Empirical Welfare Maximization with Constraints*

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Abstract

When designing eligibility criteria for welfare programs, policymakers naturally want to target the individuals who will benefit the most given their cost to the program. This paper extends the previous literature on Empirical Welfare Maximization (EWM) for selecting eligibility criteria based on data by allowing for uncertainty in estimating the budget needed to implement the criterion, in addition to its benefit. Due to the additional estimation error, the EWM rule no longer selects eligibility criteria that consistently achieve the highest benefit possible while satisfying a budget constraint uniformly. The lack of uniformity is shown to apply to any statistical rule. I also propose two new statistical rules that perform better than the EWM rule under a budget constraint, and use them to select eligibility criteria for Medicaid expansion based on experimental data, a setting with imperfect take-up and varying program costs.

Keywords: empirical risk minimization, heterogeneous treatment effects and costs, cost-benefit analysis, individualized treatment rules

JEL classification codes: C14, C44, C52

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1 Introduction

When a welfare program induces varying benefits across individuals, and when resources are scarce, policymakers naturally want to prioritize eligibility to individuals who will benefit the most. Based on experimental data, cost-benefit analysis can inform policymakers on which subpopulations to prioritize, but these subpopulations might not align with any available eligibility criterion such as an income threshold. Kitagawa and Tetenov (2018) propose a statistical rule, Empirical Welfare Maximization (EWM), that can directly select an eligibility criterion from a set of available criteria based on the experimental data. For example, if available criteria take the form of income thresholds, EWM considers the problem of maximizing the expected benefits in the population

$$\max_{t \leq \bar{t}} \; E[\text{benefit} \cdot \mathbf{1}\{\text{income} \leq t\}]$$

and approximates the optimal threshold based on benefits estimated from experimental data. Recent papers have argued the EWM approach is attractive under a wide range of data distributions, including Athey and Wager (2021), among others. Notably the average benefits under the eligibility criterion selected by EWM converges to the highest attainable benefits as the sample size grows. This property can be thought of as *uniform asymptotic welfare efficiency.*

In practice policymakers often face budget constraints, but only have imperfect information about whether a given eligibility criterion satisfies the budget constraint. First, there may be imperfect take-up: eligible individuals might not participate in the welfare program, resulting in zero cost to the government, e.g. Finkelstein and Notowidigdo (2019). Second, costs incurred by eligible individuals who participate in the welfare program may vary considerably, largely driven by individuals' different needs but also many other factors, e.g. Finkelstein et al. (2017). Both considerations are hard to predict ex ante, implying that the potential cost of providing eligibility to any given individual is unknown at the time of designing the eligibility criterion. Unobservability of the potential cost requires estimation based on experimental data, contributing to uncertainty in the budget estimate of a given eligibility criterion.

For this empirically relevant setting where the budget needed to implement an eligibility criterion involves an unknown cost, this paper introduces a new property of statistical rules, namely *asymptotic feasibility*. A statistical rule is asymptotically feasible if given a large enough experimental sample, the statistical rule is very likely to select feasible eligibility criteria that satisfy the budget constraint in the target population. Since a potential cost overrun due to an underestimate of the budget can be very costly to policy makers, I argue that *asymptotically feasibility* is an equally desirable property in the current setting, in addition to *asymptotic welfare efficiency*. This paper answers three questions in the current setting with unknown cost: whether any statistical rule can achieve uniform good performance, whether the obvious extension of the existing EWM statistical rule remains attractive, and what are the trade-offs among alternative statistical rules.

An ideal statistical rule should maintain good performance for a wide range of data distributions. I therefore define *uniform asymptotic feasibility* and *uniform asymptotic welfare efficiency*, which impose *asymptotic feasibility* and *asymptotic welfare efficiency* uniformly over a class of reasonable data distributions, respectively. Firstly as an important theoretical contribution, I prove an impossibility result that no statistical rule can satisfy these two uniform properties simultaneously.

Second, I show the direct extension of the existing EWM statistical rule is not appealing in the setting with unknown cost. For certain real-world relevant data distributions where the budget constraint is exactly binding, the direct extension is neither asymptotically feasible nor asymptotically welfare efficient. The reason is that this plug-in extension ignores the estimation error in the estimated budget constraint, which has non-negligible consequences even when the sample size is large.

Finally I show there exist alternative statistical rules that can achieve either uniform asymptotic feasibility or uniform asymptotic welfare efficiency. The first statistical rule I propose, the *mistake-controlling rule*, achieves the highest benefit possible while satisfying a budget constraint with high probability. Therefore, scaling the significance level inversely proportional to the sample size, I show the mistake-controlling rule satisfies uniform asymptotic feasibility. The second statistical rule I propose, the *trade-off rule*, can select infeasible eligibility criteria, but only if borrowing money and exceeding the budget constraint are not too costly to justify the marginal gain in benefit from violating the budget constraint. I show the trade-off rule is uniformly asymptotically welfare efficient. The choice between the two rules should align with policymakers' attitudes toward the budget constraint. If policymakers are financially conservative, then the eligibility criterion selected by the mistake-controlling rule is more suitable. If policymakers want to reach as many individuals as possible, then the eligibility criterion selected by the trade-off rule is more suitable.

To illustrate the statistical rules proposed in this paper, I consider a budget-constrained Medicaid expansion. Medicaid is a government-sponsored health insurance program intended for the low-income population in the United States. Based on the Oregon Health Insurance Experiment (OHIE) conducted in 2008, I derive Medicaid expansion criteria using the two statistical rules I propose. Oregon's current Medicaid expansion criterion is based on a threshold for household income only. In my application, I examine whether health can be improved by allowing the income threshold to vary by the number of children in the household, setting the budget constraint to the current level. The mistake-controlling rule selects an eligibility criterion that limits eligibility to be lower than the current level. This lower level occurs because Medicaid costs have high variability, making it harder to verify whether households meet the budget constraint based on the OHIE data. In contrast, the trade-off rule selects an eligibility criterion that expands eligibility for many households above the current level, especially those with children. The higher level occurs because, based on the OHIE data, Medicaid improves health for these households, and the additional health benefit from violating the budget constraint exceeds the cost of doing so, assuming a reasonable upper bound on the monetary value on the health benefit.

The rest of the paper proceeds as follows. Section 1.1 discusses related work in more detail. Section 2 presents theoretical results. Sections 3 and 4 propose two statistical rules for selecting eligibility criteria. Section 5 conducts a simulation study to illustrate the asymptotic properties of the statistical rules I propose. Section

6 considers an empirical example of designing a more flexible Medicaid expansion criterion for the low-income population in Oregon. Section 7 concludes. Proofs may be found in the Appendix. Supporting lemmas and additional results may be found in the Online Appendix.

1.1 Literature review

This paper is related to the traditional literature on cost-benefit analysis, e.g. Dhaliwal et al. (2013), and to the recent literature on EWM, e.g. Kitagawa and Tetenov (2018), Rai (2019), Athey and Wager (2021) and Mbakop and Tabord-Meehan (2021). More broadly, this paper contributes to a growing literature on statistical rules in econometrics, including Manski (2004), Dehejia (2005), Hirano and Porter (2009), Stoye (2009), Chamberlain (2011), Bhattacharya and Dupas (2012), Demirer et al. (2019), Carneiro et al. (2020), Kasy and Sautmann (2021), Sun et al. (2021), Yata (2021) and Kitagawa et al. (2022), among others.

The traditional cost-benefit analysis compares the cost and benefit of a given welfare program. The effect of program eligibility is first estimated based on a randomized control trial (RCT), and then converted to a monetary benefit for calculating the cost-benefit ratio. For example, Gelber et al. (2016) and Heller et al. (2017) compare the efficiency of various crime prevention programs based on their cost-benefit ratios. However, the cost-benefit ratio is only informative for whether this welfare program should be implemented with the fixed eligibility criterion as implemented in the RCT.

The literature on statistical rules in econometrics has also developed a definition for optimality of statistical rules. Manski (2004) considers the minimax-regret, defined to be loss in expected welfare achieved by the statistical rule relative to the welfare achieved by the theoretically optimal eligibility criterion. In the absence of any constraint, under the theoretically optimal eligibility criterion, anyone with positive benefit from the welfare program would be assigned with eligibility. The minimax regret is the upper bound on the regret that results from not knowing the data distribution. Manski (2004) argues a statistical rule is preferred if its minimax-regret converges to zero with the sample size, and analyzes the minimax-regret for the class of statistical rules that only selects eligibility criterion based on subsets of the observed covariates. Stoye (2009) shows that with continuous covariates and no functional form restrictions on the set of criteria, minimax regret does not converge to zero with the sample size because the theoretically optimal criterion can be too difficult to approximate by statistical rule. Kitagawa and Tetenov (2018) avoid this issue by imposing functional form restrictions. They propose the EWM rule, which starts with functional form restrictions on the class of available criteria, and then selects the criterion with the highest estimated benefit (empirical welfare) based on an RCT sample. They prove the optimality of EWM in the sense that its regret converges to zero at the minimax rate. Importantly, the regret is defined to be loss in expected welfare relative to the maximum achievable welfare in the constrained class, which avoids the negative results of Stoye (2009). Athey and Wager (2021) propose doubly-robust estimation of the average benefit, which leads to an optimal rule even with quasi-experimental data. Mbakop and Tabord-Meehan (2021) propose a Penalized Welfare Maximization assignment rule which relaxes restrictions of the criterion class.

The existing EWM literature has not addressed budget constraints with an unknown cost. Kitagawa and Tetenov (2018) consider a capacity constraint, which they enforce using random rationing. Random rationing is not ideal as it uses the limited resource less efficiently than accounting for the cost of providing the welfare program to an individual. When there is no restriction on the functional form of the eligibility criterion, Bhattacharya and Dupas (2012) demonstrate that given a capacity constraint, the optimal eligibility criterion is based on a threshold on the benefit of the welfare program to an individual. When the cost of providing the welfare program to an individual is heterogeneous, however, budget constraints can be more complicated than capacity constraints, and require estimation. Carneiro et al. (2020) considers the optimal choice of covariate collection in order to maximize the precision of the estimation for the average treatment effect. While they also consider a constrained decision problem, the budget constraint can be verified directly. They also allow for a more complicate trade-off between additional covariates and additional observations. Sun et al. (2021) propose a framework for estimating the optimal rule under a budget constraint when there is no functional form restriction. The theoretical contribution of this paper is to extend the literature by allowing both functional form restrictions and budget constraints with an unknown cost.

2 Theoretical results

In this paper I focus on the problem of designing welfare program eligibility criterion under budget constraints, in particular when the cost of providing eligibility to any given individual is unknown ex ante. Section 2.1 explains how this problem translates to a constrained optimization problem that policymakers aim to solve. Section 2.2 defines two desirable properties of statistical rules, which selects eligibility criteria based on experimental data. Sections 2.3 and 2.4 derive some negative implications of these difficulties on finding statistical rules with good properties.

2.1 Motivation and setup

I begin by setting up a general constrained optimization problem, which depends on the following random attributes of an individual:

$$A = (\Gamma, R, X) \in \mathcal{A} \subseteq \mathbb{R}^{2+p}. \tag{1}$$

Here Γ is her benefit from the treatment, R is the cost to the policymaker of providing her with the treatment, and $X \in \mathcal{X} \subset \mathbb{R}^p$ denotes her p-dimensional characteristics. The individual belongs to a population that can be characterized by the joint distribution P on her random attributes A. The unknown distribution $P \in \mathcal{P}$ is from a class of distributions $\mathcal{P}$.

A policy $g(X) \in \{0, 1\}$ determines the treatment status for an individual with observed characteristics X, where 1 is treatment and 0 is no treatment. Let $\mathcal{G}$ denote the class of policies policymakers can choose from. The optimization problem is to find a policy with maximal benefit while subject to a constraint on its cost:[1]

$$\max_{g \in \mathcal{G}} E_P[\Gamma \cdot g(X)] \ s.t. \ E_P[R \cdot g(X)] \leq k. \tag{2}$$

If the policymaker does not have a fixed budget but still want to account for cost, the scalar Γ can be the difference in benefit and cost. Since a fixed budget is common, I impose a harsh constraint.

The benefit-cost attributes (Γ, R) of any given individual may or may not be observed. When the benefit-cost attributes are observed, policymakers observe i.i.d. $A_i = (\Gamma_i, R_i, X_i)$ of individual i in a random sample of the target population. When the benefit-cost attributes are unobserved, the focus of this paper is the setting where policymakers can construct their estimates (Γ_i^*, R_i^*) in a random sample of sample size n along with the characteristics X_i from an experiment or quasi-experiment that satisfy Assumption (2.4) discussed later.

Applying the Law of Iterated Expectation, the constrained optimization problem (2) can be written as

$$\max_{g \in \mathcal{G}} E_P[E_P[\Gamma \mid X] \cdot g(X)] \ s.t. \ E_P[E_P[R \mid X] \cdot g(X)] \leq k. \tag{3}$$

When the eligibility criterion can be based on any characteristics whatsoever, the class of available criteria is unrestricted i.e. $\mathcal{G} = 2^{\mathcal{X}}$. In this unrestricted class, when the cost is non-negative, the above expression makes clear that the optimal eligibility criterion is based on thresholding by the benefit-cost ratio $E_P[\Gamma \mid X]/E_P[R \mid X]$ where the numerator and the denominator are respectively the average effects conditional on the observed characteristics (CATE) and the conditional average resource required. Online Appendix B provides a formal statement.

