denominator (the case we implement on real data). In Panel B, we use the same bandwidth for numerator and denominator.

Brugler, Linton, Noss, and Pedace (2018) (who advocate partial success) and Hautsch and Horvath (2019) (who advocate partial failure). In this Section, we use the LAC test to assess the efficacy of circuit breakers.

We employ two datasets: SPY (an Exchange Traded Fund replicating the S&P 500 index) and VXX (an Exchange Trading Note replicating a portfolio of VIX indices at different maturity). Data are from the TAQ database and include all trades recorded in 2020. Before applying the test, data are cleaned using standard procedures.

We first compute the robust LAC test every second around the circuit breakers using $h_n = 1$ minute for the denominator, $L_n = 100$ (the distance between two preaveraged prices, in tick time), $k_n = 50$ (the number of observations used to preaverage). We also compute the COR test with the same bandwidths. To compute the test in a given time point, we require to have $N_{eff} > 10$, where N_{eff} is the number of "effective" observations defined as

$$N_{eff} = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n} - s}{h_n}\right).$$

Results are displayed in Figure 8. We discuss separately what happens before the circuit breaker and after it. For the first two days (9 and 12 march), the LAC test is significant (especially for SPY, which is more liquid) up to 5 minutes before the circuit breaker is triggered. On March 16th, the circuit breaker is triggered immediately after the market opening and we do not have enough transaction data for reliable test measurement (N_{eff} is never greater than 10). On March 18 there is no evidence of explosions right before the circuit breaker is triggered. The LAC values are mostly negative, signaling a volatility explosion. This can be confirmed by the values of the COR test (shown in Figure 7) who tend to be inside the confidence bands, consistently with a volatility explosion (see Table 1). We note that the LAC test is significant right after the opening of the markets on 9 and 12 march, so that its high-values may be due to the spike in volatility observed at the beginning of the day (volatility intraday effect). Summarizing, according to our seismometer, there is mild evidence of market turbulence *just before* the circuit breakers are triggered (the actual trigger is based on the closing price of preceding day).

The picture is completely different at the end of the breaks. In all the eight considered

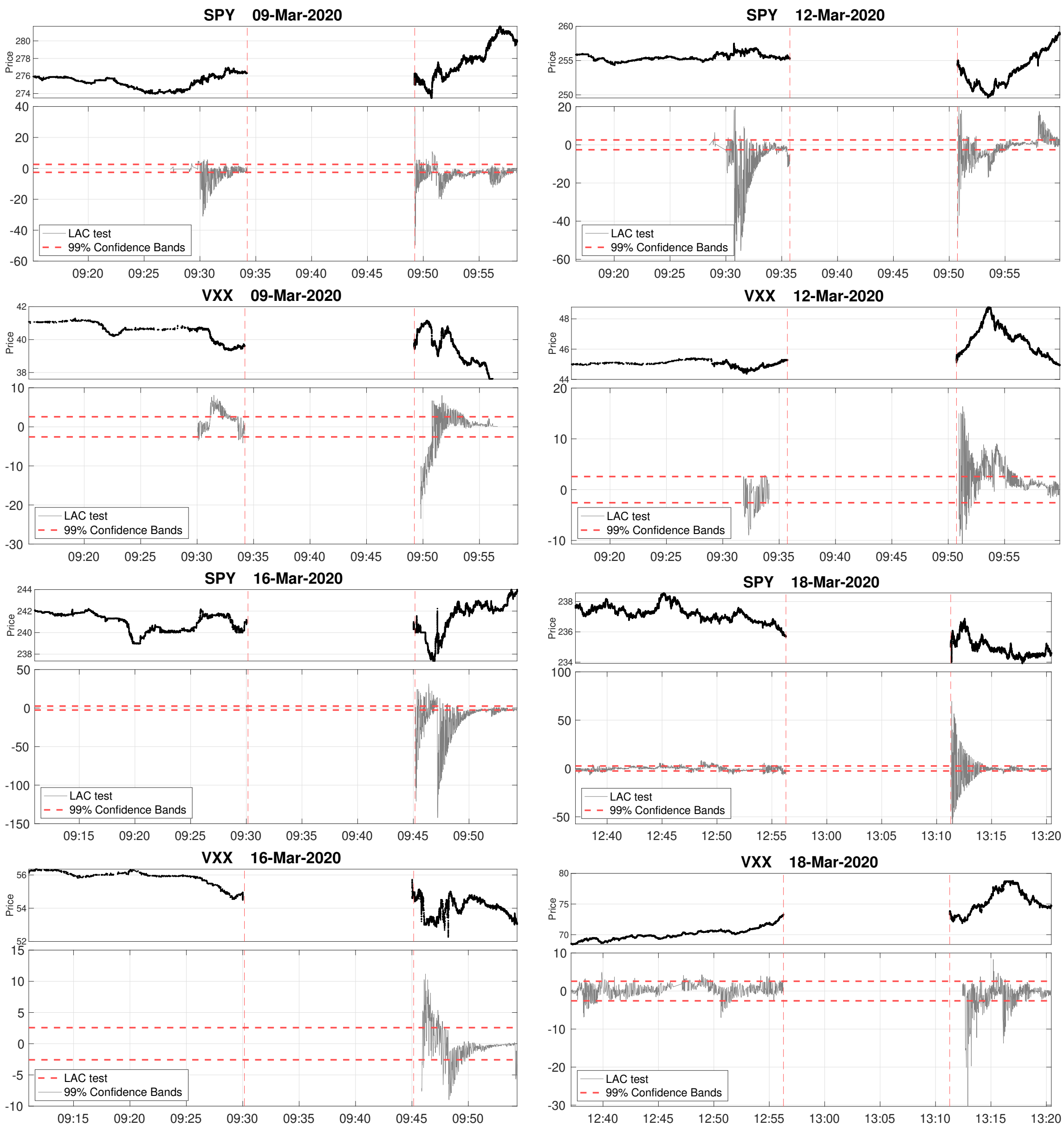


Figure 6: The LAC test computed every second before and after the market-wide circuit breakers in 2020. We consider two assets, SPY and VXX.

instances (four market wide circuit breakers, two assets), the LAC test is very significantly oscillating between negative and positive values at market re-opening, signalling a clear volatility explosion for both assets. Again, the COR test is not large with the exception of VXX on March 12, where the data display an uprise consistent with a drift explosion in VXX. In general, there is strong evidence of market turbulence *after* the circuit breaking

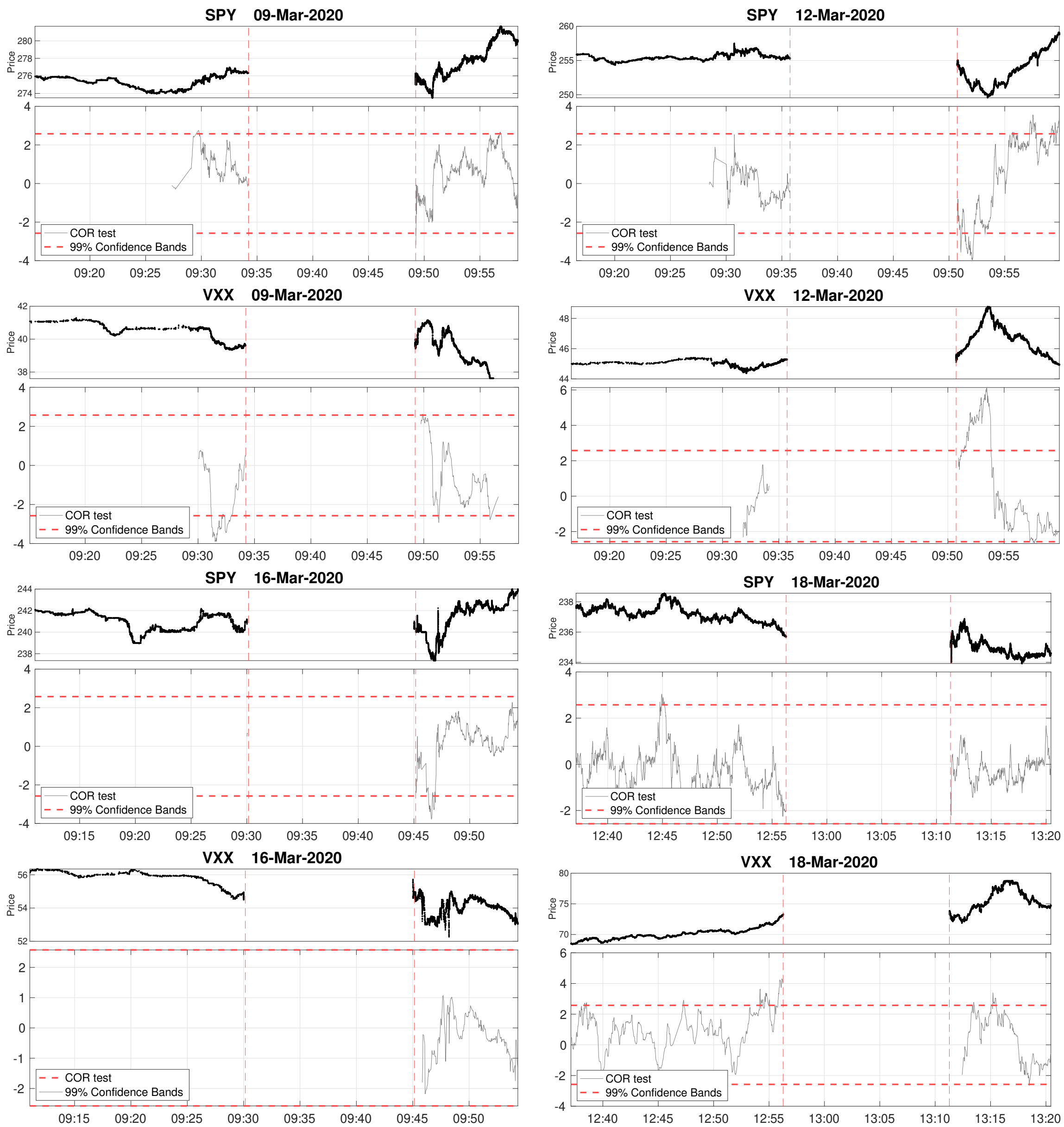


Figure 7: The COR test computed every second before and after the market-wide circuit breakers in 2020. We consider two assets, SPY and VXX.

ends. While our results needs caution the in interpretation due to the extremely small numbers of events analyzed, our results lean toward trading interruptions serving as transient volatility amplifiers, confirming the results of Hautsch and Horvath (2019).

Finally, we display the behavior of the test on 20 March 2020. This was one of the most volatile days in the sample, with a maximum drawdown of -7.12%, thus exceeding the

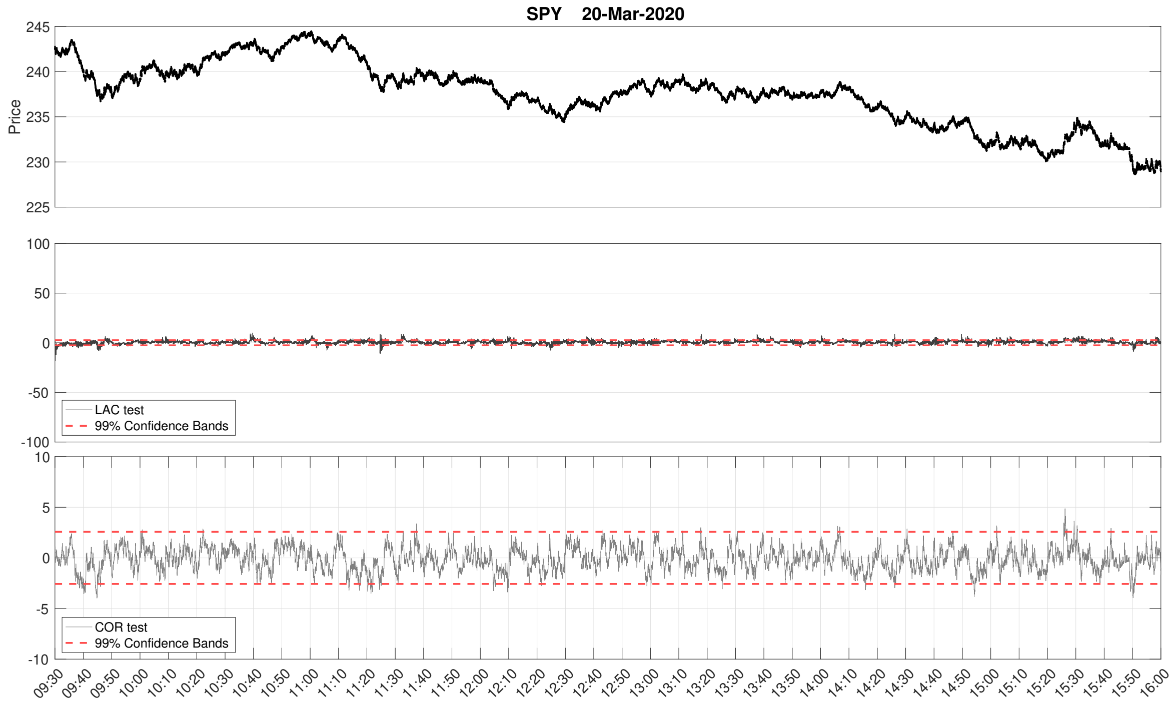


Figure 8: LAC and COR test computed every second on March 20, 2020.

circuit breaker triggering threshold if this drop was computed from the closing price of the day before. In this case, neither the LAC test nor the COR test display departure from normal market behavior. These results indicate that the LAC test responds primarily to abrupt changes in volatility, modeled through explosive volatility coefficients, rather than to elevated volatility levels alone.

6 Conclusions

We introduce a new test to determine whether financial high-frequency prices exhibit a drift or a volatility burst: the Local Auto-Covariance (LAC) test. We show that its asymptotic behavior allows for a more granular partitioning of the explosion rate parameter region than the COR test proposed in Christensen, Oomen, and Renò (2022), which is based on local averaging and devised to detect drift explosions. In particular, the LAC test can detect volatility explosions even when they are not accompanied by drift explosions. Volatility explosions are shown to occur after market-wide circuit breakers, raising questions about the effectiveness of these mechanisms. We conclude that the proposed LAC test is a valuable statistical tool for market surveillance and the assessment of market regulation.

This paper represents a first step toward the determination of the explosion rates of the drift and volatility processes. This aspect of Itō semimartingale processes, commonly used to model asset prices, remains largely unexplored in the statistical literature. However, as demonstrated by our empirical example, it may play a crucial role in enhancing market stability. In this paper, we show how to delineate the regions in which explosion rates lie. Future extensions should aim at establishing more refined statistical inference methods for these parameters using alternative techniques.

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Supplementary Material

Without loss of generality, we assume that the kernel function is such that $K(z) = 0$ for $z > 0$; proofs can be generalized to the two-sided case. For simplicity, we indicate $K\left(\frac{t_{i-1}-1}{h_n}\right)$ as K_i, omitting dependence on n. We use the notation $x_n \asymp y$ if $c^{-1}y_n \leqslant x_n \leqslant cy_n$ for a finite positive constant $c > 1$. Finite constants used in bounds are denoted by C, C' and they can vary from line to line. For simplicity we assume $\frac{\Delta_{n,i}}{\Delta_n} \longrightarrow C$ uniformly in i, which can be readily generalized using $\frac{\Delta_{n,i}}{\Delta_n} \longrightarrow H'(t_i)$ uniformly in i, as in Assumption 3. When $X_n = Y_n + o_p(Y_n)$, we write $X_n \overset{p}{\sim} Y_n$.

Lemma S-1 *Define*

$$D_{\tau,n} := \frac{1}{h_n}\sum_{i=2}^{n} K_i \left(\int_{t_{i-1}}^{t_i} \mu_v dv \int_{t_{i-2}}^{t_{i-1}} \frac{\mu_v}{\sqrt{\Delta_{n,i-1}}} dv\right). \tag{S-1}$$

Under Assumptions 1-2, the following estimate holds:

$$D_{\tau,n} - \sqrt{\Delta_n}\mu_1^2 = O_p\left(\Delta_n^{1/2} + \sqrt{\Delta_n} h_n^{\min\{\Gamma/2,B\}}\right). \tag{S-2}$$

Proof. Consider the decomposition $D_{\tau,n} = D^1_{\tau,n} + D^2_{\tau,n} + D^3_{\tau,n}$, where

$$D^1_{\tau,n} := \frac{1}{h_n}\sum_{i=2}^{n} K_i \left(\int_{t_{i-1}}^{t_i} \mu_v dv \int_{t_{i-2}}^{t_{i-1}} \frac{\mu_v}{\sqrt{\Delta_{n,i-1}}} dv\right) - \frac{1}{\sqrt{\Delta_n} h_n}\sum_{i=2}^{n} K_i \left(\int_{t_{i-1}}^{t_i} \mu_v dv\right)^2;$$
$$D^2_{\tau,n} := \frac{1}{\sqrt{\Delta_n} h_n}\sum_{i=2}^{n} K_i \left(\int_{t_{i-1}}^{t_i} \mu_v dv\right)^2 - \sqrt{\Delta_n}\int_0^t \frac{1}{h_n} K\left(\frac{v-\tau}{h_n}\right)\mu_v^2 dv;$$
$$D^3_{\tau,n} := \sqrt{\Delta_n}\int_0^t \frac{1}{h_n} K\left(\frac{v-\tau}{h_n}\right)\mu_v^2 dv - \sqrt{\Delta_n}\mu_1^2.$$

For the first term, write $\xi_{n,i} = \Delta_{n,i}/\Delta_n$. By assumptions, $c_1 \leq \xi_{n,i} \leq c_2$. We have:

$$\begin{aligned} D^1_{\tau,n} =& \frac{1}{h_n}\sum_{i=2}^{n} K_i \int_{t_{i-1}}^{t_i} \mu_v dv \int_{t_{i-2}}^{t_{i-1}} \mu_v \left(\frac{1}{\sqrt{\Delta_{n,i-1}}} - \frac{1}{\sqrt{\Delta_n}}\right) dv \\ =& \frac{1}{h_n\sqrt{\Delta_n}}\sum_{i=2}^{n} K_i \int_{t_{i-1}}^{t_i} \mu_v dv \int_{t_{i-2}}^{t_{i-1}} \mu_v \left(\sqrt{\xi_{n,i}} - 1\right) dv \end{aligned}$$