Given a random sample, to approximate the optimal eligibility criterion g_P^*, one can estimate the benefit-cost ratio based on the estimated CATE and the estimated conditional average resource required. The resulting statistical rule selects eligibility criteria that are thresholds based on the estimated benefit-cost ratio. The challenge

[1]Following Kitagawa and Tetenov (2018), I implicitly assume the maximizer exists in $\mathcal{G}$ with the notation in (2).

is that the selected eligibility criterion can be hard to implement when the estimated benefit-cost ratio is a complicated function of X. Restrictions on the criterion class $\mathcal{G}$ address this issue. A common restriction is to consider thresholds based directly on X, e.g. assigning eligibility when an individual's income is below a certain value.

Restrictions on the criterion class $\mathcal{G}$ mean that there might not be closed-form solutions to the population problem (2). In particular, the constrained optimal eligibility criterion g_P^* might not be an explicit function of the CATE and the conditional average resource required. Therefore it might be difficult to directly approximate g_P^* based on the estimated CATE and the estimated conditional average resource required. However, this is not an obstacle to deriving the statistical rules I propose. As I demonstrate later, the derivation does not require the knowledge of the functional form of the constrained optimal eligibility criterion g_P^*.

I next specialize the constrained optimization problem to selecting eligibility criterion for welfare programs with the example of Medicaid expansion. In the example of implementing welfare programs, policies take the form of eligibility criteria. I restrict attention to non-randomized policies as in the leading example of welfare programs, deterministic policies such as income thresholds are more relevant. While the optimization problem policymakers face in practice can take on different forms, constrained optimization problems of this mathematical structural are ubiquitous in social science. Theoretically oriented readers may proceed directly to Section 2.2.

Example 2.1. Welfare program eligibility criterion under budget constraint

Suppose the government wants to implement some welfare program. The treatment in this example is eligibility for such welfare program. Due to a limited budget, the government cannot make eligibility universal and can only provide eligibility to a subpopulation. To use the budget efficiently, policymakers consider the constrained optimization problem (2). In this example, the policy $g(X)$ assigns an individual to eligibility based on her observed characteristics X, and is usually referred to as an eligibility criterion. I denote Γ to be the benefit experienced by an individual after receiving eligibility for the welfare program. Specifically, let (Y_1, Y_0) denote the potential outcomes that would have been observed if an individual were assigned with and without eligibility, respectively. The benefit from eligibility criterion is therefore defined as $\Gamma := Y_1 - Y_0$. Note that maximizing benefit is equivalent to maximizing the outcomes (welfare) under the utilitarian social welfare function: $E_P[Y_1 \cdot g(X) + Y_0(1 - g(X))]$. I denote R to be the potential cost from providing an individual with eligibility for the welfare program. Both Γ and R are unobserved at the time of assignment and will need to be estimated.

Policymakers might be interested in multiple outcomes for an in-kind transfer program. Hendren and Sprung-Keyser (2020) capture benefits by the willingness to pay (WTP). Assuming eligible individuals make optimal choices across multiple outcomes, the envelope theorem allows policymakers to focus on benefit in terms of one particular outcome.

Medicaid Expansion Medicaid is a government-sponsored health insurance program intended for the low-income population in the United States. Up till 2011, many states provided Medicaid eligibility to able-bodied adults with income up to 100% of the federal poverty level. The 2011 Affordable Care Act (ACA) provided resources for states to expand Medicaid eligibility for all adults with income up to 138% of the federal poverty level starting in 2014.

Suppose policymakers want to maximize the health benefit of Medicaid by adopting a more flexible expansion criterion. Specifically, the more flexibility criterion relaxes the uniform income threshold of 138% and allows the income thresholds to vary with the number of children in the household. It may be infeasible to implement a new expansion criterion if it costs more than the current one. Therefore I impose the constraint that the more flexible criterion must cost no more than the current one.

Correspondingly, the constrained optimization problem (2) sets Γ to be the health benefit from Medicaid, R to be the *excess* per-enrollee cost of Medicaid relative to the current expansion criterion, and the appropriate threshold to be $k = 0$. The characteristics X include both income and number of children in the household.

The criterion class in this example includes income thresholds that can vary with the number of children in the household:

$$\mathcal{G} = \left\{ g(x) = \begin{cases} \mathbf{1}\{\text{income} \leq \beta_1\} & , \ \text{numchild} = 1 \\ \vdots & \\ \mathbf{1}\{\text{income} \leq \beta_j\} & , \ \text{numchild} = j \end{cases} \right\} \tag{4}$$

for characteristics $x = (\text{income}, \ \text{numchild})$ and $\beta_j \geq 0$. $\triangle$

2.2 Desirable properties for statistical rules

To simplify the notation, I define the *welfare* function and the *budget* function:

$$W(g; P) = E_P[\Gamma \cdot g(X)], \ B(g; P) = E_P[R \cdot g(X)]. \tag{5}$$

As explained in Example 2.1 from Section 2, under a utilitarian social welfare function, maximizing the benefit with respect to eligibility criterion is equivalent to maximizing the welfare, which is why I refer to $W(g; P)$ as the welfare function. The welfare function and the budget function are both deterministic functions from $\mathcal{G} \to \mathbb{R}$. The index by the distribution P highlights that welfare and budget of criterion g vary with P, and in particular, whether a criterion g satisfies the budget constraint depends on which distribution P is of interest.

When the benefit-cost attributes (Γ, R) are unobserved and the distribution P is unknown, both the welfare function and the budget function are unknown functions. Denote by $\widehat{g}$ a statistical rule that selects an eligibility criterion after observing some experimental data of sample size n distributed according to P^n. This section provides formal definitions for two desirable properties of $\widehat{g}$.

Definition 2.1. A statistical rule $\widehat{g}$ is *pointwise asymptotically welfare-efficient* under the data distribution P if for any $\epsilon > 0$

$$\lim_{n \to \infty} Pr_{P^n} \left\{ W(\widehat{g}; P) - W(g^*_P; P) < -\epsilon \right\} = 0, \tag{6}$$

and *uniformly asymptotically welfare-efficient* over the class of distributions $\mathcal{P}$ if for any $\epsilon > 0$

$$\lim_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n} \{W(\widehat{g}; P) - W(g^*_P; P) < -\epsilon\} = 0.$$

A statistical rule is *pointwise asymptotically feasible* under the data distribution P if

$$\lim_{n\to\infty} Pr_{P^n}\{B(\widehat{g}; P) > k\} = 0,$$

and *uniformly asymptotically feasible* over the class of distributions $\mathcal{P}$ if

$$\lim_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n}\{B(\widehat{g}; P) > k\} = 0. \tag{7}$$

■

The above two properties build on the existing EWM literature. For the first property, the current EWM literature considers statistical rules that select eligibility criteria that attain the same value as g^*_P in expectation over repeated sample draws as $n \to \infty$. Instead, I strengthen the convergence in mean to convergence in probability, where the probability of $\widehat{g}$ selecting eligibility criteria that achieve strictly lower welfare than g^*_P approaches zero as $n \to \infty$.[2] In the setting of the existing EWM literature, the constrained optimal criterion g^*_P is also the unconstrained optimal criterion $\mathcal{G}$. Therefore it is impossible for any statistical rule $\widehat{g}$ to select a criterion that achieves higher value than g^*_P. In my setting, however, the constrained optimal criterion g^*_P is not necessarily the unconstrained optimal criterion. Thus, I allow the statistical rule $\widehat{g}$ to select a criterion that achieves higher welfare than g^*_P for all data distributions, albeit only at the cost of violating the budget constraint.

The second property is new to the EWM literature. It imposes that given a large enough sample size, the statistical rule $\widehat{g}$ is unlikely to select infeasible eligibility criteria that violate the budget constraint, so that it is "asymptotically feasible".[3] Asymptotic feasibility of statistical rules is specific to the current setting where the

[2] A more precise definition replaces the probability in (6) with $Pr_{P^n}\{\omega : W(\widehat{g}(\omega); P) - W(g^*_P; P) < -\epsilon\}$, the probability over repeated samples drawn from P^n that $\widehat{g}$ selects eligibility criteria that achieves welfare strictly less than maximum feasible level in the population distributed according to P.

[3] A more precise definition replaces the probability in (7) with $Pr_{P^n}\{\omega : B(\widehat{g}(\omega); P) > k\}$, the probability over repeated samples drawn from P^n that $\widehat{g}$ selects eligibility criteria that violate the budget constraint in the population distributed according to P.

budget constraint involves unknown cost.

While both are desirable properties, the next section shows a negative result that it is impossible for a statistical rule to satisfy both properties when the data distribution is unknown and belongs to a sufficiently rich class of distributions $\mathcal{P}$.

2.3 Impossibility result

Theorem 2.1 presents an impossibility result that no statistical rule can be both uniformly asymptotically welfare-efficient and uniformly asymptotically feasible in a sufficiently rich class of distributions $\mathcal{P}$. Assumption 2.1 explains the notion of richness: the sets of feasible eligibility criteria differ only marginally at nearby pairs of distributions. Assumptions 2.2 and 2.3 characterize distributions where such richness can be problematic for statistical rules to simultaneously achieve uniform asymptotic welfare-efficiency and uniform asymptotic feasibility.

Assumption 2.1. Contiguity. *There exists a distribution $P_0 \in \mathcal{P}$ under which a non-empty set of eligibility criteria satisfies the constraint exactly $\mathcal{G}_0 = \{g : B(g; P_0) = k\}$. Furthermore, the class of distributions $\mathcal{P}$ includes a sequence of data distributions $\{P_{h_n}\}$ contiguous to P_0, under which for all $g \in \mathcal{G}_0$, there exists some $C > 0$ such that*

$$\sqrt{n} \cdot (B(g; P_{h_n}) - k) > C.$$

Assumption 2.2. Binding constraint. *Under the data distribution P_0, the constraint is satisfied exactly at the constrained optimum i.e. $B(g^*_{P_0}; P_0) = k$.*

Assumption 2.3. Separation. *Under the data distribution P_0, $\exists \epsilon > 0$ such that for any feasible criterion g, whenever*

$$\left| B(g; P_0) - B(g^*_{P_0}; P_0) \right| > 0,$$

we have

$$W(g^*_{P_0}; P_0) - W(g; P_0) > \epsilon.$$

Equivalently, $W(g^*_{P_0}; P_0)$ is separated from that of other feasible criteria with different $B(g; P_0)$.

In Online Appendix B.2, I give more primitive assumptions under which Assumption 2.1 is guaranteed to hold. These primitive assumptions are relatively weak. Furthermore, it is not implausible for real-world distributions to satisfy both Assumptions 2.2 and 2.3. Hendren and Sprung-Keyser (2020) estimate fourteen welfare programs (out of 133) to have negative or zero net cost to the government, which implies these programs "pay for themselves". To see how these program can satisfy both Assumptions 2.2 and 2.3, consider a one-dimensional criterion class $\mathcal{G}$, e.g. income thresholds. Figure 2.1 illustrates a distribution that satisfies both Assumptions 2.2 and 2.3, while both the welfare function $W(g; P_0)$ and the budget function $B(g; P_0)$ are still continuous in g. Importantly, there exists a neighborhood around $g^*_{P_0}$ where feasible criteria can achieve welfare gain without any effect on the budget. This suggests the impossibility results stated below are relevant in real-world settings.

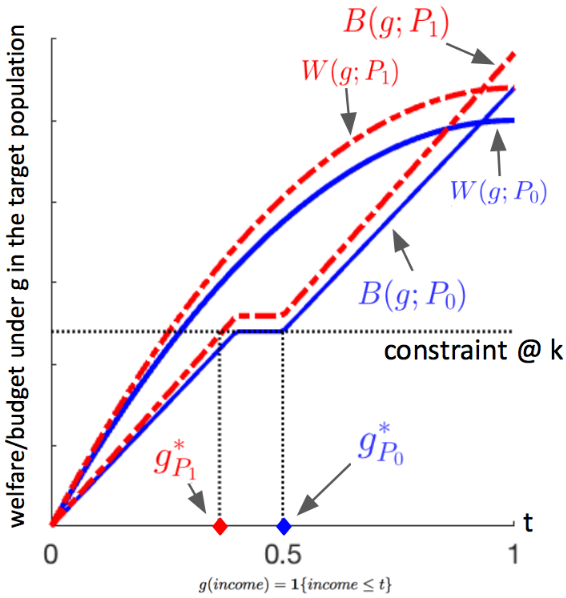
Theorem 2.1. *Suppose Assumption 2.1 holds for the class of data distributions $\mathcal{P}$. For $P_0 \in \mathcal{P}$ considered in Assumption 2.1, suppose it also satisfies Assumptions 2.2 and 2.3. Then no statistical rule can be both uniformly asymptotically welfare-efficient and uniformly asymptotically feasible. In particular, if a statistical rule $\widehat{g}$ is pointwise asymptotically welfare-efficient and pointwise asymptotically feasible under P_0, then it is not uniformly asymptotically feasible.*

Figure 2.1 provides some intuition for Theorem 2.1. The pictured distribution P_0 satisfies both Assumptions 2.2 and 2.3: the constrained optimal criterion $g^*_{P_0}$ satisfies the budget constraint exactly, i.e. $B(g^*_{P_0}; P_0) = k$, and is separated from the rest of feasible eligibility criteria. Note that if a statistical rule $\widehat{g}$ is pointwise asymptotically welfare-efficient and pointwise asymptotically feasible under P_0, then it has to select eligibility criteria close to $g^*_{P_0}$ with high probability over repeated sample draws distributed according to P_0^n as $n \to \infty$.