Now, using the stochastic continuity of μ_t in Eq. (2) and the fact that

$$\frac{1}{h_n}\sum_{i=2}^{n} K_i \Delta_{n,i} \to 1,$$

we obtain $D^1_{\tau,n} = O_p(\sqrt{\Delta_n})$. Using similar computations as in Lemma 1 and Lemma 2 of Christensen, Oomen, and Renò (2022), the kernel properties and Assumption 1, we obtain $D^2_{\tau,n} = O_p(\frac{1}{n^{3/2}h_n})$ and $D^3_{\tau,n} = O_p(\sqrt{\Delta_n} h_n^{\min\{\Gamma/2,B\}})$, with B defined as in Assumption 2. Then it is enough to sum the orders to conclude. □

Proof. [Proof of Theorem 2.1] We write

$$X_t = X'_t + X''_t$$

where $X'_t = \int_0^t \mu_u(1-u)^{-\alpha} du + \int_0^t \sigma_u (1-u)^{-\beta} dW_u$ and X''_t is the jump part. We first show that, without loss of generality, we can set $X''_t = 0$. This is a consequence of Theorem 13.3.3, part (b) in Jacod and Protter (2012), henceforth JP12, which is warranted by the condition $h_n n^c \to c' \in (0, +\infty)$ (since k_n in Theorem 13.3.3 of JP12 is nh_n in our case, and this assumption implies Eq. (13.3.14) and in turn Eq. (13.3.13) with $\beta = 0$ in JP12). Application of Theorem 13.3.3 in JP12 requires conditions (iii), (iv) and (v) in Assumption 1. Theorem 13.3.3 in JP12 is for the localized variance estimator (the denominator of LAC) with a uniform kernel, but it can be generalized to the numerator as in Remark 13.3.6 in JP12 (using the globally even function $F(x, y) = xy$ instead of the function $F(x, y) = |xy|$ considered there), to a kernel satisfying Assumption 2 and to observation times as in Assumption 3 following the proof of Lemma 3 in Christensen, Oomen, and Renò (2022). The condition $r < \frac{2}{2-c}$ is Eq. (13.3.15) in JP12 as derived from Lemma 13.2.6 in JP12. We can use the same Lemma in our case, applying it to the function $F(x, y) = xy$ instead of the function $F(x) = x^2$ (used for spot variance estimation in Theorem 13.3.3 in JP12). The only change would be the value of $s' = 1$ (instead of $s' = 2$), but s' does not appear in the first rate of Eq. (13.2.21) in JP12 so the restrictions on r are the exact same. The term s' appears in the the second rate of Eq. (13.2.21) in JP12, but the negligibility of this term with respect to the leading order $n^{\frac{1-c}{2}}$ (which is exactly $\sqrt{k_n}$ in the notation of Theorem 13.3.3 in JP12) is automatically satisfied by a suitable truncation rate $\bar{\omega}$, which in our case is sufficient to neglect this term since we do not truncate.

We then set $X''_t = 0$. We then divide the proof in steps for clarity.

1. First, we study the case where $\alpha = \beta = 0$. Let us consider the following decomposition of the test statistic:

$$\sqrt{\frac{h_n}{K_2}} \frac{\mathrm{LAC}_1^n - \sqrt{\Delta_n}\mu_1^2}{\widehat{\sigma^2}_1^n} + \sqrt{\frac{h_n}{K_2}} \frac{\sqrt{\Delta_n}\mu_1^2}{\widehat{\sigma^2}_1^n}. \tag{S-3}$$

The second term is $O_p(\sqrt{\Delta_n h_n})$, since:

$$\sqrt{\frac{h_n}{K_2}} \frac{\sqrt{\Delta_n}\mu_1^2}{\widehat{\sigma^2}_1^n} \leq C\sqrt{\Delta_n h_n}$$

due to the boundedness of the processes μ, σ and the assumptions on kernel. Now we concentrate on the first term. Note that:

$$\frac{1}{h_n}\sum_{i=2}^n K_i \left(\Delta_i^n X \frac{\Delta_{i-1}^n X}{\sqrt{\Delta_{n,i-1}}} \right) = D_{\tau,n} + V_{\tau,n} + C_{\tau,n},$$

where $D_{\tau,n}$ is defined in Lemma S-1, and

$$V_{\tau,n} := \frac{1}{h_n}\sum_{i=2}^n K_i \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \frac{\int_{t_{i-2}}^{t_{i-1}} \sigma_s dW_s}{\sqrt{\Delta_{n,i-1}}} \right);$$

$$C_{\tau,n} := \frac{1}{h_n}\sum_{i=2}^n K_i \left[\left(\int_{t_{i-1}}^{t_i} \mu_s ds \frac{\int_{t_{i-2}}^{t_{i-1}} \sigma_s dW_s}{\sqrt{\Delta_{n,i-1}}} \right) + \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \frac{\int_{t_{i-2}}^{t_{i-1}} \mu_s ds}{\sqrt{\Delta_{n,i-1}}} \right) \right].$$

By Lemma S-1:

$$\sqrt{h_n}(\mathrm{LAC}_1^n(1) - \sqrt{\Delta_n}\mu_1^2) = \sqrt{h_n}(C_{\tau,n} + V_{\tau,n}) + \underbrace{O_p\left(\Delta_n^{1/2} h_n^{1/2} + \sqrt{\Delta_n} h_n^{\min\{\Gamma,B\}+\frac{1}{2}} \right)}_{o_p(1)}.$$

For the term $\sqrt{h_n}C_{\tau,n}$, we have:

$$\sqrt{h_n}C_{\tau,n} = \frac{1}{\sqrt{h_n}}\sum_{i=2}^n K_i \left[\left(\int_{t_{i-1}}^{t_i} \mu_s ds \frac{\int_{t_{i-2}}^{t_{i-1}} \sigma_s dW_s}{\sqrt{\Delta_{n,i-1}}} \right) + \left(\int_{t_{i-1}}^{t_i} \sigma_s dW_s \frac{\int_{t_{i-2}}^{t_{i-1}} \mu_s ds}{\sqrt{\Delta_{n,i-1}}} \right) \right],$$

which, up to a negligible part due to the stochastic continuity of the process μ in Eq. (2), together with $h_n n^{1/2} \to 0$, and neglecting end effects, can be rearranged as $\sum_{i=2}^{n-1} v_{i,n}$, with:

$$v_{i,n} = \mu_{t_{i-1}} \sigma_{t_{i-1}} \frac{1}{\sqrt{h_n}} K_i \frac{1}{\sqrt{\Delta_{n,i-1}}} \left(\int_{t_{i-1}}^{t_i} dW_s \right) \left(\int_{t_{i-2}}^{t_{i-1}} ds + \int_{t_i}^{t_{i+1}} ds \right).$$

Now, we apply Theorem 2.2.14 of Jacod and Protter (2012). Note that $\sum_{i=2}^{n-1} \mathbb{E}[v_{i,n} | \mathcal{F}_{t_{i-1}}] = 0$, and

$$\sum_{i=2}^{n-1} \mathbb{E}[v_i^2 | \mathcal{F}_{t_{i-1}}] = \sum_{i=2}^{n-1} (\mu_{t_{i-1}} \sigma_{t_{i-1}})^2 \frac{\Delta_{n,i}}{\Delta_{n,i-1}} \frac{1}{h_n} K_i^2 \left(\Delta_{n,i-1} + \Delta_{n,i+1}\right)^2 = O_p\left(\Delta_n\right) \tag{S-4}$$

since μ, σ are bounded and $\sum_{i=2}^{n-1} \frac{\Delta_{n,i}}{h_n} K_i^2 \to 1$. We conclude that

$$C_{\tau,n} = O_p\left(\frac{\sqrt{\Delta_n}}{\sqrt{h_n}} \right)$$

which implies that $C_{\tau,n}$ is dominated by $V_{\tau,n}$. Indeed, proceeding in the same way,

$$\sqrt{h_n} V_{\tau,n} = \sum_{i=2}^{n} \tilde{v}_i + o_p(1),$$

with

$$\tilde{v}_i = \frac{\sigma_{t_{i-1}}^2}{\sqrt{h_n}} K_i \left[\frac{\int_{t_{i-2}}^{t_{i-1}} dW_s}{\sqrt{\Delta_{n,i-1}}} \int_{t_{i-1}}^{t_i} dW_s \right].$$

Therefore, in order to apply again Theorem 2.2.14 of Jacod and Protter (2012), note that $\sum_{i=2}^{n} \mathbb{E}[\tilde{v}_i | \mathcal{F}_{t_{i-1}}] = 0$, and by Itō's lemma:

$$\sum_{i=2}^{n} \mathbb{E}[\tilde{v}_i^2 | \mathcal{F}_{t_{i-1}}] = \sum_{i=2}^{n} \sigma_{t_{i-1}}^4 \frac{1}{h_n} K_i^2 \left[\frac{\left(\int_{t_{i-2}}^{t_{i-1}} dW_s \right)^2}{\Delta_{n,i-1}} \mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} dW_s \right)^2 | \mathcal{F}_{t_{i-1}} \right] \right]$$
$$= \sum_{i=2}^{n} \frac{1}{h_n} K_i^2 \frac{\Delta_{n,i}}{\Delta_{n,i-1}} \sigma_{t_{i-1}}^4 \left[\left(\int_{t_{i-2}}^{t_{i-1}} dW_s \right)^2 \right].$$

Taking again the conditional expectation it follows that:

$$\sqrt{h_n} V_{\tau,n} \to \mathcal{N}(0, K_2 \sigma_1^4). \tag{S-5}$$

Therefore (S-3) reduces to:

$$\sqrt{\frac{h_n}{K_2}}\frac{V_{\tau,n}}{\widehat{\sigma^2}_1^n} + O_p\left(\Delta_n^{1/2}\sqrt{h_n} + \Delta_n^{1/2}h_n^{\min\{\Gamma,B\}+\frac{1}{2}}\right) + O_p(\sqrt{\Delta_n}).$$

The second and third term are $o_p(1)$, and the first term converges stably in law to a standard normal distribution by (S-5) and the consistency of $\widehat{\sigma^2}_1^n$ when $\alpha = \beta = 0$.

2. Now we consider $0 < \alpha < 1$ and $0 < \beta < 1/2$. For simplicity, from now on we omit the subscript 1. We begin by decomposing the numerator of Eq. (6) into the same three components:

$$\begin{aligned}
D_n &= \frac{1}{h_n}\sum_{i=2}^n K_i\left(\int_{t_{i-1}}^{t_i}\mu_s(1-s)^{-\alpha}ds\int_{t_{i-2}}^{t_{i-1}}\frac{\mu_s}{\sqrt{\Delta_{n,i-1}}}(1-s)^{-\alpha}ds\right);\\
C_n &= \frac{1}{h_n}\sum_{i=2}^n K_i\left[\left(\int_{t_{i-1}}^{t_i}\mu_s(1-s)^{-\alpha}ds\int_{t_{i-2}}^{t_{i-1}}\frac{\sigma_s}{\sqrt{\Delta_{n,i-1}}}(1-s)^{-\beta}dW_s\right)\right.\\
&+\left.\left(\int_{t_{i-1}}^{t_i}\sigma_s(1-s)^{-\beta}dW_s\int_{t_{i-2}}^{t_{i-1}}\frac{\mu_s}{\sqrt{\Delta_{n,i-1}}}(1-s)^{-\alpha}ds\right)\right];\\
V_n &= \frac{1}{h_n}\sum_{i=2}^n K_i\left(\int_{t_{i-1}}^{t_i}\sigma_s(1-s)^{-\beta}dW_s\int_{t_{i-2}}^{t_{i-1}}\frac{\sigma_s}{\sqrt{\Delta_{n,i-1}}}(1-s)^{-\beta}dW_s\right).
\end{aligned}$$

For the first term we have:

$$\begin{aligned}
D_n &\overset{p}{\sim}\frac{1}{h_n}\sum_{i=2}^n K_i\mu_{i-1}^2\left(\int_{t_{i-1}}^{t_i}(1-s)^{-\alpha}ds\int_{t_{i-2}}^{t_{i-1}}\frac{1}{\sqrt{\Delta_{n,i-1}}}(1-s)^{-\alpha}ds\right)\\
&=\frac{1}{h_n}\sum_{i=2}^n K_i\mu_{i-1}^2\frac{1}{(1-\alpha)^2\sqrt{\Delta_{n,i-1}}}((1-t_i)^{1-\alpha}-(1-t_{i-1})^{1-\alpha})((1-t_{i-1})^{1-\alpha}-(1-t_{i-2})^{1-\alpha})\\
&=\frac{1}{h_n}\sum_{i=2}^n K_i\mu_{i-1}^2\frac{1}{\sqrt{\Delta_{n,i-1}}}(1-\xi_i)^{-\alpha}(1-\xi_{i-1})^{-\alpha}\Delta_{n,i}\Delta_{n,i-1},
\end{aligned}$$

where in the first line we used the stochastic continuity of μ_t and in the third line we used the mean value theorem for suitable numbers $t_{i-1} < \xi_i < t_i$. Now, since $\xi_i < 1$ for all i, we can Taylor expand:

$$(1-\xi_i)^{-\alpha} = (1-\xi_{i-1}-(\xi_i-\xi_{i-1}))^{-\alpha} \approx (1-\xi_{i-1})^{-\alpha} + \alpha(\xi_i-\xi_{i-1})(1-\xi_{i-1})^{-\alpha-1}. \quad \text{(S-6)}$$

Consequently, we write $D_n = D_{1,n} + D_{2,n}$, where

$$D_{1,n} = \frac{1}{h_n}\sum_{i=2}^{n} K_i \mu_{i-1}^2 \frac{1}{\sqrt{\Delta_{n,i-1}}}(1-\xi_{i-1})^{-2\alpha}\Delta_{n,i}\Delta_{n,i-1}$$

$$D_{2,n} = \frac{1}{h_n}\sum_{i=2}^{n} K_i \mu_{i-1}^2 \frac{1}{\sqrt{\Delta_{n,i-1}}}\alpha(\xi_i-\xi_{i-1})(1-\xi_{i-1})^{-2\alpha-1}\Delta_{n,i}\Delta_{n,i-1}$$

For the term $D_{1,n}$, we first notice that, by assumption on sampling times and the boundedness of μ_t,

$$D_{1,n} \overset{p}{\sim} \frac{\Delta_n^{1/2}}{h_n}\sum_{i=2}^{n} K_i (1-\xi_{i-1})^{-2\alpha}\Delta_{n,i-1}.$$

When $\alpha < 1/2$, we just use Riemann integration and after a change of variable (as for the term A_n in the proof of Theorem 2 of Christensen, Oomen, and Renò, 2022), we have

$$D_{1,n} \overset{p}{\sim} n^{-1/2} h_n^{-2\alpha} m_K(-2\alpha),$$

where $m_K(-2\alpha) = \int_{-\infty}^{0} K(x)(-x)^{-2\alpha}$. For the case $\alpha > 1/2$, we first notice that, using the assumptions on sampling times, and since $t_{i-1} < \xi_i < t_i$,

$$D_{1,n-1} \le \frac{1}{h_n}\sum_{i=2}^{n-1} K_i \mu_{i-1}^2 \frac{1}{\sqrt{\Delta_{n,i-1}}}(1-t_i)^{-2\alpha}\Delta_{n,i}\Delta_{n,i-1} \le C' \frac{\Delta_n^{3/2}}{h_n}\sum_{i=2}^{n} K_i(1-t_{i-1})^{-2\alpha}$$

$$D_{1,n} \ge \frac{1}{h_n}\sum_{i=2}^{n} K_i \mu_{i-1}^2 \frac{1}{\sqrt{\Delta_{n,i-1}}}(1-t_{i-1})^{-2\alpha}\Delta_{n,i}\Delta_{n,i-1} \ge C \frac{\Delta_n^{3/2}}{h_n}\sum_{i=2}^{n} K_i(1-t_{i-1})^{-2\alpha}$$

Now we split the previous summation into two parts. Denote by $\mathcal{C}_n$ the set of indexes i such that $K_i \ge c'_n > 0$ for a positive sequence c'_n: the existence of such a sequence is guaranteed by the assumptions on the kernel. We have:

$$\begin{aligned} D_{1,n} \geqslant & \frac{C c'_n \Delta_n^{3/2}}{h_n}\sum_{j\in\mathcal{C}_n}(1-t_{j-1})^{-2\alpha} + \frac{C\Delta_n^{3/2}}{h_n}\sum_{j\in\mathcal{C}_n^c} K_j(1-t_{j-1})^{-2\alpha} \\ \geqslant & \frac{C c'_n \Delta_n^{3/2}}{h_n}\sum_{j\in\mathcal{C}_n}(1-t_{j-1})^{-2\alpha} \\ \geqslant & C\frac{\Delta_n^{3/2-2\alpha}}{h_n}\sum_{j\in\mathcal{C}_n}\left(1-\frac{j-1}{n}\right)^{-2\alpha}, \end{aligned}$$

where the second inequality derives from the positivity of the terms $j \in \mathcal{C}_n^c$, and the third from the assumption on times. Now, as $c_n' \to 0$, the series is convergent when $\alpha > 1/2$, and repeating the same reasoning for the upper bound, we conclude that $D_{1,n} \asymp \frac{\Delta_n^{3/2-2\alpha}}{h_n}$. Similarly, when $\alpha = 1/2$, the properties of the harmonic series imply $D_{1,n} \asymp \dfrac{\Delta_n^{1/2}}{h_n} \log(n)$. For the second term we have, in the same way, $D_{2,n} \asymp \dfrac{\Delta_n^{3/2-2\alpha}}{h_n}$ for $0 < \alpha < 1$. This is dominated by $D_{1,n}$ when $0 < \alpha < 1/2$.