Under Assumption 2.1, the class of distributions $\mathcal{P}$ is sufficiently rich so that along a sequence of data distributions $\{P_{h_n}\}$ that is contiguous to P_0 as $n \to \infty$, the budget functions $B(g; P_{h_n})$ converge to $B(g; P_0)$ while $B(g^*_{P_0}; P_{h_n}) > k$, i.e. $g^*_{P_0}$ is not feasible under P_{h_n}. Figure 2.1 showcases P_1 as one distribution from this sequence. The contiguity between $\{P_{h_n}\}$ and P_0 implies that the statistical rule $\widehat{g}$

must select criteria close to $g^*_{P_0}$ with high probability under $P^n_{h_n}$ as well. However, the criterion $g^*_{P_0}$ is infeasible under P_{h_n}, and therefore the statistical rule $\widehat{g}$ cannot be asymptotically feasible under P_{h_n}.

Figure 2.1: Illustration for Assumptions 2.1-2.3 underlying Theorem 2.1



Notes: This figure plots welfare functions $W(g; P)$ and budget functions $B(g; P)$ for populations distributed according to P_0 (blue solid lines) or P_1 (red dashed lines), where P_1 is a distribution from the sequence of distributions $\{P_{h_n}\}$ that is contiguous to P_0 under Assumption 2.1. The distribution P_0 satisfies Assumptions 2.2 and 2.3. The x-axis indexes a one-dimensional criterion class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \leq t\}\}$ for a one-dimensional continuous characteristic X_i with support on $[0, 1]$, e.g. eligibility criteria based on income thresholds. The black dotted line marks the budget threshold k. The bold blue dot marks $g^*_{P_0}$, the constrained optimal eligibility criterion under P_0. The bold red dot marks $g^*_{P_1}$, the constrained optimal eligibility criterion under P_1.

2.4 Non-uniformity of the sample-analog rule

The previous negative result implies that no statistical rule can be both uniformly asymptotically welfare-efficient and uniformly asymptotically feasible. Thus, policymakers might want to consider statistical rules that satisfy one of these two properties. This section demonstrates why a direct extension to the existing approach in the EWM literature, the sample-analog rule, does not satisfy either property.

I first describe the EWM approach (Kitagawa and Tetenov, 2018) and its direct extension. To propose a feasible approach, Kitagawa and Tetenov (2018) construct individuals' benefit and cost estimates (Γ_i^*, R_i^*) for (Γ, R) as described below in Section 2.5. Since (Γ, R) involves potential outcomes, they are often unobserved and require estimation based on RCT that introduces estimation errors in addition to sampling errors. To highlight the drawback of the direct extension to EWM, in this section, I consider settings where we observe an experimental data of sample size n where (Γ, R) is directly observable, i.e. $(\Gamma_i^*, R_i^*) = (\Gamma_i, R_i)$. The goal of the simplification is to highlight that the non-uniformity I show below can arise from sampling errors alone.

One can estimate the welfare function and the budget function using their sample-analog versions:

$$\widehat{W}_n(g) := \frac{1}{n}\sum_i \Gamma_i^* \cdot g(X_i), \ \widehat{B}_n(g) := \frac{1}{n}\sum_i R_i^* \cdot g(X_i). \tag{8}$$

A direct extension to the existing approach in the EWM literature is a statistical rule that solves the sample version of the population constrained optimization problem (2):

$$\widehat{g}_{\text{sample}} \in \arg \max_{\widehat{B}_n(g) \leq k} \widehat{W}_n(g). \tag{9}$$

The subscript "sample" emphasizes how this approach verifies whether a criterion satisfies the constraint by comparing the *sample analog* $\widehat{B}_n(g)$ with k directly, i.e. imposes a sample-analog constraint. If no criterion satisfies the constraint, then I set $\widehat{g}_{\text{sample}}$ to not assign any eligibility, i.e. $\widehat{g}_{\text{sample}}(x) = 0$ for all $x \in \mathcal{X}$.

A key insight from Kitagawa and Tetenov (2018) is that without a constraint, the sample-analog rule is uniformly asymptotically welfare-efficient. Unfortunately this intuition does not extend to the current setting where the constraint involves an unknown cost. Consider a one-dimensional criterion class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \leq t\}\}$, which is based on thresholds of a one-dimensional continuous characteristic X. Suppose the policymakers know benefit is positive for everyone so that welfare function is strictly increasing, and only need to estimate whether a given threshold satisfies a capacity constraint due to imperfect take-up. Furthermore, suppose the experiment sample observes take-up R_i so that the only uncertainty arises from sampling errors. These settings satisfy Assumptions 2.2 and 2.3. Even in these simple settings, the sample-analog rule selects infeasible criteria half the time, and achieves strictly suboptimal level of welfare the other half the time as sample size gets large. Proposition 2.1 formalizes these settings where the sample-analog rule is neither pointwise asymptotically welfare efficient nor pointwise asymptotically feasible.

Proposition 2.1. One-dimensional threshold and imperfect take-up. *Consider the special case where under distribution P, the benefit $\Gamma > 0$ almost surely and the cost R is binary with $R = 0$ for $X \in [\underline{t}, \overline{t}]$, but $Pr_P\{R = 1 \mid X\} \in (0, 1)$ otherwise. Then for some budget threshold $k = E_P[R \cdot \mathbf{1}\{X \leq \underline{t}\}]$, the budget constraint binds. The sample-analog rule $\widehat{g}_{sample}$, as the sample size $n \to \infty$, selects infeasible criteria half the time, and achieves strictly suboptimal level of welfare the other half the time.*

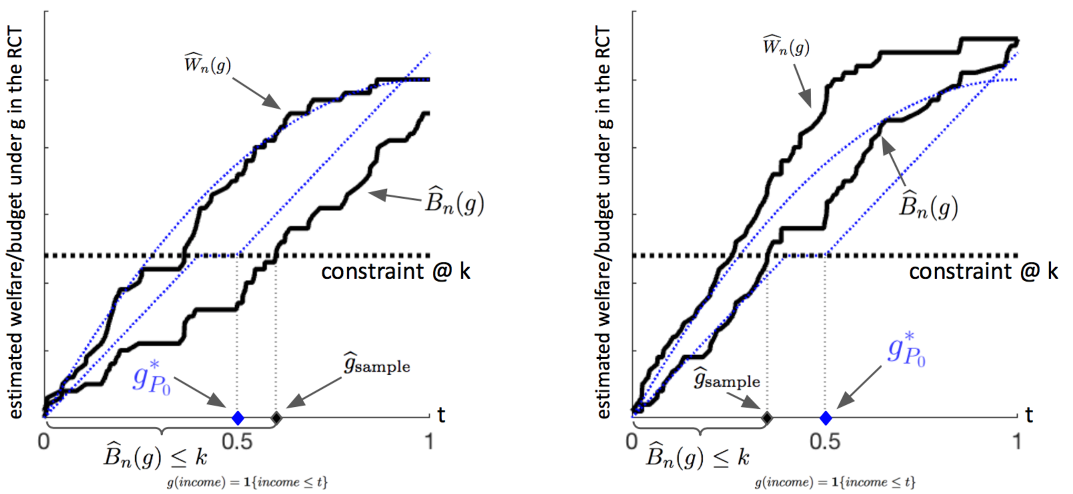
Figure 2.2 illustrates the setup of Proposition 2.1, where the sampling uncertainty can be particularly problematic for the sample-analog rule. Since policymakers know the benefit is positive for everyone, $\widehat{g}_{\text{sample}}$ takes a simple form of the highest threshold where the sample-analog constraint is satisfied exactly. The driving force behind the failure of $\widehat{g}_{\text{sample}}$ as described in Proposition 2.1 is that due to sampling uncertainty, whether a criterion satisfies the sample-analog constraint is an imperfect measure of whether it satisfies the constraint in the population.

Since the sample-analog rule $\widehat{g}_{\text{sample}}$ restricts attention to criteria that satisfy the sample-analog constraint, there is no guarantee the selected criterion is actually feasible. This is very likely to happen when there is welfare gain in exceeding the budget constraint as in the setup of Proposition 2.1 where $W(g; P)$ is strictly increasing in g. Therefore, the sample-analog rule $\widehat{g}_{\text{sample}}$ is not asymptotically feasible under P. As illustrated in Figure 2.2, after observing a sample depicted in panel (a), the sample-analog rule picks an infeasible threshold because the sample-analog constraint is still satisfied there.

Similarly, there is uncertainty about whether the sample-analog constraint is satisfied at the constrained optimum g_P^*, and it is possible for the sample-analog rule $\widehat{g}_{\text{sample}}$ to miss g_P^*. In the setup of Proposition 2.1, when the sample-analog rule $\widehat{g}_{\text{sample}}$ misses g_P^*, it is guaranteed to select a suboptimal criterion and therefore is not asymptotically welfare-efficient under P. As illustrated in Figure 2.2, after observing a sample depicted in panel (b), the sample-analog rule picks a suboptimal threshold because the sample-analog constraint is violated at the constrained optimum g_P^*.

Figure 2.2: Illustration for Proposition 2.1

(a) infeasible (b) suboptimal



Notes: This figure plots the budget function $B(g; P)$ (blue dotted line) and its sample-analogs $\widehat{B}_n(g)$ (black wriggly line) based on two different observed samples in panel (a) and (b) respectively. The x-axis indexes a one-dimensional criterion class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \leq t\}\}$ for a one-dimensional continuous characteristic X_i with support on $[0, 1]$. The black dotted line marks the budget threshold k. The constrained optimal threshold g_P^* is $t = 0.5$. The sample-analog rule $\widehat{g}_{\text{sample}}$ selects an infeasible threshold in panel (a) and selects a suboptimal threshold in panel (b).

Proposition 2.1 describes special settings where we can infer the asymptotic probability of selecting infeasible criteria and achieving strictly suboptimal welfare. Under additional assumption on the estimation quality of $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$, it is possible to study how large is the violation to the budget constraint in more general settings. This assumption is also needed for establishing the statistical properties of the alternative statistical rules I propose in Sections 3 and 4, which I state in Section 2.5.

Let $\hat{\mathcal{G}} = \{g \in \mathcal{G} : \widehat{B}_n(g) \leq k\}$ denote the set of criteria that $\widehat{g}_{\text{sample}}$ can choose from, which contains criteria that do not violate the sample-analog of the budget constraint. We are interested in bounding the probability that $\widehat{g}_{\text{sample}}$ selects a criterion that violates the population budget constraint by a fixed amount c by the follows:

$$Pr_{P^n}\left\{B(\widehat{g}_{\text{sample}};P)-k>c\right\}\leq Pr_{P^n}\left\{\exists g:g\in\hat{\mathcal{G}}\ and\ B(g;P)>k+c\right\} \tag{10}$$

The corollary stated below shows the chance that a large amount of budget violation occurs is smaller when there is less variability in $\widehat{B}_n(g)$. The chance also vanishes to zero as the sample size gets larger. The bound can also be estimated based on the experimental data.

Corollary 2.1. *Under Assumption 2.4, the probability (10) is upper bounded by the CDF of* $\inf\limits_{g\in\mathcal{G}}\frac{G_P^B(g)}{\Sigma_P^B(g,g)^{1/2}}$ *evaluated at* $\frac{-\sqrt{n}\cdot c}{\max_{g\in\mathcal{G}}\Sigma^B(g,g)^{1/2}}$.

The amount of welfare loss of $\widehat{g}_{\text{sample}}$ depends on the specific data distribution. However, there are many common data distributions under which the amount of welfare loss does not vanish even as the sample size gets larger. Corollary 2.2 formalizes the welfare loss for data distributions described in Proposition 2.1, which can be large even for these simple settings where the benefit Γ is known to be positive and sampling error of the take-up R is the only source of uncertainty.

Corollary 2.2. *For the data distributions described in Proposition 2.1, let* $\inf_{B(g;P)<k}W(g^*_{P_0};P_0)-W(g;P_0)=\epsilon>0$ *be the smallest amount of welfare loss from missing the binding solution* $g^*_{P_0}$. *Then as* $n\to\infty$, *the sample-analog rule* $\widehat{g}_{sample}$ *would achieve at least* ϵ *loss in welfare half the time:*

$$Pr_{P^n}\left\{W(g^*_P;P)-W(\widehat{g}_{sample};P)\geq\epsilon\right\}\to 50\%.$$

Given the non-uniformity issues with the sample-analog rule $\widehat{g}_{\text{sample}}$, in Sections 3 and 4 I propose alternative statistical rules that have attractive uniformity properties. In particular, the uniformity holds for a large class of distributions for which the welfare function $W(g;P)$ and the budget function $B(g;P)$ can be estimated reasonably well as described in Section 2.5.