We now deal with the covariance term. In order to apply Theorem 2.2.14 of Jacod and Protter (2012), we employ the notation $C_n = \sum_{i=2}^n n_{i,2}$ with:

$$\begin{aligned} n_{i,2} := \frac{1}{h_n} K_i \Bigg[& \left(\int_{t_{i-1}}^{t_i} \mu_s (1-s)^{-\alpha} ds \int_{t_{i-2}}^{t_{i-1}} \frac{1}{\sqrt{\Delta_{n,i-1}}} \sigma_s (1-s)^{-\beta} dW_s \right) \\ & + \left(\int_{t_{i-1}}^{t_i} \sigma_s (1-s)^{-\beta} dW_s \int_{t_{i-2}}^{t_{i-1}} \frac{1}{\sqrt{\Delta_{n,i-1}}} \mu_s (1-s)^{-\alpha} ds \right) \Bigg]. \end{aligned}$$

The last term can be rearranged as:

$$n_{i,2} \overset{p}{\sim} \frac{1}{h_n \sqrt{\Delta_n}} K_i \mu_{t_{i-1}} \sigma_{t_{i-1}} \left(\int_{t_{i-1}}^{t_i} (1-s)^{-\beta} dW_s \right) \left(\int_{t_{i-2}}^{t_{i-1}} (1-s)^{-\alpha} ds + \int_{t_i}^{t_{i+1}} (1-s)^{-\alpha} ds \right),$$

so that we have $\sum_{i=2}^{n-1} \mathbb{E}[n_{i,2} | \mathcal{F}_{t_{i-1}}] = o_p(1)$, and

$$\sum_{i=2}^{n-1} \mathbb{E}[n_{i,2}^2 | \mathcal{F}_{t_{i-1}}] \overset{p}{\sim} \frac{1}{h_n^2 \Delta_n} \sum_{i=2}^{n-1} K_i^2 \mu_{t_{i-1}}^2 \sigma_{t_{i-1}}^2 \Delta_{n,i-1} (1-t_{i-1})^{-2\beta} \left(\Delta_{n,i-2} (1-t_{i-2})^{-\alpha} + \Delta_{n,i} (1-t_i)^{-\alpha} \right)^2.$$

Now we use similar expansions as in (S-6) for the terms $(1-t_i)^{-\alpha}$ and $(1-t_{i-2})^{-\alpha}$, and we obtain, when $\alpha + \beta < \frac{1}{2}$,

$$\sum_{i=2}^{n-1} \mathbb{E}[n_{i,2}^2 | \mathcal{F}_{t_{i-1}}] \overset{p}{\sim} \frac{C \Delta_n}{h_n^{2\alpha+2\beta+1}} \int_{-\infty}^{0} K(z) |z|^{-2\alpha-2\beta} \, dz + \frac{C \Delta_n^4}{h_n^2} \sum_{i=2}^{n-1} K_i^2 (1-t_{i-1})^{-2\beta-2\alpha-2},$$

which is $O_p\left(\frac{\Delta_n}{h_n^{2\alpha+2\beta+1}} \right)$, therefore C_n is of order $\frac{\Delta_n^{1/2}}{h_n^{\alpha+\beta+1/2}}$. On the other hand, when $\alpha + \beta >$

$\frac{1}{2}$, and neglecting the $o_p(1)$ term,

$$\sum_{i=2}^{n-1} \mathbb{E}[n_{i,2}^2|\mathcal{F}_{t_{i-1}}] \leqslant C\Delta_n^3 \frac{1}{h_n^2} \sum_{i=1}^{n} (1-t_i)^{-2\beta-2\alpha};$$
$$\sum_{i=2}^{n-1} \mathbb{E}[n_{i,2}^2|\mathcal{F}_{t_{i-1}}] \geqslant c\Delta_n^3 \frac{1}{h_n^2} \sum_{i=2}^{n-1} (1-t_{i-2})^{-2\beta-2\alpha},$$

so that, applying the same steps used for the term $D_{1,n}$, we conclude that $C_n \asymp \frac{\Delta_n^{1-\alpha-\beta}}{h_n}$ and, when $\alpha+\beta=\frac{1}{2}$, $C_n \asymp \frac{\Delta_n^{1/2}}{h_n}\sqrt{\log(n)}$.

Finally, for the variance term $V_n = \sum_{i=2}^n n_{i,3}$, with:

$$n_{i,3} := \frac{1}{h_n} K_i \left(\int_{t_{i-1}}^{t_i} \sigma_s (1-s)^{-\beta} dW_s \int_{t_{i-2}}^{t_{i-1}} \frac{1}{\sqrt{\Delta_{n,i-1}}} \sigma_s (1-s)^{-\beta} dW_s \right),$$

we have

$$\sum_{i=2}^{n} \mathbb{E}[n_{i,3}|\mathcal{F}_{t_{i-1}}] = 0,$$

and

$$\sum_{i=2}^{n} \mathbb{E}[n_{i,3}^2|\mathcal{F}_{t_{i-1}}] = \frac{1}{h_n^2} \sum_{i=2}^{n} K_i^2 \mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} \sigma_s (1-s)^{-\beta} dW_s \int_{t_{i-2}}^{t_{i-1}} \frac{1}{\sqrt{\Delta_{n,i-1}}} \sigma_s (1-s)^{-\beta} dW_s \right)^2 |\mathcal{F}_{t_{i-1}} \right]. \tag{S-7}$$

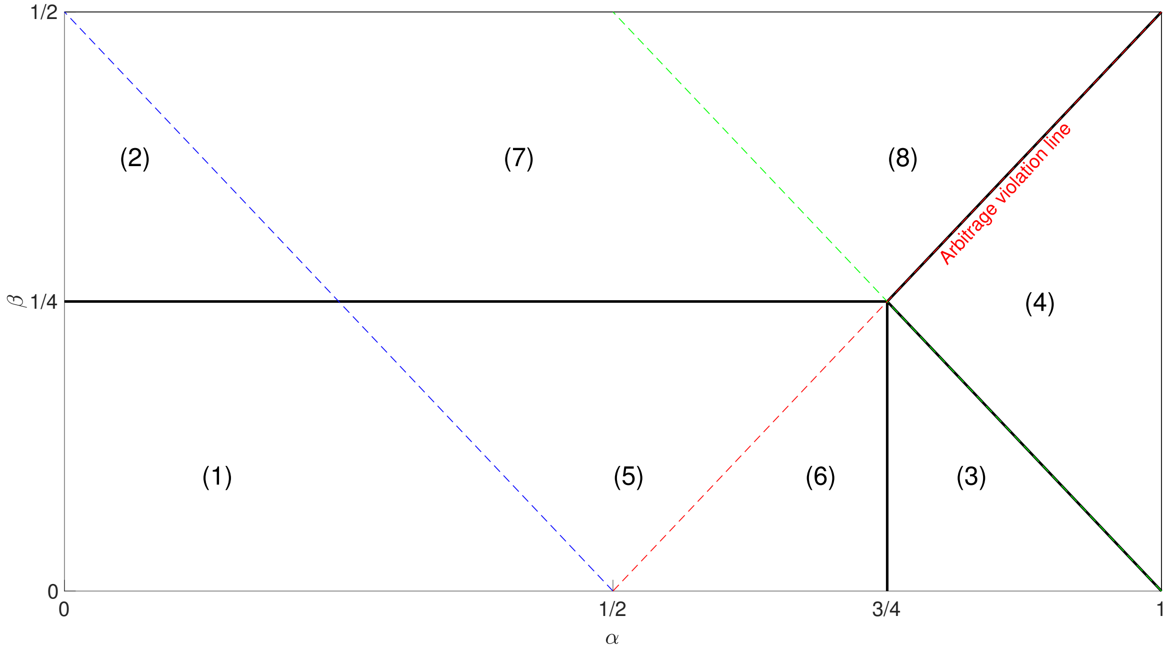
Using similar computations as for the $D_{1,n}$ term, the last expression can be decomposed into two parts, and the order of V_n is $\frac{1}{h_n^{2\beta+1/2}}$ when $\beta < \frac{1}{4}$, since in this case the dominating term of (S-7) is equivalent in probability to $\frac{C}{h_n^{4\beta+1}} \sigma_s^4 \int_{-\infty}^{0} K^2(x)|x|^{-4\beta}$.

When $\beta > \frac{1}{4}$, the order of V_n is $\frac{\Delta_n^{1/2-2\beta}}{h_n}$, and when $\beta = \frac{1}{4}$, the order is $\frac{\sqrt{\log(n)}}{h_n}$.

3. Now we turn to the denominator. Under Assumption 3, following similar steps to those in the proof of Theorem 2 in Christensen, Oomen, and Renò (2022), we obtain the order of the denominator: the rate is $\Delta_n^{2-2\alpha}/h_n + h_n^{-2\beta}$ when $\alpha - \beta > 1/2$, and it is $h_n^{-2\beta}$ when $\alpha - \beta \leqslant 1/2$.

4. Finally we combine the results of the different settings. Below, we enumerate the zones

Figure 9: (α, β)-plane



Note. The different regions in the (α, β) plane which characterize the different zones in the proof of Theorem 2.1.

accordingly to Figure 9, which is split in the relevant zones to analyze the rates (the splitting with final results being that of Figure 1, where the zones of Figure 9 that yield the same asymptotic results are merged).

(Zone 1) Here $\alpha + \beta < 1/2$ and $\beta < 1/4$. The rates to compare for the three terms D_n, C_n, V_n are, respectively, $\frac{\sqrt{\Delta_n}}{h_n^{2\alpha}}$, $\frac{\sqrt{\Delta_n}}{h_n^{\alpha+\beta+1/2}}$ and $\frac{1}{h_n^{2\beta+1/2}}$. Comparing the first and the second term, the latter dominates:

$$\frac{1}{h_n^{2\alpha}} \cdot h_n^{\alpha+\beta+1/2} = h_n^{-\alpha+\beta+1/2} \to 0$$

since $-\alpha + \beta + 1/2 > 0$. Then we compare the third term with the second term:

$$\frac{1}{h_n^{2\beta+1/2}} \cdot \frac{h_n^{\alpha+\beta+1/2}}{\Delta_n^{1/2}} = \frac{h_n^{\alpha-\beta}}{\Delta_n^{1/2}} \to \infty.$$

If $\alpha - \beta < 0$ this is immediate. If $\alpha - \beta > 0$, then we write

$$\frac{(h_n n)^{\alpha-\beta}}{\Delta_n^{1/2-\alpha+\beta}} \to \infty,$$

since $1/2 - \alpha + \beta > 0$. Therefore V_n dominates, and we get that the test is distributed as a normal distribution with mean 0 and standard deviation:

$$\sqrt{\frac{h_n}{K_2}}\frac{\sqrt{\frac{1}{h_n^{4\beta+1}}\sigma_1^4\int_{-\infty}^0 K^2(x)|x|^{-4\beta}}}{\sigma_1^2 h_n^{-2\beta}\int_{-\infty}^0 K^2(x)|x|^{-2\beta}} = \frac{\sqrt{K_2^{-1}\int_{-\infty}^0 K^2(x)|x|^{-4\beta}}}{\int_{-\infty}^0 K^2(x)|x|^{-2\beta}}. \tag{S-8}$$

(Zone 2) Here $\alpha+\beta<1/2$ and $\beta>1/4$. Compared to the previous zone, the only rate that changes is V_n, which becomes $\frac{\Delta_n^{1/2-2\beta}}{h_n}$ and now it dominates the rate of C_n since:

$$\frac{\Delta_n^{1/2-2\beta}}{h_n}\cdot\frac{h_n^{\alpha+\beta+1/2}}{\Delta_n^{1/2}} = \frac{\Delta_n^{-2\beta}}{h_n^{-\alpha-\beta+1/2}} \to \infty.$$

Therefore V_n dominates and $T_{n,1}^{\text{LAC}} \to \infty$, since the test statistic is of order

$$\sqrt{h_n}\frac{\Delta_n^{1/2-2\beta}}{h_n}h_n^{2\beta} = \frac{\Delta_n^{1/2-2\beta}}{h_n^{1/2-2\beta}} \to \infty$$

due to $\beta > 1/4$.

(Zone 3) Consider the case $\alpha+\beta<1$ jointly with $\alpha>3/4$ and $\beta<1/4$, which readily implies $\alpha+\beta>\alpha-\beta>1/2$. In this case, the rates to compare at the numerator are: $\frac{\Delta_n^{3/2-2\alpha}}{h_n}$ for D_n, $\frac{\Delta_n^{1-\alpha-\beta}}{h_n}$ for C_n, and $\sqrt{\frac{1}{h_n^{4\beta+1}}}$ for V_n. Comparing the rates, it turns out that D_n dominates, and similar computations as for the previous zone show that $T_{n,1}^{\text{LAC}} \to \infty$.

(Zone 4) Additionally, if $\alpha-\beta>1/2$ and $\alpha+\beta>1$, which implies $\alpha>1/2$, the rate for V_n is $\sqrt{\frac{\Delta_n^{1-4\beta}}{h_n^2}}$ if $\beta>1/4$. Therefore D_n still dominates the numerator, and again $T_{n,1}^{\text{LAC}} \to \infty$ for any possible rate of the denominator. For instance, when the denominator has rate $h_n^{-2\beta}$, the test statistic is of order

$$\sqrt{h_n}\frac{\Delta_n^{3/2-2\alpha}}{h_n}h_n^{2\beta} = \frac{\Delta_n^{3/2-2\alpha}}{h_n^{1/2-2\beta}} = \frac{\Delta_n^{3/2-2\alpha}}{h_n^{1/2-2\beta}}\cdot\frac{n^{1/2-2\beta}}{n^{1/2-2\beta}},$$

and the denominator converges to zero due to $nh_n \to \infty$, while the numerator converges to infinity due to $n^{-1-2\beta+2\alpha} \to \infty$.

(Zone 5) Here $\alpha+\beta>1/2$; $\beta<1/4$; $\alpha-\beta<1/2$. The rates to compare for the three terms are: $\frac{\Delta_n^{3/2-2\alpha}}{h_n}$ for D_n, $\frac{\Delta_n^{1-\alpha-\beta}}{h_n}$ for C_n, and $\sqrt{\frac{1}{h_n^{4\beta+1}}}$ for V_n if $\alpha>1/2$. If $\alpha<1/2$ the rate becomes $\frac{\Delta_n^{1/2}}{h_n^{2\alpha}}$ for D_n and the others are unchanged. In both cases V_n dominates over

C_n and D_n, hence the test statistic is distributed as in zone 1 since $\beta < 1/4$.

(Zone 6) Here $3/4 > \alpha > 1/2$; $\beta < 1/4$; $\alpha - \beta > 1/2$. The rates to compare for the three terms are the same as for zone 3. The condition $\alpha - \beta > 1/2$ implies that D_n dominates C_n. Therefore we compare D_n and V_n:

$$\frac{\Delta_n^{3/2-2\alpha}}{h_n} \cdot h_n^{2\beta+1/2} = \frac{\Delta_n^{3/2-2\alpha}}{h_n^{-2\beta+1/2}}; \tag{S-9}$$

the limit of the previous ratio is indeterminate, therefore either D_n or V_n dominate. To study the behavior of $T_{n,1}^{\mathrm{LAC}}$, we note that the denominator of $T_{n,1}^{\mathrm{LAC}}$ has a rate either of $\Delta_n^{2-2\alpha}/h_n$ or $h_n^{-2\beta}$.

If the former rate dominates in the denominator it implies that:

$$\frac{\Delta_n^{2-2\alpha}}{h_n^{1-2\beta}} = \frac{\Delta_n^{3/2-2\alpha}}{h_n^{1/2-2\beta}} \frac{\Delta_n^{1/2}}{h_n^{1/2}} \to \infty, \tag{S-10}$$

and since $\Delta_n^{1/2}/h_n^{1/2} \to 0$, it is the first term $\frac{\Delta_n^{3/2-2\alpha}}{h_n^{1/2-2\beta}}$ which goes to infinity. This condition implies that D_n dominates and the test explodes. If instead $\frac{\Delta_n^{2-2\alpha}}{h_n^{1-2\beta}} \to 0$ and $\frac{\Delta_n^{3/2-2\alpha}}{h_n^{1/2-2\beta}} \to 0$ then V_n dominates and the test is distributed as in zone 1. Finally, if $\frac{\Delta_n^{2-2\alpha}}{h_n^{1-2\beta}} \to 0$ but $\frac{\Delta_n^{3/2-2\alpha}}{h_n^{1/2-2\beta}} \to \infty$, D_n dominates and the test explodes again.

Note that the behaviour of the test, i.e. which term dominates asymptotically, depends on the bandwidth choice. Indeed, for a bandwidth $h_n \propto \Delta_n^c$ with $c > \frac{3/2-2\alpha}{1/2-2\beta}$, the ratio (S-9) becomes:

$$\frac{\Delta_n^{3/2-2\alpha}}{h_n^{1/2-2\beta}} = \Delta_n^{3/2-2\alpha-c(1/2-2\beta)} \to \infty,$$

and $T_{n,1}^{\mathrm{LAC}} \to \infty$. Instead, for $c < \frac{3/2-2\alpha}{1/2-2\beta}$ the previous ratio goes to 0 and the test is distributed as in zone 1 due to the condition $\beta < 1/4$.

(Zone 7) Here $1/2 < \alpha + \beta < 1$; $\beta > 1/4$. Both for $\alpha > 1/2$ and $\alpha < 1/2$ the dominating term is V_n . In both cases similar computations as above show that the test explodes.

(Zone 8) Here $\alpha - \beta < 1/2$; $\beta > 1/4$; $\alpha + \beta > 1$, and V_n dominates. As for the previous zone, the test explodes.

□

Proof. [Proof of Corollary 2.1]

The consistency of $\begin{pmatrix} T_{n,1}^{\text{LAC}} \\ \\ T_{n,1}^{\text{COR}} \end{pmatrix}$ follows from Theorem 1 of Christensen, Oomen, and Renò (2022) and from Theorem 2.1.