2.5 Estimates for benefit and cost

The appropriate expressions for these estimates depend on the type of observed data. Below I state the estimates formed based an RCT that randomly assigns the eligibility, which is the leading case of Kitagawa and Tetenov (2018). The observed data $\{A_i^*\}_{i=1}^n$ consists of i.i.d. observations $A_i^* = (Y_i, Z_i, D_i, X_i) \in \mathcal{A}^*$. The distribution of A_i^* is induced by the distribution of (Γ, R, X) as in the population, as well as the sampling design of the RCT. Here D_i is an indicator for being in the eligibility arm of the RCT, Y_i is the observed outcome and Z_i is the observed cost of providing eligibility to an individual participating in the RCT. The observed cost is mechanically zero if an individual is not randomized into the eligibility arm. The estimates for (Γ, R) are

$$\Gamma_i^* = \alpha(X_i, D_i) \cdot Y_i, \; R_i^* = \frac{D_i}{p(X_i)} \cdot Z_i, \tag{11}$$

where $\alpha(X_i, D_i) = \frac{D_i}{p(X_i)} - \frac{1-D_i}{1-p(X_i)}$ and $p(X_i)$ is the propensity score, the probability of receiving eligibility conditional on the observed characteristics. Since the sampling design of an RCT is known, the propensity score is a known function of the observed characteristics.

Assumption 2.4. Estimation quality. *The recentered empirical processes $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$ defined in (8) converge to mean-zero Gaussian processes G_P^W and G_P^B uniformly over $g \in \mathcal{G}$, with covariance functions $\Sigma_P^W(\cdot,\cdot)$ and $\Sigma_P^B(\cdot,\cdot)$ respectively:*

$$\left\{\sqrt{n} \cdot \left(\frac{1}{n}\sum_i \Gamma_i^* \cdot g(X_i) - W(g;P)\right)\right\}_{g\in\mathcal{G}} \to_d G_P^W$$
$$\left\{\sqrt{n} \cdot \left(\frac{1}{n}\sum_i R_i^* \cdot g(X_i) - B(g;P)\right)\right\}_{g\in\mathcal{G}} \to_d G_P^B$$

Moreover, the convergence holds uniformly over $P \in \mathcal{P}$. The covariance functions are uniformly bounded, with diagonal entries bounded away from zero uniformly over $g \in \mathcal{G}$. There is a uniformly consistent estimator $\widehat{\Sigma}^B(\cdot,\cdot)$ of the covariance function $\Sigma_P^B(\cdot,\cdot)$.

Online Appendix B.3 gives primitive assumptions under which Assumption 2.4

is guaranteed to hold for $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$ constructed using an RCT such as in (11) or an observational study. As standard in the literature, I need to restrict the complexity of the criterion class $\mathcal{G}$, and assume unconfoundedness and strong overlap.

3 New statistical rule that ensures uniform asymptotic feasibility

This section describes how to impose the constraint in the estimation problem (9) differently from the sample-analog rule $\widehat{g}_{\text{sample}}$ to derive a statistical rule that is uniformly asymptotically feasible. Specifically, the statistical rule proposed in this section tightens the sample-analog constraint by taking into account estimation error as explained in Theorem 3.1. However, a tighter sample-analog constraint can lead to lower welfare if the original constraint binds at the constrained optimal rule. Theorem 3.2 formalizes this intuition.

Due to sampling uncertainty, the sample analog $\widehat{B}_n(g)$ provides an imperfect measure of the expected cost of g in the population. Thus, an infeasible criterion may be mistakenly classified as feasible by checking whether it meets the sample-analog constraint, i.e. comparing $\widehat{B}_n(g)$ directly with the constraint k. Tightening the sample-analog constraint results in a more conservative estimate for the class of feasible eligibility criteria, and therefore reduces the chance that the selected criterion mistakenly exceeds the constraint in the population. The amount of tightening is chosen to bound the chance of a mistake uniformly over the class of distributions.

Theorem 3.1. *Suppose Assumption 2.4 holds for the class of data distributions $\mathcal{P}$. Collect eligibility criteria*

$$\hat{\mathcal{G}}_\alpha = \left\{ g : g \in \mathcal{G} \text{ and } \frac{\sqrt{n}\left(\widehat{B}_n(g) - k\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha \right\}, \tag{12}$$

where c_α is the α-quantile from $\inf_{g \in \mathcal{G}} \frac{G_P^B(g)}{\Sigma_P^B(g,g)^{1/2}}$ for G_P^B the Gaussian process defined in Assumption 2.4, and $\widehat{\Sigma}^B(\cdot,\cdot)$ the consistent estimator for its covariance function.

Then

$$\limsup_{n\to\infty}\sup_{P\in\mathcal{P}} Pr_{P^n}\{\hat{\mathcal{G}}_\alpha \cap \mathcal{G}^+ \neq \emptyset\} < \alpha, \tag{13}$$

where $\mathcal{G}^+ = \{g : B(g;P) > k\}$ *is the set of infeasible eligibility criteria.*

Note that the sample-analog constraint is tightened by $c_\alpha \cdot \frac{\widehat{\Sigma}^B(g,g)^{1/2}}{\sqrt{n}}$ where c_α is negative, which means the class $\hat{\mathcal{G}}_\alpha$ only includes eligibility criteria where the constraint is slack in the sample. The sample-analog constraint is tightened proportionally to the standard deviation to reflect that $\widehat{B}_n(g)$ might be particularly noisy for some g, and inversely proportional to the (square root of) sample size because intuitively larger sample size reduces the sampling uncertainty.

3.1 Mistake-controlling rule

I propose a mistake-controlling rule that maximizes the sample welfare $\widehat{W}_n(g)$ within the class of eligibility criteria $\hat{\mathcal{G}}_\alpha$ as defined in Equation (12).

$$\widehat{g}_{\text{mistake}} \in \arg\max_{g\in\hat{\mathcal{G}}_\alpha} \widehat{W}_n(g). \tag{14}$$

Here the subscript "mistake" highlights that with high probability this statistical rule selects feasible criterion. If the set $\hat{\mathcal{G}}_\alpha$ is empty, then I set $\widehat{g}_{\text{mistake}}$ to not assign any eligibility: $\widehat{g}_{\text{mistake}}(x) = 0$ for all $x \in \mathcal{X}$. Unlike $\widehat{g}_{\text{sample}}$, as a direct consequence of Theorem 3.1, with probability at least $1-\alpha$ this statistical rule is guaranteed to not mistakenly choose infeasible eligibility criteria. The next theorem details the improvement by $\widehat{g}_{\text{mistake}}$ over $\widehat{g}_{\text{sample}}$ in terms of uniform asymptotic feasibility.

Theorem 3.2. *Suppose Assumption 2.4 holds for the class of data distributions* $\mathcal{P}$*. If* $\alpha_n \to 0$ *as* $n \to \infty$*, then the mistake-controlling rule* $\widehat{g}_{mistake}$ *defined in (14) is uniformly asymptotically feasible over* $\mathcal{P}$*.*

For distributions where the constraint is slack at the constrained optimal eligibility criteria, as long as the sample-analog constraint is tightened slower than the square root of the sample size, the mistake-controlling rule $\widehat{g}_{\text{mistake}}$ is pointwise asymptotically welfare-efficient and pointwise asymptotically feasible. Corollary 3.1 formalizes the rate of tightening below.

Corollary 3.1. *If the data distribution P induces a constrained optimal criterion at which the constraint is slack, then the mistake-controlling rule $\widehat{g}_{mistake}$ is asymptotically welfare-efficient and asymptotically feasible under P for $\alpha_n \to 0$ at a rate such that $c_{\alpha_n} = o(n^{1/2})$.*

4 New statistical rule that ensures uniform asymptotic welfare-efficiency

Exceeding the budget constraint in the population might be desirable if it achieves higher welfare and does not cost too much. How to model such trade-off depends on the specific welfare program. Section 4.1 first explores possible forms of the trade-off that are of practical relevance, and then focuses on the trade-off where the marginal cost per unit of violating the constraint is constant with a known upper bound. The upper bound is context specific and as an example, Section 6 discusses how to set it for Medicaid expansion. Relaxing the constraint results in weakly higher welfare than that of the constrained optimal criterion g_P^*. Section 4.2 derives a statistical rule that implements such trade-off in the sample, and therefore is uniformly asymptotically welfare-efficient.

4.1 Form of the trade-off

Note that the original constrained optimization problem (9) may be reformulated as

$$\max_{g \in \mathcal{G}} \inf_{\lambda \geq 0} \; W(g; P) - \lambda \cdot (B(g; P) - k). \tag{15}$$

where λ measures the marginal gain of relaxing constraint. In many settings where the benefit is monetary, it may be natural to think for every dollar spent right above the budget constraint, there is a welfare gain of λ dollar.

This formulation implies that policymakers are willing to enforce the constraint at all costs since λ is unbounded. This formulation may not reflect the real objective, however, as policymakers might only be willing to trade off violations of constraint against gains in welfare to certain extent, bounding $\lambda \in [0, \overline{\lambda}]$ and resulting in a new objective function

$$\max_{g\in\mathcal{G}} \min_{\lambda\in[0,\overline{\lambda}]} W(g;P) - \lambda \cdot (B(g;P) - k). \tag{16}$$

This objective function might be natural if the government finances the deficit by borrowing at a fixed interest rate. The trade-off can also be non-linear. Below I provide two examples. The first example allows the cost of exceeding the budget constraint to vary with the amount of deficit, perhaps because government debt can have an effect on long-term interest rates. In this setting, the violation of constraint can be modeled as a piecewise linear function:

$$\max_{g\in\mathcal{G}} \min_{\lambda_1\in[0,\overline{\lambda}_1],\lambda_2\in[0,\overline{\lambda}_2]} W(g;P) - \lambda_1 \cdot (B(g;P) - k_1) - \lambda_2 \cdot (B(g;P) - k_2)$$

where $k_2 > k_1$ demarcate ranges of deficit, which correspond to different interest rate λ_1 and λ_2. The second example allows the cost of exceeding the budget constraint to be constant, perhaps because violation results in a one-time penalty:

$$\max_{g\in\mathcal{G}} \min_{\lambda\in[0,\overline{\lambda}]} W(g;P) - \lambda \cdot \mathbf{1}\{B(g;P) > k\}.$$

This paper focuses on a linear trade-off as in (16). Using the notation $(x)_+$ to denote the positive part of $x \in \mathbb{R}$, the inner optimization problem of (16) can be written as

$$\min_{\lambda\in[0,\overline{\lambda}]} W(g;P) - \lambda \cdot (B(g;P) - k) = W(g;P) - \overline{\lambda} \cdot (B(g;P) - k)_+.$$

Note, however, this reformulated objective does not involve optimization over λ: the solution to (16) is equivalent to the solution to a non-smooth but piecewise linear optimization problem:

$$\widetilde{g}_P \in \arg\max_{g\in\mathcal{G}} W(g;P) - \overline{\lambda} \cdot (B(g;P) - k)_+. \tag{17}$$

By construction, this reformulation is able to relax the constraint. Therefore, the solution $\widetilde{g}_P$ achieves weakly higher welfare than the constrained optimal criterion

g_P^* for any data distribution P. The next lemma formalizes this observation.

Lemma 4.1. *For any $\overline{\lambda} \in [0, \infty]$, the solution to the trade-off problem (17) achieves weakly higher welfare than the constrained optimum: $W(\widetilde{g}_P; P) \geq W(g_P^*; P)$ for any distribution P. Moreover, the violation to the budget constraint is upper bounded by $\frac{W(\widetilde{g}_P;P)-W(g_P^*;P)}{\overline{\lambda}}$.*

4.2 Trade-off rule

Given a new objective function (17) that trades off the gain and the cost from violating the constraint, the goal is to derive a statistical rule that is likely to select eligibility criteria that maximize the new objective function. Consider the trade-off statistical rule defined as

$$\widehat{g}_{\text{tradeoff}} \in \arg\max_{g \in \mathcal{G}} \ \widehat{W}_n(g) - \overline{\lambda} \cdot (\widehat{B}_n(g) - k)_+, \tag{18}$$

where the subscript "tradeoff" highlights that this statistical rule is able to relax the constraint by trading off the gain and the cost from violating the constraint.

Theorem 4.1. *Suppose Assumption 2.4 holds for the class of data distributions $\mathcal{P}$. Then the trade-off rule $\widehat{g}_{tradeoff}$ defined in (18) is uniformly asymptotically welfare-efficient under $\mathcal{P}$. Moreover, with probability approaching one, the violation to the budget constraint is upper bounded by $\frac{W(\widehat{g}_{tradeoff};P)-W(g_P^*;P)}{\overline{\lambda}}$ uniformly over $\mathcal{P}$.*

To gain intuition for the above results, I note that the trade-off rule $\widehat{g}_{\text{tradeoff}}$ is very likely to select eligibility criteria that achieve the same welfare as $\widetilde{g}_P$ over repeated sample draws as $n \to \infty$ uniformly over $\mathcal{P}$. Furthermore, Lemma 4.1 shows the trade-off solution $\widetilde{g}_P$ achieves weakly higher welfare than the constrained optimal criterion g_P^* for any data distribution P. Therefore, the trade-off rule $\widehat{g}_{\text{tradeoff}}$ is asymptotically welfare-efficient uniformly over $\mathcal{P}$.