The covariance between the tests can be found by applying again Theorem 2.2.14 of Jacod and Protter (2012). The crucial part is the variance, since the other three conditions are immediate: therefore we study $\sum_{i=2}^{n} \mathbb{E}[(\tilde{M}_i)^2 | \mathcal{F}_{t_{i-1}}]$ with $\tilde{M}_i := \frac{K_i^2}{h_n^2 \sqrt{\Delta_{n,i-1}}} (\Delta_i^n X)^2 \left(\Delta_{i-1}^n X\right)$. Now, similarly to Proof of Theorem 2.1:

$$\sum_{i=2}^{n} \mathbb{E}[(\tilde{M}_i)^2 | \mathcal{F}_{t_{i-1}}] = h_n^{-4} \sum_{i=2}^{n} K_i^4 \left[\frac{\left(\int_{t_{i-2}}^{t_{i-1}} dW_s\right)^2}{\Delta_{i-1}^n} \mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} dW_s \right)^4 | \mathcal{F}_{t_{i-1}} \right] \right]$$

$$= C h_n^{-4} \sum_{i=2}^{n} K_i^4 \Delta_n^2 \frac{\left(\int_{t_{i-2}}^{t_{i-1}} dW_s\right)^2}{\Delta_{i-1}^n} = C \Delta_n h_n^{-3} \int K_s^4 ds.$$

Finally, we consider the standardization of the two test, applied to the previous term:

$$\frac{h_n^2}{K_2^2 \left(\widehat{\sigma^2}_1^n\right)^3} \sum_{i=2}^{n} \mathbb{E}[(\tilde{M}_i)^2 | \mathcal{F}_{t_{i-1}}] = C \Delta_n h_n^{-1} \int K_s^4 ds \to 0,$$

due to the boundedness of $\widehat{\sigma^2}_1^n$ and K_2, and to $h_n n \to \infty$.

The proof readily extends to the case of $\beta < 1/4$ and $\alpha - \beta < 1/2$, using the same computations of the proof of Theorem 2.1 for zones 1 and 5, where the rates of the numerator and denominator compensate, see (S-8), and the results of Theorem 2 of Christensen, Oomen, and Renò (2022). □

Proof. [Proof of Theorem 3.1]

Note that:

$$\overline{Y}_{i,n}^{+} = X_{t_i} + k_n^{-1} \sum_{j=0}^{k_n - 1} \left(X_{t_{i+j}} - X_{t_i} \right) + k_n^{-1} \sum_{j=0}^{k_n - 1} \epsilon_{t_{i+j}}, \tag{S-11}$$

and similarly

$$\overline{Y}_{i,n}^{-} = X_{t_i} + k_n^{-1} \sum_{j=0}^{k_n-1} \left(X_{t_{i-j}} - X_{t_i}\right) + k_n^{-1} \sum_{j=0}^{k_n-1} \epsilon_{t_{i-j}}. \tag{S-12}$$

In view of this we readily get the following decomposition (for $s = 1$):

$$\overline{\mathrm{LAC}}_1^n = \mathrm{LAC}_1^{L_n} + \sum_{j=1}^{8} R_{n,1}^j \tag{S-13}$$

where $\mathrm{LAC}_1^{L_n}$ is defined as:

$$\mathrm{LAC}_1^{L_n} = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) \left(X_{t_{(i+1)L_n}} - X_{t_{iL_n}}\right) \frac{\left(X_{t_{iL_n}} - X_{t_{(i-1)L_n}}\right)}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}} \tag{S-14}$$

and

$$R_{n,1}^1 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) k_n^{-1} \left(\sum_{j=0}^{k_n-1} \epsilon_{t_{(i+1)L_n-j}} - \epsilon_{t_{iL_n+j}}\right) \frac{k_n^{-1}\sum_{j=0}^{k_n-1} \epsilon_{t_{iL_n-j}} - \epsilon_{t_{(i-1)L_n+j}}}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}};$$

$$R_{n,1}^2 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) \left(k_n^{-1} \sum_{j=0}^{k_n-1} \left(X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}}\right) - \left(X_{t_{iL_n+j}} - X_{t_{iL_n}}\right)\right)$$
$$\cdot \frac{\left(k_n^{-1}\sum_{j=0}^{k_n-1} \left(X_{t_{iL_n-j}} - X_{t_{iL_n}}\right) - \left(X_{t_{(i-1)L_n+j}} - X_{t_{(i-1)L_n}}\right)\right)}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}};$$

$$R_{n,1}^3 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right)$$
$$\cdot \left(k_n^{-1} \sum_{j=0}^{k_n-1} \left(X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}}\right) - \left(X_{t_{iL_n+j}} - X_{t_{iL_n}}\right)\right) \frac{k_n^{-1}\sum_{j=0}^{k_n-1} \epsilon_{t_{iL_n-j}} - \epsilon_{t_{(i-1)L_n+j}}}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}};$$

$$R_{n,1}^4 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right)$$
$$\cdot \left(k_n^{-1} \sum_{j=0}^{k_n-1} \epsilon_{t_{(i+1)L_n-j}} - \epsilon_{t_{iL_n+j}}\right) \frac{\left(k_n^{-1}\sum_{j=0}^{k_n-1} \left(X_{t_{iL_n-j}} - X_{t_{iL_n}}\right) - \left(X_{t_{(i-1)L_n+j}} - X_{t_{(i-1)L_n}}\right)\right)}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}};$$

$$R_{n,1}^5 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) \left(X_{t_{(i+1)L_n}} - X_{t_{iL_n}}\right) \frac{k_n^{-1}\sum_{j=0}^{k_n-1} \epsilon_{t_{iL_n-j}} - \epsilon_{t_{(i-1)L_n+j}}}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}};$$

$$R_{n,1}^6 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) k_n^{-1} \left(\sum_{j=0}^{k_n-1} \epsilon_{t_{(i+1)L_n-j}} - \epsilon_{t_{iL_n+j}}\right) \frac{\left(X_{t_{iL_n}} - X_{t_{(i-1)L_n}}\right)}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}};$$

$$R_{n,1}^7 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right)\left(X_{t_{(i+1)L_n}} - X_{t_{iL_n}}\right) \frac{\sum_{j=0}^{k_n-1}\left(X_{t_{iL_n-j}} - X_{t_{iL_n}}\right) - \left(X_{t_{(i-1)L_n+j}} - X_{t_{(i-1)L_n}}\right)}{k_n\sqrt{t_{iL_n} - t_{(i-1)L_n}}};$$

$$R_{n,1}^8 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) \frac{\sum_{j=0}^{k_n-1}\left(X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}}\right) - \left(X_{t_{iL_n+j}} - X_{t_{iL_n}}\right)}{k_n} \frac{\left(X_{t_{iL_n}} - X_{t_{(i-1)L_n}}\right)}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}}.$$

Notice that given the assumptions on n, L_n, k_n and h_n, the proof of Theorem 2.1 immediately extends to $\mathrm{LAC}_1^{L_n}$ for the product of the returns $\left(X_{t_{(i+1)L_n}} - X_{t_{iL_n}}\right)\left(X_{t_{iL_n}} - X_{t_{(i-1)L_n}}\right)$, with the filtration $\mathcal{F}_i$ replaced by $\mathcal{F}_{iL_n}$ and $\widehat{\sigma^2}_1^n$ replaced by:

$$(\tilde{\sigma}_1^n)^2 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right)\left(\overline{Y}_{(i+1)L_n,n}^- - \overline{Y}_{iL_n,n}^+\right)^2.$$

Therefore, to prove the statement it is enough to show that the dominating term of $\overline{\mathrm{LAC}}_1^n$ in (S-13) is $\mathrm{LAC}_1^{L_n}$; and that the dominating term of $(\tilde{\sigma}_1^n)^2$ is:

$$(\tilde{\sigma}_1^{n,1})^2 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right)\left(X_{t_{(i+1)L_n}} - X_{t_{iL_n}}\right)^2.$$

We start to prove the theorem under the null hypothesis of no bursts.

In this case, $\mathrm{LAC}_1^{L_n}$ has order $h_n^{-1/2}$. Now prove that the order of the reminders are negligible compared to the order of $\mathrm{LAC}_1^{L_n}$. Starting from $R_{n,1}^1$, due to $L_n > k_n$ and the assumptions on the noise, $\mathbb{E}\left[R_{n,1}^1\right] = 0$. Moreover, there exists a $C > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[(R_{n,1}^1)^2\right] \\
&= \frac{1}{h_n^2} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K^2\left(\frac{t_{iL_n}-1}{h_n}\right) \cdot \mathbb{E}\left[\left(k_n^{-1}\sum_{j=0}^{k_n-1}\left(\epsilon_{t_{(i+1)L_n-j}} - \epsilon_{t_{iL_n+j}}\right) \cdot \frac{k_n^{-1}\sum_{j=0}^{k_n-1}\left(\epsilon_{t_{iL_n-j}} - \epsilon_{t_{(i-1)L_n+j}}\right)}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}}\right)^2\right] \\
&= \frac{1}{h_n^2 k_n^2}\frac{C}{L_n\Delta_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K^2\left(\frac{t_{iL_n}-1}{h_n}\right).
\end{aligned}$$