5 Medicaid Expansion: Monte Carlo Simulations

This section presents a simulation study calibrated to the distribution underlying the data from the OHIE. The simulation results confirm the negative result on the sample-analog rule $\widehat{g}_{\text{sample}}$ discussed in Section 2.2, and illustrate the improvement by the statistical rules proposed in Sections 3 and 4.

To ensure the practical relevance of the simulation, I attempt to preserve the distribution of the data from the OHIE. As explained later in Section 6, I can construct benefit and cost estimates of Medicaid eligibility using the OHIE. The benefit is defined to be the increase in the probability of reporting good subjective health after receiving Medicaid eligibility, and the cost is defined to be health care expenditure that needs to be reimbursed by Medicaid, in excess to the current expansion criterion. I defer the details on the construction of the estimates (Γ_i^*, R_i^*) to Section 6.1. For the purpose of this simulation study, the OHIE represents the population P, and therefore I can take these estimates as the true benefit and cost (Γ, R).

Table 5.1 presents the basic summary statistics for the estimates (Γ_i^*, R_i^*). Under the current definition of the cost, a criterion is feasible if it incurs a negative cost.

Table 5.1: Summary statistics of the OHIE sample by number of children

Number of children	Sample size	Sample mean of Γ_i^*	Sample mean of R_i^*
0	5,758	3.1%	\$651
1	1,736	10.3%	\$348
≥ 2	2,641	1.5%	-\$275
-	10,135	3.9%	\$358

Notes: This table presents summary statistics on the sample of individuals who responded to both the initial and the main surveys from the Oregon Health Insurance Experiment (the OHIE sample). The first three rows represent individuals living with different number of children (family members under age 19), and the last row is the aggregate. The estimate for benefit Γ_i^* is an estimate for the increase in the probability of an individual reporting "excellent/very good/good" on self-reported health (as opposed to "poor/fair") after receiving Medicaid eligibility. The estimate for cost R_i^* is an estimate for individual's health care expenditure that needs to be reimbursed by Medicaid, in excess of the current expansion criteria.

I randomly draw observations from the OHIE sample to form a random sample. Under this simulation design, I can solve for the constrained optimal criterion as

$$g_P^* \in \arg \max_{g \in \mathcal{G},\ B(g;P) \leq 0} W(g; P)$$

where $(W(g;P), B(g;P))$ are the sample-analog in the OHIE sample. The criterion class $\mathcal{G}$ includes income thresholds that can vary with the number of children. The constrained optimal criterion is

$$g_P^*(x) = \begin{cases} \mathbf{1}\{\text{income} \leq 75\%\}, & \text{numchild} = 0 \\ \mathbf{1}\{\text{income} \leq 343\%\}, & \text{numchild} = 1 \\ \mathbf{1}\{\text{income} \leq 160\%\}, & \text{numchild} \geq 2 \end{cases}$$

for characteristics $x = (\text{income}, \text{ numchild})$. The maximum feasible welfare is given by $W(g_P^*; P) = 3.8\%$, an increase of 3.8% in reporting good subjective health. The cost associated with the constrained optimal criterion is $B(g_P^*; P) = -\$0.6$, meaning per enrollee g_P^* costs \$0.6 less than the current expansion criterion.

5.1 Simulation results

Table 5.2 compares the performance of various statistical rules $\widehat{g}$ through 500 Monte Carlo iterations. At each iteration, I randomly draw observations from the OHIE sample to form a random sample. I simulate with the same sample size as the original sample to hold the amount of sampling uncertainty constant. Given the random sample, I collect eligibility criteria chosen by each of following statistical rules:

- sample-analog rule $\widehat{g}_{\text{sample}}$.
- mistake-controlling rule $\widehat{g}_{\text{mistake}}$ with critical value c_α described in Section 3 (α is set to 5%.)
- trade-off rule $\widehat{g}_{\text{tradeoff}}$ with trade-off coefficient $\bar{\lambda}$ described in Section 6.2.2.

I evaluate the welfare function and the budget function $(W(g;P), B(g;P))$ for a given criterion in the original OHIE sample. Averages over 500 iterations provide simulation evidence on the asymptotic properties of the above statistical rules, as shown in Table 5.2.

Table 5.2: Simulation results: asymptotic properties of statistical rules $\widehat{g}$

Statistical rule	sample-analog $\widehat{g}_{\text{sample}}$	mistake-controlling $\widehat{g}_{\text{mistake}}$	trade-off $\widehat{g}_{\text{tradeoff}}$
Prob. of selecting infeasible criteria	35.4%	8%	79.6%
Prob. of selecting suboptimal criteria	87.0%	98.6%	37.6%
Average welfare loss	0.06	0.60	-0.02
Average cost	-\$3	-\$57	\$105

Notes: This table reports asymptotic properties of statistical rules $\widehat{g}$, as averaged over 500 simulations. Row 1 reports the probability that the rule selects an eligibility criterion that violates the budget constraint, i.e. $Pr_{P^n}\{B(\widehat{g};P) > 0\}$. Row 2 reports the probability that the rule achieves strictly less welfare than the constrained optimal criterion g_P^*, i.e. $Pr_{P^n}\{W(\widehat{g};P) < W(g_P^*;P)\}$. Row 3 reports the average welfare loss of the rule relative to the maximum feasible welfare, i.e. $\frac{E_{P^n}\left[W(g_P^*;P)-W(\widehat{g};P)\right]}{W(g_P^*;P)}$. Row 4 reports the average cost of the criteria selected by the rule, i.e. $E_{P^n}\left[B(\widehat{g};P)\right]$.

Row 1 of Table 5.2 illustrates that it is possible for all three statistical rule $\widehat{g}$ to select infeasible criteria. A lower probability of selecting infeasible criteria suggests the rule is closer to achieving asymptotic feasibility. Proposition 2.1 describes distributions where the sample-analog rule $\widehat{g}_{\text{sample}}$ is not asymptotic feasible. In the distribution calibrated to the OHIE sample, the sample-analog rule $\widehat{g}_{\text{sample}}$ might not be asymptotically feasible, either, as it can select infeasible eligibility criteria in 19.4% of the draws. In contrast, Theorem 3.1 guarantees that the mistake-controlling rule $\widehat{g}_{\text{mistake}}$ selects infeasible eligibility criteria in less than 5% of the draws, regardless of the distribution. Simulation confirms such guarantee as the mistakes only happen 8% of the time.

Row 2 of Table 5.2 illustrates that it is possible for all three statistical rule $\widehat{g}$ to achieve weakly higher welfare than the constrained optimal criterion g_P^*. This can happen when $\widehat{g}$ selects an infeasible criterion. A lower probability of selecting suboptimal criteria suggests the rule is closer to achieving asymptotic welfare-efficiency. Theorem 4.1 implies that the trade-off rule $\widehat{g}_{\text{tradeoff}}$ is uniformly asymptotically welfare efficient while there is no such guarantee for the sample-analog rule $\widehat{g}_{\text{sample}}$.

In the distribution calibrated to the OHIE, the trade-off rule $\widehat{g}_{\text{tradeoff}}$ on average achieves higher welfare than the sample-analog rule $\widehat{g}_{\text{sample}}$. As shown in row

3 of Table 5.2, the welfare loss of $\widehat{g}_{\text{tradeoff}}$ is -2% of the maximum feasible welfare $W(g^*_P; P)$, compared to 6% for $\widehat{g}_{\text{sample}}$. However, its improvement can be at the cost of violating the budget constraint more often than $\widehat{g}_{\text{sample}}$, at a rate of 79.6%. Comparing the distribution of the cost $B(\widehat{g}_{\text{sample}}; P)$ and $B(\widehat{g}_{\text{tradeoff}}; P)$, I note whenever the trade-off rule $\widehat{g}_{\text{tradeoff}}$ selects an infeasible criterion, the amount of violation is small, so that on average the budget constraint would not be violated as shown in row 4 of Table 5.2. As a result, even though the trade-off rule $\widehat{g}_{\text{tradeoff}}$ is more likely to select infeasible eligibility criteria than the sample-analog rule $\widehat{g}_{\text{sample}}$, the cost to these violations is limited.

6 Medicaid expansion: empirical illustration

In this empirical example, I illustrate how to maximize the benefit of Medicaid eligibility by allowing a more flexible expansion criterion using the statistical rules proposed in this paper based on data from the Oregon Medicaid Health Insurance Experiment (OHIE). Section 6.1 overviews the OHIE data and Section 6.2 describes how to implement the two statistical rules I propose to select a more flexible criterion based the OHIE data. Specifically, the more flexible criterion would allow the income thresholds to vary with the number of children as explained in Example 2.1 of Section 2.

6.1 Data

I use the experimental data from the OHIE, where Medicaid eligibility (D_i) was randomized in 2007 among Oregon residents who were low-income adults, but previously ineligible for Medicaid, and who expressed interest in participating in the experiment. Finkelstein et al. (2012) include a detailed description of the experiment and an assessment of the average effects of Medicaid on health and health care utilization. I include a cursory explanation here for completeness. The original OHIE sample consists of 74,922 individuals (representing 66,385 households). Of these, 26,423 individuals responded to the initial mail survey, which collects information on income as percentage of the federal poverty level and number of children, which are the characteristics X_i of interest for targeting.[4] After one year, the main survey collects data related to health (Y_i), health care utilization (C_i) and actual enrollment in Medicaid (M_i), which allows me to construct estimates for the benefit and cost of Medicaid eligibility (Γ, R). Therefore I further exclude individuals who did not respond to the main survey from my sample.

For health (Y_i), I follow the binary measurement in Finkelstein et al. (2012) based on self-reported health, where an answer of "poor/fair" is coded as $Y_i = 0$ and "excellent/very good/good" is coded as $Y_i = 1$. For health care utilization (C_i), the study collected measures of utilization of prescription drugs, outpatient visits, ER visits, and inpatient hospital visits. Finkelstein et al. (2012) annualize these utilization measures to turn these into spending estimates, weighting each type by its average cost (expenditures) among low-income publicly insured non-elderly adults in the Medical Expenditure Survey (MEPS). Note that health and health care utilization are not measured at the same scale, which requires rescaling when I consider the trade-off between the two. I address this issue in Section 6.2. Lastly, since the enrollment in Medicaid still requires an application, not everyone eligible in the OHIE eventually enrolled in Medicaid, which implies $M_i \leq D_i$.

Given the setup of the OHIE, Medicaid eligibility (D_i) is random conditional on household size (number of adults in the household) entered on the lottery sign-up form and survey wave. While the original experimental setup would ensure randomization given household size, the OHIE had to adjust randomization for later waves of survey respondents (see the Appendix of Finkelstein et al. (2012) for more details). Denote the confounders (household size and survey wave) with V_i, and define the propensity score as $p(V_i) = Pr\{D_i = 1 \mid V_i\}$. If the propensity score is known, then the construction of the estimates follows directly from the formula (11). However, the adjustment for later survey waves means I need to estimate the propensity score, and I adapt the formula (11) following Athey and Wager (2021)

[4]More accurately, I follow Sacarny et al. (2020) to approximate number of children by the number of family members under age 19 living in house as reported on the initial mail survey. I exclude individuals who did not respond to the initial survey from my sample, which differs from the sample analyzed in Finkelstein et al. (2012) as I focus on individuals who responded both to the initial and the main surveys from the OHIE. Due to this difference, the expansion criteria selected using my sample do not directly carry their properties to the population underlying the original OHIE sample, as the distributions of X differ.

to account for the estimated propensity score.

Specifically, define the conditional expectation function (CEF) of a random variable U_i as $\gamma^U = E[U_i \mid V_i, D_i]$. Since V_i in my case is discrete, I use a fully saturated model to estimate the propensity score $\widehat{p}(V_i)$ and the CEF $\widehat{\gamma}^U(V_i, D_i)$. I then form the estimated Horvitz-Thompson weight with the estimated propensity score as $\widehat{\alpha}(V_i, D_i) = \frac{D_i}{\widehat{p}(V_i)} - \frac{1-D_i}{1-\widehat{p}(V_i)}$. For health benefit due to Medicaid eligibility, define the estimate $\Gamma_i^* = \widehat{\gamma}^Y(V_i, 1) - \widehat{\gamma}^Y(V_i, 0) + \widehat{\alpha}(V_i, D_i) \cdot \big(Y_i - \widehat{\gamma}^Y(V_i, D_i)\big)$. The 2014 Medicaid spending was roughly \$6,000 per adult enrollee in Oregon, according to the expenditure information obtained from MACPAC (2019). To formalize the budget constraint that the per enrollee cost of the proposed criterion cannot exceed the 2014 criterion, I need to account for imperfect take-up because not everyone eligible for Medicaid would enroll. For the per enrollee *excess* cost relative to the current level, define the estimate $R_i^* = \widehat{\gamma}^Z(V_i, 1) + \frac{D_i}{\widehat{p}(W_i)} \cdot \big(Z_i - \widehat{\gamma}^Z(V_i, D_i)\big)$ where $Z_i = C_i - \$6,000 \cdot M_i$. As shown in the Online Appendix B.3, the estimation errors in Γ_i^* and R_i^* are asymptotically negligible.