Therefore, since

$$\left|\frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K^2\left(\frac{t_{iL_n}-1}{h_n}\right)\left(t_{(i+1)L_n} - t_{iL_n}\right)\right| \leq C \qquad \text{(S-15)}$$

we conclude that

$$R^1_{n,1} = O_p\left(\frac{1}{h_n^{1/2} k_n L_n \Delta_n}\right). \tag{S-16}$$

To study the order of $R^2_{n,1}$, we apply again Theorem 2.2.14 of Jacod and Protter (2012); call

$$\begin{aligned} r^2_{n,1} := \frac{1}{h_n} K\left(\frac{t_{iL_n} - 1}{h_n}\right)\left(k_n^{-1}\sum_{j=0}^{k_n-1}\left(X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}}\right) - \left(X_{t_{iL_n+j}} - X_{t_{iL_n}}\right)\right) \\ \cdot \frac{\left(k_n^{-1}\sum_{j=0}^{k_n-1}\left(X_{t_{iL_n-j}} - X_{t_{iL_n}}\right) - \left(X_{t_{(i-1)L_n+j}} - X_{t_{(i-1)L_n}}\right)\right)}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}}, \end{aligned}$$

then the order of $R^2_{n,1}$ can be derived from the conditional variance:

$$\begin{aligned} &\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} \mathbb{E}[r^2_{n,1} \mid \mathcal{F}_{iL_n}] = \\ &\frac{1}{h_n^2}\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K^2\left(\frac{t_{iL_n}-1}{h_n}\right)\cdot \mathbb{E}\left[\left(k_n^{-1}\sum_{j=0}^{k_n-1}\left(X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}}\right) - \left(X_{t_{iL_n+j}} - X_{t_{iL_n}}\right)\right.\right. \\ &\left.\left.\cdot\frac{k_n^{-1}\sum_{j=0}^{k_n-1}\left(X_{t_{iL_n-j}} - X_{t_{iL_n}}\right) - \left(X_{t_{(i-1)L_n+j}} - X_{t_{(i-1)L_n}}\right)}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}}\right)^2 \middle| \mathcal{F}_{iL_n}\right]. \end{aligned}$$

Due to the conditional independence of the two products in the previous summation, and using $|X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}}| = O_p(\sqrt{k_n\Delta_n})$ we obtain:

$$\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} \mathbb{E}[r^2_{n,1} \mid \mathcal{F}_{iL_n}] \leq \frac{C}{h_n^2 L_n \Delta_n}\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K^2\left(\frac{t_{iL_n}-1}{h_n}\right)\frac{k_n^2(k_n\Delta_n)^2 + k_n^4(k_n\Delta_n)^2}{k_n^4}.$$

Now using again (S-15) we get:

$$R^2_{n,1} = O_p\left(\frac{k_n}{h_n^{1/2} L_n}\right). \tag{S-17}$$

Similar computations applied to the components of $R^3_{n,1}$, together with the independence of the price and noise processes, lead to:

$$R^3_{n,1} = O_p\left(\frac{1}{h_n^{1/2} L_n \sqrt{\Delta_n}}\right).$$

The previous result holds replacing $R^3_{n,s}$ with $R^4_{n,1}$ due to symmetry. A similar argument also gives:

$$R^5_{n,1} = O_p\left(\frac{1}{\sqrt{h_n k_n L_n \Delta_n}}\right). \tag{S-18}$$

and the same applies to $R^6_{n,1}$. Finally, we have:

$$R^7_{n,1} = O_p\left(\frac{\sqrt{k_n}}{\sqrt{h_n L_n}}\right), \tag{S-19}$$

and similarly for $R^8_{n,1}$.

It is immediate to verify that, given the conditions $L_n k_n \Delta_n \to \infty$ and $L_n/k_n \to \infty$, the order of the reminders is asymptotically negligible compared to the order of $\mathrm{LAC}_1^{L_n}$.

Moving to the denominator, employing again (S-11) and (S-12), the order of $(\tilde{\sigma}_1^n)^2$ can be studied from the order of following components:

$$(\tilde{\sigma}_1^{n,1})^2 = \frac{1}{h_n}\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right)\left(X_{t_{(i+1)L_n}} - X_{t_{iL_n}}\right)^2;$$

$$(\tilde{\sigma}_1^{n,2})^2 = \frac{1}{h_n}\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) k_n^{-2}\left(\sum_{j=0}^{k_n-1} \epsilon_{t_{(i+1)L_n-j}} - \epsilon_{t_{iL_n+j}}\right)^2;$$

$$(\tilde{\sigma}_1^{n,3})^2 = \frac{1}{h_n}\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) k_n^{-2}\left(\sum_{j=0}^{k_n-1} \left(X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}}\right) - \left(X_{t_{iL_n+j}} - X_{t_{iL_n}}\right)\right)^2,$$

and by the cross-products

$$(\tilde{\sigma}_1^{n,4})^2 = \frac{1}{h_n}\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right)\left(X_{t_{(i+1)L_n}} - X_{t_{iL_n}}\right) k_n^{-1}\left(\sum_{j=0}^{k_n-1} \epsilon_{t_{(i+1)L_n-j}} - \epsilon_{t_{iL_n+j}}\right);$$

$$(\tilde{\sigma}_1^{n,5})^2 = \frac{1}{h_n}\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) k_n^{-2}\left(\sum_{j=0}^{k_n-1} \epsilon_{t_{(i+1)L_n-j}} - \epsilon_{t_{iL_n+j}}\right) \cdot \left(\sum_{j=0}^{k_n-1} \left(X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}}\right) - \left(X_{t_{iL_n+j}} - X_{t_{iL_n}}\right)\right);$$

$$(\tilde{\sigma}_1^{n,6})^2 = \frac{1}{h_n}\sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-1}{h_n}\right) k_n^{-1}\left(\sum_{j=0}^{k_n-1} \left(X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}}\right) - \left(X_{t_{iL_n+j}} - X_{t_{iL_n}}\right)\right) \cdot \left(X_{t_{(i+1)L_n}} - X_{t_{iL_n}}\right).$$

The order of $(\tilde{\sigma}_1^{n,1})^2$ is 1. Similar computations as for the numerator show that $(\tilde{\sigma}_1^{n,2})^2$ is $O_p(\frac{1}{k_n L_n \Delta_n})$ which is negligible due to $k_n L_n \Delta_n \to \infty$. Similarly, $(\tilde{\sigma}_1^{n,3})^2$ is $O_p(\frac{k_n}{L_n})$, which is also $o(1)$. Moreover, the orders of the cross-terms are $(\tilde{\sigma}_1^{n,4})^2 = O_p(\frac{1}{\sqrt{h_n k_n}})$, $(\tilde{\sigma}_1^{n,5})^2 = O_p(\frac{1}{\sqrt{L_n h_n}})$ and $(\tilde{\sigma}_1^{n,6})^2 = O_p\left(\sqrt{\frac{k_n \Delta_n}{h_n}}\right)$. Due to the assumptions on the h_n, k_n and L_n the cross terms are negligible and we conclude.

Now we consider the case $0 < \alpha < 1$ and $0 < \beta < 1/2$, when the drift and/or the volatility burst. In this case, referring to the zones of Figure 9:

$$\begin{aligned}
\mathrm{LAC}_s^{L_n} &\asymp \frac{1}{h_n^{2\beta+\frac{1}{2}}}, && \text{for zones 1, 5 and 6;} \\
\mathrm{LAC}_s^{L_n} &\asymp \frac{\left(L_n \Delta_n\right)^{\frac{1}{2}-2\beta}}{h_n}, && \text{for zones 2, 7 and 8;} \\
\mathrm{LAC}_s^{L_n} &\asymp \frac{\left(L_n \Delta_n\right)^{\frac{3}{2}-2\alpha}}{h_n}, && \text{for zones 3, 4 and 6.}
\end{aligned}$$

Since the rates above are dominating $h_n^{-1/2}$, which is the rate of $\mathrm{LAC}_s^{L_n}$ under no bursts, it follows that the reminder term $R_{n,s}^1$ is again negligible compared to $\mathrm{LAC}_s^{L_n}$. In order to study the other reminders, note that using the notation:

$$\tilde{X}_{n,i}^2 = \left(k_n^{-1} \sum_{j=0}^{k_n-1} \left(X_{t_{(i+1)L_n-j}} - X_{t_{(i+1)L_n}} \right) - \left(X_{t_{iL_n+j}} - X_{t_{iL_n}} \right) \right)$$

we can write

$$R_{n,s}^2 = \frac{1}{h_n} \sum_{i=1}^{[\frac{n-k_n+1}{L_n}-1]} K\left(\frac{t_{iL_n}-s}{h_n}\right) \tilde{X}_{n,i}^2 \frac{\tilde{X}_{n,i-1}^2}{\sqrt{t_{iL_n} - t_{(i-1)L_n}}}$$

Compared to $\mathrm{LAC}_s^{L_n}$, the reminder above contains the product $\tilde{X}_{n,i}^2 \tilde{X}_{n,i-1}^2$ inside the summation, instead of $\left(X_{t_{(i+1)L_n}} - X_{t_{iL_n}}\right)\left(X_{t_{iL_n}} - X_{t_{(i-1)L_n}}\right)$. The order of the former is $(k_n\Delta_n)^{2-2\alpha} + (k_n\Delta_n)^{1-\beta}$, which is dominated by the order of the latter, which is $(L_n\Delta_n)^{2-2\alpha} + (L_n\Delta_n)^{1-\beta}$, due to $L_n/k_n \to \infty$. Therefore $\mathrm{LAC}_s^{L_n}$ dominates $R_{n,s}^2$. A similar argument applies to the other reminders, as well as to the denominator: the comparison in this cases is immediate, using the previous results together with $k_n^{-1} \sum_{j=0}^{k_n-1} \epsilon_{t_{iL_n-j}} - \epsilon_{t_{(i-1)L_n+j}} = O_p(k_n^{-1/2})$. □