6.2 Budget-constrained Medicaid expansion

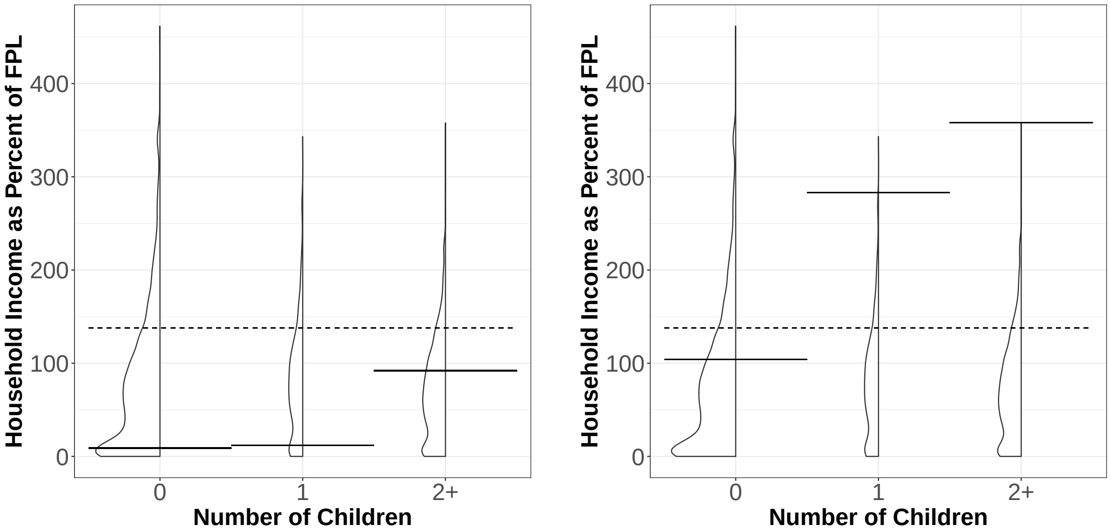
This section reports the Medicaid expansion criteria selected by the two statistical rules proposed in Sections 3 and 4 based on the OHIE data. The following subsections provide details on implementing these statistical rules.

Figure 6.1 summarizes the selected expansion criteria, which are income thresholds specific to the number of children. The mistake-controlling rule $\widehat{g}_{\text{mistake}}$ chooses to restrict Medicaid eligibility, especially lowering the income threshold for childless individuals far below the current level. This reflects the large variation in the cost estimates, which results in uncertainty about whether it would be feasible to provide eligibility to many individuals in the population. As a result, the budget estimate for the selected criterion is -\$140, far below the threshold of zero. In contrast, the trade-off rule $\widehat{g}_{\text{tradeoff}}$ chooses to assign Medicaid eligibility to more individuals, and to raise the income thresholds above the current level. The higher level occurs because on average the benefit estimates are positive, as illustrated in Table 5.1, which suggests many individuals still exhibit health benefit from being eligible for Medicaid. Under a reasonable assumption for an upper bound on the monetary value on the health benefit specified in Section 6.2.2, the trade-off rule finds that the additional health benefit from violating the budget constraint exceeds the cost of doing so. Therefore the budget estimate for the selected criterion is \$110, slightly above the threshold of zero.

Figure 6.1: More flexible Medicaid expansion criteria

(a) criterion selected by the mistake-controlling rule

(b) criterion selected by the trade-off rule



Notes: This figure plots the more flexible Medicaid expansion criteria selected by the two statistical rules proposed in Sections 3 and 4 based on results from the OHIE. The horizontal dashed line marks the income thresholds under the current expansion criterion, which is 138% regardless of the number of children in a household. The horizontal solid lines mark the more flexible criterion selected by various statistical rules, i.e. income thresholds that can vary with number of children. For each number of children, I also plot the underlying income distribution to visualize individuals below the thresholds. Panel (a) plots the criterion selected by the mistake-controlling rule $\widehat{g}_{\text{mistake}}$. Panel (b) plots the criterion selected by the trade-off rule $\widehat{g}_{\text{tradeoff}}$.

6.2.1 Implementing the mistake-controlling rule

To construct the mistake-controlling rule $\widehat{g}_{\text{mistake}}$ as proposed in Section 3, I maximize the sample welfare function among eligibility criteria in $\hat{\mathcal{G}}_{\alpha}$, which are guaranteed to contain only feasible eligibility criteria with probability approaching $1-\alpha$. For a conventional level of $\alpha = 5\%$, constructing $\hat{\mathcal{G}}_{\alpha}$ requires an estimate for the critical value c_α, the 5%-quantile from $\inf\limits_{g\in\mathcal{G}} \frac{G_P^B(g)}{\Sigma_P^B(g,g)^{1/2}}$, the infimum of the Gaussian process $G_P^B(\cdot) \sim \mathcal{GP}(0, \Sigma_P^B(\cdot,\cdot))$. In practice, I construct a grid on $\mathcal{G}$ as

$$\widetilde{\mathcal{G}} = \{g(x) = \mathbf{1}\{income \cdot 1\{numchild = j\} \leq y_j\} : j \in \{0, 1, \geq 2\}, y_j \in \{0, 50, 100, \ldots, 500\}\} \tag{19}$$

for characteristics $x = (income,\ numchild)$. This grid thus consists of income thresholds every 50% of the federal poverty level, and the thresholds can vary with number of children. I then approximate the infimum over infinite-dimensional $\mathcal{G}$ by the minimum over $\widetilde{\mathcal{G}}$ with estimated covariance i.e. $\min\limits_{g\in\widetilde{\mathcal{G}}} \frac{\widetilde{h}(g)}{\widehat{\Sigma}^B(g,g)^{1/2}}$. Here $\widetilde{h}(\cdot) \sim \mathcal{N}(0, \widehat{\Sigma}^B)$ is a Gaussian vector indexed by $g \in \widetilde{\mathcal{G}}$, with $\widehat{\Sigma}^B$ is sub-matrix of the covariance estimate $\widehat{\Sigma}^B(\cdot,\cdot)$ for $g \in \widetilde{\mathcal{G}}$. Based on 10,000 simulation draws I estimate c_α to be -2.56. The validity of this approximation is given by the uniform consistency of the covariance estimator under Assumption 2.4.

6.2.2 Implementing the trade-off rule

To construct the trade-off rule $\widehat{g}_{\text{tradeoff}}$ as proposed in Section 4, I need to choose $\overline{\lambda}$, the upper bound on the marginal gain from violating the constraint. In my empirical illustration, the budget constraint is in terms of monetary value. The objective function, however, is measured based on self-reported health, which does not directly translate to a monetary value. Following Finkelstein et al. (2019), I convert self-reported health into value of a statistical life year (VSLY) based on existing estimates. Specifically, a conservative measure for the increase in quality-adjusted life year (QALY) when self-reported health increases from "poor/fair" to "excellent/very good/good" is roughly 0.6. The "consensus" estimate for the VSLY for one unit of QALY from Cutler (2004) is $100,000 for the general US population. Taken these estimates together, I set $\overline{\lambda} = 1/(0.6 \cdot 10^5)$.

7 Conclusion

In this paper, I focus on properties of statistical rules when the cost of implementing any given eligibility criterion needs to be estimated. The existing EWM rule selects an eligibility criterion that maximizes a sample analog of the social welfare function, and only accounts for constraints that can be verified with certainty in the population. In reality, the cost of providing eligibility to any given individual might be unknown ex-ante due to imperfect take-up and heterogeneity. Therefore, in addition to asymptotic welfare-efficiency that has been studied by the EWM literature, I introduce a new desirable property of statistical rules in the setting of unknown cost, namely asymptotic feasibility. Unlike the setting of known cost, I show the direct extension to the existing EWM approach is no longer asymptotically welfare efficient nor asymptotically feasible for certain real-world relevant data distributions. I prove a stronger impossibility result where no statistical rule can satisfy both asymptotic feasibility and asymptotic welfare-efficiency uniformly. In light of this negative result, I propose two alternative statistical rules: the mistake-controlling rule and the trade-off rule that perform uniformly well based on the two properties, respectively. I illustrate their asymptotic properties and implementation details using experimental data from the OHIE. A calibrated simulation exercise also confirms better performance of the two alternative rules relative to the EWM rule.

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A Proofs of theorems

Proof. **Proof of Theorem 2.1.** The population is a probability space $(\Omega, \mathcal{A}, P)$, which induces the sampling distribution P^n that governs the observed sample. A statistical rule $\widehat{g}$ is a mapping $\widehat{g}(\cdot) : \Omega \to \mathcal{G}$ that selects a policy from the policy class $\mathcal{G}$ based on the observed sample. Note that the selected policy $\widehat{g}(\omega)$ is still deterministic because the policy class $\mathcal{G}$ is restricted to be deterministic policies, though the proof below extends to random policies. When no confusion arises, I drop the reference to event ω for notational simplicity.

Suppose $\widehat{g}$ is asymptotically welfare-efficient and asymptotically feasible under P_0. We want to prove there is non-vanishing chance that $\widehat{g}$ selects policies that are infeasible some other distributions:

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n} \{\omega : B(\widehat{g}(\omega); P) > k\} > 0. \tag{20}$$

Asymptotical welfare-efficiency under P_0 implies for any $\epsilon > 0$ we have

$$\limsup_{n\to\infty} Pr_{P_0^n} \left\{\omega : W(g^*_{p_0} P_0) - W(\widehat{g}(\omega); P_0) > \epsilon\right\} = 0. \tag{21}$$

Asymptotical feasibility under P_0 implies $Pr_{P_0^n}\{\omega : B(\widehat{g}(\omega); P_0) \leq k\} \to 1$.

Consider the event ω' where $|W(\widehat{g}(\omega'); P) - W(g^*_P; P)| < \epsilon$ and $\widehat{g}(\omega')$ is feasible. Asymptotic welfare-efficiency and asymptotic feasibility imply imply $Pr_{P_0^n} \{\omega'\} \to 1$. To see this, note the probability of such event has an asymptotic lower bound of one

$$\begin{aligned}
&Pr_{P_0^n} \left\{\omega' : W(g^*_{p_0} P_0) - W(\widehat{g}(\omega'); P_0) \leq \epsilon \textit{ and } B(\widehat{g}(\omega'); P_0) \leq k\right\} \\
\geq &Pr_{P_0^n} \left\{\omega : W(g^*_{p_0} P_0) - W(\widehat{g}(\omega); P_0) \leq \epsilon\right\} + Pr_{P_0^n} \{\omega : B(\widehat{g}(\omega); P_0) \leq k\} - 1
\end{aligned}$$

where the first two terms converge to one as $n \to \infty$ respectively under asymptotic welfare-efficiency and asymptotic feasibility. For ϵ satisfying Assumption 2.3, we have $B(\widehat{g}; P_0) = B(g^*_{P_0}; P_0) = k$ under the event ω', since the constraint is exactly satisfied at $g^*_{P_0}$ under Assumption 2.2. By Law of Total Probability, we have

$$Pr_{P_0^n}\{\omega : B(\widehat{g}(\omega); P_0) = k\} \geq Pr_{P_0^n}\{\omega'\} \cdot Pr_{P_0^n}\{B(\widehat{g}; P_0) = k \mid \omega'\}$$

Then the above argument shows $Pr_{P_0^n}\{\omega : B(\widehat{g}(\omega); P_0) = k\} \to 1$ as $n \to \infty$.

Following the notation in Assumption 2.1, denote the set of policies where the constraints bind exactly under the limit distribution P_0 by

$$\mathcal{G}_0 = \{g \in \mathcal{G} : B(g; P_0) = k\}. \tag{22}$$

Under Assumption 2.1, the sequence $P^n_{h_n}$ is contiguous with respect to the sequence P^n_0 , which means $P^n_0(A_n) \to 0$ implies $P^n_{h_n}(A_n) \to 0$ for every sequence of measurable sets A_n on $\mathcal{A}^n$. Then $Pr_{P_0^n}\{A_n : B(\widehat{g}(A_n); P_0) = k\} \to 1$ implies there exists an $N(u)$ such that for all $n \geq N(u)$, we have $Pr_{P^n_{h_n}}\{A_n : B(\widehat{g}(A_n); P_0) = k\} \geq 1 - u$. That is, with high probability, the statistical rule $\widehat{g}$ selects policies from $\mathcal{G}_0$ based on the observed sample distributed according to $P^n_{h_n}$. Recall Assumption 2.1 implies for all $g \in \mathcal{G}_0$, for any sample size n, we have $B(g; P_{h_n}) - k > C/\sqrt{n}$. Thus this statistical rule cannot uniformly satisfy the constraint since with sample size $n \geq N(u)$, we have

$$\sup_{P \in \mathcal{P}} Pr_{P^n}\{B(\widehat{g}; P) > k\} \geq Pr_{P^n_{h_n}}\{B(\widehat{g}; P_{h_n}) > k\} \geq 1 - u \tag{23}$$

□

Proof. **Proof of Proposition 2.1.** Consider a one-dimensional policy class $\mathcal{G} = \{g : g(x) = \mathbf{1}\{x \leq t\}\}$, which includes thresholds for a one-dimensional continuous characteristic X. Since $\Gamma > 0$ almost surely, the welfare function $W(t; P) := E_P[\Gamma \cdot \mathbf{1}\{X \leq t\}]$ is strictly increasing in t.

Suppose the budget constraint takes the form of a capacity constraint, involving a binary take-up decision R (that may or may not be independent of X). By assumption, the probability an individual takes up the treatment for $X \in [\underline{t}, \overline{t}]$ is zero but between zero and one otherwise. Then the budget function $B(t; P) := E_P[R \cdot \mathbf{1}\{X \leq t\}]$ is flat in the interval $[\underline{t}, \overline{t}]$ but also strictly increasing otherwise.

The population problem is

$$\max_t \; W(t;P) \; s.t. \; B(t;P) \leq k,$$

where $B(t;P) = k$ for $t \in [\underline{t}, \bar{t}]$ by assumption. The constrained optimal threshold is therefore the highest threshold where the constraint is satisfied exactly i.e. $t^* = \bar{t}$. This also implies $R \cdot \mathbf{1}\{X \leq t\} \sim Bernoulli(k)$ for $t \in [\underline{t}, \bar{t}]$.

The sample-analog rule solves the following sample problem

$$\max_t \frac{1}{n} \sum_i \Gamma_i \cdot \mathbf{1}\{X_i \leq t\} \; s.t. \; \widehat{B}_n(t) \leq k$$

for $\widehat{B}_n(t) := \frac{1}{n} \sum_i R_i \cdot \mathbf{1}\{X_i \leq t\}$. Given that $\Gamma > 0$ almost surely, the sample-analog rule equivalently solves $\max_t \widehat{B}_n(t) \; s.t. \; \widehat{B}_n(t) \leq k$. However, the solution is not unique because $\widehat{B}_n(t)$ is a step function. To be conservative, let the sample-analog rule be the smallest possible threshold to maximize $\widehat{B}_n(t)$:

$$\widehat{t} = \min \left\{ \arg\max_t \{ \widehat{B}_n(t) \; s.t. \; \widehat{B}_n(t) \leq k \} \right\}.$$

Note that we can also write $\widehat{B}_n(t) = \frac{1}{n} \sum_{R_i = 1} \mathbf{1}\{X_i \leq t\}$, which makes it clear that $\widehat{t}$ corresponds to ranking X_i among individuals with $R_i = 1$, and then picking the lowest threshold such that we assign treatment to the first $\lfloor k \cdot n \rfloor$ individuals. This also means if in the sample few individuals take up the treatment such that $\frac{1}{n} \sum_i R_i \leq k$, we can have a sample-analog rule that treats everyone up to $\max_{R_i=1} X_i$. Taken together both scenarios, we note the sample-analog rule implies the treated share in the sample is equal to

$$\widehat{B}_n(\widehat{t}) := \frac{1}{n} \sum_i R_i \cdot \mathbf{1}\{X_i \leq \widehat{t}\} = \min \left\{ \frac{1}{n} \sum_i R_i, \frac{\lfloor k \cdot n \rfloor}{n} \right\}.$$

Note that

$$\begin{cases} \widehat{t} > \bar{t} \Leftrightarrow & B(\widehat{t};P) > B(\bar{t};P) \\ \widehat{t} < \underline{t} \Leftrightarrow & B(\widehat{t};P) < B(\underline{t};P) \Leftrightarrow W(\widehat{t};P) < W(\underline{t};P) \end{cases}$$

which means whenever $\widehat{t} > \bar{t}$, the sample-analog rule violates the constraint in the population as $B(\bar{t}; P) = k$; whenever $\widehat{t} < \underline{t}$, the sample-analog rule achieves strictly less welfare than t^* in the population because $W(\underline{t}; P)$ is strictly less than $W(t^*; P)$. We next derive the limit probability for these two events. Applying Law of Total Probability, we have

$$
\begin{aligned}
& Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \widehat{B}_n(\widehat{t})\right\} \\
=& Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \frac{\lfloor k \cdot n \rfloor}{n} \text{ } and \text{ } \frac{\lfloor k \cdot n \rfloor}{n} < \frac{1}{n}\sum_i R_i\right\} \\
& + Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \frac{1}{n}\sum_i R_i \text{ } and \text{ } \frac{\lfloor k \cdot n \rfloor}{n} \geq \frac{1}{n}\sum_i R_i\right\} \qquad (24) \\
\geq& Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \frac{\lfloor k \cdot n \rfloor}{n} \text{ } and \text{ } \frac{\lfloor k \cdot n \rfloor}{n} < \frac{1}{n}\sum_i R_i\right\} \\
\geq& Pr_{P^n}\left\{\widehat{B}_n(\bar{t}) < \frac{\lfloor k \cdot n \rfloor}{n}\right\} + Pr_{P^n}\left\{\frac{\lfloor k \cdot n \rfloor}{n} < \frac{1}{n}\sum_i R_i\right\} - 1 \qquad (25)
\end{aligned}
$$

For the first term in (25), we have the following lower bound

$$
\begin{aligned}
& Pr_{P^n}\left\{\frac{1}{n}\sum_i R_i \cdot \mathbf{1}\{X_i \leq \bar{t}\} \leq \frac{k \cdot n - 1}{n}\right\} \\
=& Pr_{P^n}\left\{\sqrt{n}\left(\frac{1}{n}\sum_i R_i \cdot \mathbf{1}\{X_i \leq \bar{t}\} - k\right) \leq -\frac{1}{\sqrt{n}}\right\} \to 0.5
\end{aligned}
$$

To see the convergence, we apply the Central Limit Theorem to the LHS, and note that $-\frac{1}{\sqrt{n}}$ converges to zero. Denote $p_R = Pr\{R = 1\}$. For the second term in (25), we have the following lower bound

$$
\begin{aligned}
& Pr_{P^n}\left\{\frac{1}{n}\sum_i R_i \geq \frac{k \cdot n}{n}\right\} \\
=& Pr_{P^n}\left\{\sqrt{n}\left(\frac{1}{n}\sum_i R_i - p_R\right) \geq \sqrt{n} \cdot (k - p_R)\right\} \to 1
\end{aligned}
$$

To see the convergence, we apply the Central Limit Theorem to the LHS, and note that $\sqrt{n}\cdot(k-p_R)$ diverges to $-\infty$ for $p_R > k$. We thus conclude

$$\lim_{n\to\infty} Pr_{P^n}\left\{B(\widehat{t};P) > B(\bar{t};P)\right\} \geq 0.5$$

which proves $\widehat{t}$ is not pointwise asymptotically feasible under the distribution P.

Similar argument shows $Pr_{P^n}\left\{\widehat{B}_n(\underline{t}) > \widehat{B}_n(\widehat{t})\right\}$ has a limit of at least one half. We thus conclude

$$\lim_{n\to\infty} Pr_{P^n}\left\{B(\widehat{t};P) < B(\underline{t};P)\right\} \geq 0.5 \Leftrightarrow \lim_{n\to\infty} Pr_{P^n}\left\{W(\widehat{t};P) < W(\underline{t};P)\right\} \geq 0.5$$

which proves that $\widehat{t}$ is not pointwise asymptotically welfare-efficient under the distribution P. Since $B(\underline{t};P) = B(\bar{t};P)$, we actually have

$$\lim_{n\to\infty} Pr_{P^n}\left\{B(\widehat{t};P) > B(\bar{t};P)\right\} = \lim_{n\to\infty} Pr_{P^n}\left\{W(\widehat{t};P) < W(\underline{t};P)\right\} = 0.5$$

Proof of Corollary 2.1. The event in (10) is equivalent to the event that $\hat{\mathcal{G}}$ includes at least one criteria that violates the budget constraint by c. The derivation for the bound therefore is based on such an event:

$$\begin{aligned}
&Pr_{P^n}\left\{\min_{B(g;P)>k+c} \frac{\sqrt{n}\left(\widehat{B}_n(g)-k\right)}{\Sigma^B(g,g)^{1/2}} \leq 0\right\} \\
=&Pr_{P^n}\left\{\min_{B(g;P)>k+c}\left\{\frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} + \frac{\sqrt{n}\left(B(g;P)-k\right)}{\Sigma^B(g,g)^{1/2}}\right\} \leq 0\right\} \\
\leq&Pr_{P^n}\left\{\min_{B(g;P)>k+c} \frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} \leq -\min_{B(g;P)>k+c} \frac{\sqrt{n}\left(B(g;P)-k\right)}{\Sigma^B(g,g)^{1/2}}\right\} \\
\leq&Pr_{P^n}\left\{\min_{B(g;P)>k+c} \frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} \leq -\frac{\min_{B(g;P)>k+c}\sqrt{n}\left(k-B(g;P)\right)}{\max_{B(g;P)>k+c}\Sigma^B(g,g)^{1/2}}\right\} \\
<&Pr_{P^n}\left\{\min_{B(g;P)>k+c} \frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} \leq \frac{-\sqrt{n}\cdot c}{\max_{B(g;P)>k+c}\Sigma^B(g,g)^{1/2}}\right\} \\
<&Pr_{P^n}\left\{\min_{g\in\mathcal{G}} \frac{\sqrt{n}\left(\widehat{B}_n(g)-B(g;P)\right)}{\Sigma^B(g,g)^{1/2}} \leq \frac{-\sqrt{n}\cdot c}{\max_{g\in\mathcal{G}}\Sigma^B(g,g)^{1/2}}\right\}
\end{aligned}$$

Under Assumption 2.4, the empirical process $\left\{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)\right\}$ converges to a Gaussian process G_P^B for $G_P^B(\cdot) \sim \mathcal{GP}(0, \Sigma_P^B(\cdot,\cdot))$ and we have a consistent covariance estimate $\widehat{\Sigma}^B(\cdot,\cdot)$, which proves the corollary. □

Proof. **Proof of Corollary 2.2.** The proof is identical to the second part of the proof of Proposition 2.1. □

Proof. **Proof of Theorem 3.1.** By construction, the limit probability for any policy in $\hat{\mathcal{G}}_\alpha$ to violate the budget constraint is

$$
\begin{aligned}
&Pr_{P^n}\{\exists g : g \in \hat{\mathcal{G}}_\alpha \text{ and } B(g;P) > k\} = Pr_{P^n}\{\min_{B(g;P)>k} \frac{\sqrt{n}\left(\widehat{B}_n(g) - k\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\} \\
\leq & Pr_{P^n}\{\min_{B(g;P)>k} \frac{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\} \\
\leq & Pr_{P^n}\{\min_{g\in\mathcal{G}} \frac{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\}
\end{aligned}
$$

Under Assumption 2.4, uniformly over $P \in \mathcal{P}$, the empirical process $\left\{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)\right\}$ converges to a Gaussian process G_P^B for $G_P^B(\cdot) \sim \mathcal{GP}(0, \Sigma_P^B(\cdot,\cdot))$ and we have a consistent covariance estimate $\widehat{\Sigma}^B(\cdot,\cdot)$. Then by the definition of c_α, we have

$$
\begin{aligned}
&\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n}\{\min_{g\in\mathcal{G}} \frac{\sqrt{n}\left(\widehat{B}_n(g) - B(g;P)\right)}{\widehat{\Sigma}^B(g,g)^{1/2}} \leq c_\alpha\} \\
=&\sup_{P\in\mathcal{P}} Pr_{P^n}\{\inf_{g\in\mathcal{G}} \frac{G_P^B(g)}{\Sigma_P^B(g,g)^{1/2}} \leq c_\alpha\} = \alpha.
\end{aligned}
$$

□

Proof. **Proof of Theorem 3.2 and Corollary** 3.1**.** Theorem 3.2 follows from Theorem 3.1, replacing α with α_n in its proof. Specifically, by construction we have

$$
\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n}\{B(\hat{g}_{\text{mistake}};P) > k\} \leq \limsup_{n\to\infty} \sup_{P\in\mathcal{P}} Pr_{P^n}\{\exists g : g \in \hat{\mathcal{G}}_{\alpha_n} \text{ and } B(g;P) > k\} \leq \alpha_n.
$$

For Corollary 3.1, we decompose the welfare loss into

$$\begin{aligned}
&W(g_P^*;P) - W(\hat{g}_{\text{mistake}};P)\\
=&W(g_P^*;P) - \widehat{W}_n(\hat{g}_{\text{mistake}}) + \widehat{W}_n(\hat{g}_{\text{mistake}}) - W(\hat{g}_{\text{mistake}};P)\\
\leq&\sup_{g\in\mathcal{G}} |W(g;P) - \widehat{W}_n(g)| + \widehat{W}_n(g_P^*) - \widehat{W}_n(\hat{g}_{\text{mistake}})
\end{aligned}$$

Under the event that $g_P^* \in \hat{\mathcal{G}}_{\alpha_n}$, we are guaranteed that the last term is non-positive. This happens with probability

$$Pr_{P^n}\{\frac{\sqrt{n}\left(\widehat{B}_n(g_P^*) - k\right)}{\widehat{\Sigma}^B(g_P^*, g_P^*)^{1/2}} \leq c_{\alpha_n}\} \overset{a}{\sim} \Phi\left(c_{\alpha_n} + \frac{\sqrt{n}\,(k - B(g_P^*;P))}{\Sigma_P^B(g_P^*, g_P^*)^{1/2}}\right) \to 1$$

for $B(g_P^*;P)$ strictly below the threshold. Since c_{α_n} diverges to $-\infty$ at a rate slower $\sqrt{n}$, the term in the parenthesis diverges to ∞ as $n \to \infty$. We then apply Lemma B.4 and B.6 in the Online Appendix (proved in Section B.3 under primitive conditions that lead to Assumption 2.4), which shows $\sup_{g\in\mathcal{G}} |W(g;P) - \widehat{W}_n(g)|$ converges to zero in probability. We then conclude $\hat{g}_{\text{mistake}}$ is asymptotically welfare-efficient under distribution P. □

Proof. **Proof of Lemma 4.1.** By definition

$$\begin{aligned}
W(g_P^*;P) = \max_{B(g;P)\leq k} W(g;P) &= \max_{B(g;P)\leq k} W(g;P) - \overline{\lambda}\cdot(B(g;P)-k)_+\\
&\leq \max_{g\in\mathcal{G}} W(g;P) - \overline{\lambda}\cdot(B(g;P)-k)_+\\
&= W(\widetilde{g}_P;P) - \overline{\lambda}\cdot(B(\widetilde{g}_P;P)-k)_+ \leq W(\widetilde{g}_P;P)
\end{aligned}$$

and suppose $B(\widetilde{g}_P;P) > k$, we have the following upper bound for violations to the budget constraint

$$B(\widetilde{g}_P;P) - k \leq \frac{W(\widetilde{g}_P;P) - W(g_P^*;P)}{\overline{\lambda}}.$$

□

Proof. **Proof of Theorem 4.1.** We first prove $\widehat{g}_{\text{tradeoff}}$ is asymptotically welfare-efficient. Denote the new objective function with $V(g;P) = W(g;P) - \overline{\lambda} \cdot (B(g;P) - k)_+$, whose maximizer is $\widetilde{g}_P$. Denote $\widehat{V}_n(g) = \widehat{W}_n(g) - \overline{\lambda} \cdot (\widehat{B}_n(g) - k)_+$ to be the sample-analog, whose maximizer is $\widehat{g}_{\text{tradeoff}}$. Apply uniform deviation bound to the difference between the value under $\widehat{g}_{\text{tradeoff}}$ and $\widetilde{g}_P$, we have

$$\begin{aligned}
&V(\widetilde{g}_P;P) - V(\widehat{g}_{\text{tradeoff}};P) \\
=&V(\widetilde{g}_P;P) - \widehat{V}_n(\widehat{g}_{\text{tradeoff}}) + \widehat{V}_n(\widehat{g}_{\text{tradeoff}}) - V(\widehat{g}_{\text{tradeoff}};P) \\
\leq&V(\widetilde{g}_P;P) - \widehat{V}_n(\widetilde{g}_P) + \widehat{V}_n(\widehat{g}_{\text{tradeoff}}) - V(\widehat{g}_{\text{tradeoff}};P) \\
\leq&2\sup_g \left| V(g;P) - \widehat{V}_n(g) \right| \\
=&2\sup_g \left| W(g;P) - \widehat{W}_n(g) - \overline{\lambda} \cdot (\max\{B(g;P) - k, 0\} - \max\{\widehat{B}_n(g) - k, 0\}) \right| \\
\leq&2\sup_g \left| \widehat{W}_n(g) - W(g;P) \right| + 2\overline{\lambda} \cdot \sup_g \left| \max\{\widehat{B}_n(g) - k, 0\} - \max\{B(g;P) - k, 0\} \right| \\
\leq&2\sup_g \left| \widehat{W}_n(g) - W(g;P) \right| + 2\overline{\lambda} \cdot \sup_g \left| \widehat{B}_n(g) - B(g;P) \right|
\end{aligned}$$

The last line uses the fact that $|\max\{a,0\} - \max\{b,0\}| \leq |a - b|$. Both terms, $\sup_g \left| \widehat{W}_n(g) - W(g;P) \right|$ and $\sup_g \left| \widehat{B}_n(g) - B(g;P) \right|$ converge to zero in probability under Assumption 2.4. Specifically, Lemma B.4 and B.6 in the Online Appendix (proved in Section B.3 under primitive conditions that lead to Assumption 2.4) imply the uniform convergence in probability

$$\sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \widehat{W}_n(g) - W(g;P) \right| \to_p 0, \ \sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \widehat{B}_n(g) - B(g;P) \right|_p \to 0.$$

At the same time, by definition we have $V(\widetilde{g}_P;P) - V(\widehat{g}_{\text{tradeoff}};P) \geq 0$. We thus conclude

$$\sup_{P \in \mathcal{P}} |V(\widehat{g}_{\text{tradeoff}};P) - V(\widetilde{g}_P;P)| \to_p 0.$$

Furthermore, we have $W(g_P^*;P) \leq V(\widetilde{g}_P;P) \leq W(\widetilde{g}_P;P)$ as a by-product of Lemma

4.1 for any $P \in \mathcal{P}$. Putting these together, we have

$$\begin{aligned}
&\inf_{P\in\mathcal{P}} \{W(\widehat{g}_{\text{tradeoff}}; P) - W(g_P^*; P)\} \\
\geq &\inf_{P\in\mathcal{P}} \{V(\widehat{g}_{\text{tradeoff}}; P) - W(g_P^*; P)\} \\
\geq &\inf_{P\in\mathcal{P}} \{V(\widehat{g}_{\text{tradeoff}}; P) - V(\widetilde{g}_P; P)\} + \inf_{P\in\mathcal{P}} \{V(\widetilde{g}_P; P) - W(g_P^*; P)\} \\
\geq &\inf_{P\in\mathcal{P}} \{V(\widehat{g}_{\text{tradeoff}}; P) - V(\widetilde{g}_P; P)\} \to_p 0.
\end{aligned}$$

which proves uniform asymptotic welfare-efficiency.

Also by the uniform deviation bound shown before, we have that uniformly with probability approaching one, to the difference between the value under $\widehat{g}_{\text{tradeoff}}$ and $\widetilde{g}_P$, we have

$$V(g_P^*; P) - V(\widehat{g}_{\text{tradeoff}}; P) \leq 0$$

which is equivalent to $(B(\widehat{g}_{\text{tradeoff}}; P) - k)_+ \leq \frac{W(\widehat{g}_{\text{tradeoff}};P)-W(g_P^*;P)}{\overline{\lambda}}$. □

Online Appendix to Empirical Welfare Maximization with Constraints

This Online Appendix contains proofs of supporting lemmas and additional results stated in the paper.

B Primitive assumptions and auxiliary lemmas

I first prove the optimal rule that solves the population constrained optimization problem takes the form of threshold. In Section B.2, I first provide primitive assumptions on the class of DGPs. I then prove in Lemma B.1, which establishes that these primitive assumptions imply Assumption 2.1.

In Section B.3, I verify Assumption 2.4 for settings where the observed sample comes from an RCT or an observational study, and the propensity score can be estimated efficiently based on parametric regressions.

B.1 Constrained optimal rule without functional form restriction

The population problem is to find rules based on X_i that solves

$$\max_{g:\mathcal{X}\to\{0,1\}} E[\Gamma_i g(X_i)] \ s.t. \ E[R_i g(X_i)] < k$$

By Law of Iterated Expectation, we can write the constrained optimization problem as

$$\max_{g:\mathcal{X}\to\{0,1\}} E[\gamma(X_i)g(X_i)] \ s.t. \ E[r(X_i)g(X_i)] < k$$

where $\gamma(X_i) = E[\Gamma_i \mid X_i]$ and $r(X_i) = E[R_i \mid X_i]$.

Claim B.1. Let $d\mu = r(x)f(x)dx$ denote the positive measure. The constrained optimization problem is equivalent to

$$\max_{g:\mathcal{X}\to\{0,1\}} \int \frac{\gamma(x)}{r(x)} g(x) d\mu \ s.t. \int g(x) d\mu = k$$

Let X^* be the support of the solution g^*. It will take the form of $X^* = \{x : \frac{\gamma(x)}{r(x)} > c\}$ where c is chosen so that $\mu(X^*) = k$.

Proof. Let X be the support of any $g \neq g^*$ with $\mu(X) = k$. Then the objective function associated g is

$$\begin{aligned}\int_{X^*} \frac{\gamma}{r} d\mu - \int_X \frac{\gamma}{r} d\mu &= \int \frac{\gamma(x)}{r(x)} \mathbf{1}\{x \in X^*\} d\mu - \int \frac{\gamma(x)}{r(x)} \mathbf{1}\{x \in X\} d\mu \\ &= \int \frac{\gamma(x)}{r(x)} \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu\end{aligned}$$

By definition of X^*, we have $\frac{\gamma(x)}{r(x)} > c$ for $x \in X^* \backslash X$ and $\frac{\gamma(x)}{r(x)} < c$ for $x \in X \backslash X^*$. Also note that μ is a positive measure. Then the above difference is lower bounded by

$$\int \frac{\gamma(x)}{r(x)} \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu \geq c \int \left(\mathbf{1}\{x \in X^* \backslash X\} - \mathbf{1}\{x \in X \backslash X^*\}\right) d\mu \geq 0$$

as by construction, we have $\int \mathbf{1}\{x \in X^* \backslash X\} d\mu = 1$ since $\mu(X^*) = \mu(X) = k$. □

B.2 Primitive assumptions for contiguity

Assumption B.1. *Assume the class of DGPs $\{P_\theta : \theta \in \Theta\}$ has densities p_θ with respect to some measure μ. Assume P_θ is DQM at P_0 i.e. $\exists \dot{\ell}_0$ s.t. $\int [\sqrt{p_h} - \sqrt{p_0} - \frac{1}{2} h' \dot{\ell}_0 \sqrt{p_0}]^2 d\mu = o(\|h^2\|)$ for $h \to 0$.*

Assumption B.2. *For all policies g, $B(g; P_\theta)$ is twice continuously differentiable in θ at 0, and the derivatives are bounded from above and away from zero within an open neighborhood $\mathcal{N}_\theta$ of zero uniformly over $g \in \mathcal{G}$.*

Lemma B.1. *Under Assumption B.1, the class $\mathcal{P}$ includes a sequence of data distribution $\{P_{h_n}\}$ that is contiguous to P_0 for every h_n satisfying $\sqrt{n} h_n \to h$ e.g. take*

$h_n = h/\sqrt{n}$. This proves the first part of Assumption 2.1. Suppose further Assumption B.2 holds, then there exists some h for the second part of Assumption 2.1 to hold,.

Proof. **Proof of Lemma B.1.** By Theorem 7.2 of Vaart (1998), the log likelihood ratio process converges under P_0 (denoted with $\overset{p_0}{\rightsquigarrow}$) to a normal experiment

$$\log \prod_{i=1}^{n} \frac{p_{h_n}}{p_0}(A_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h' \dot{\ell}_0(A_i) - \frac{1}{2} h' I_0 h + o_{P_0}(1) \tag{26}$$

$$\overset{p_0}{\rightsquigarrow} \mathcal{N}\left(-\frac{1}{2} h' I_0 h, h' I_0 h\right) \tag{27}$$

where $\dot{\ell}_0$ is the score and $I_{P_0} = E_{P_0}[\dot{\ell}_0(A_i)\dot{\ell}_0(A_i)']$ exists. The convergence in distribution of the log likelihood ratio to a normal with mean equal to $-\frac{1}{2}$ of its variance in (27) implies mutual contiguity $P_0^n \lhd\rhd P_{h_n}^n$ by Le Cam's first lemma (see Example 6.5 of Vaart (1998)). This proves the first part of the lemma.

By Taylor's theorem with remainder we have for each policy g

$$B(g; P_{h/\sqrt{n}}) - B(g; P_0) = \frac{h'}{\sqrt{n}} \frac{\partial B(g; P_0)}{\partial \theta} + \frac{1}{2} \frac{1}{n} h' \frac{\partial^2 B(g; P_{\tilde{\theta}_n})}{\partial \theta \partial \theta'} h$$

where $\tilde{\theta}_n$ is a sequence of values with $\tilde{\theta}_n \in [0, h/\sqrt{n}]$ that can depend on g. Take h so that the first term is positive for policies with $B(g; P_0) = k$. Such h exists because we assume $\frac{\partial B(g;P_0)}{\partial \theta}$ is bounded away from zero. For $g \in \mathcal{G}_0$ where $\mathcal{G}_0 = \{g : B(g; P_0) = k\}$, the constraints are violated under P_{h_n} and furthermore (multiplying by $\sqrt{n}$)

$$\sqrt{n} \cdot (B(g; P_{h_n}) - k) > h' \frac{\partial B(g; P_0)}{\partial \theta} > 0$$

for every n. This proves the second part of the lemma for $C = \inf_{g \in \mathcal{G}_0} \left| h' \frac{\partial B(g;P_0)}{\partial \theta} \right|$
. □

B.3 Primitive assumptions and proofs for estimation quality

In this section, I verify that $\widehat{W}_n(g)$ and $\widehat{B}_n(g)$ satisfy Assumption 2.4 under primitive assumptions on the policy class and the OHIE.

Assumption B.3. *VC-class: The policy class $\mathcal{G}$ has a finite VC-dimension $v < \infty$.*

To introduce the assumptions on the OHIE, I first recall the definitions of components in $\widehat{W}_n(g)$ and $\widehat{B}_n(g)$:

$$\widehat{W}_n(g) := \frac{1}{n}\sum_i \Gamma_i^* \cdot g(X_i), \ \ \widehat{B}_n(g) := \frac{1}{n}\sum_i R_i^* \cdot g(X_i)$$

for the doubly-robust scores

$$\begin{aligned}
\Gamma_i^* &= \widehat{\gamma}^Y(V_i, 1) - \widehat{\gamma}^Y(V_i, 0) + \widehat{\alpha}(V_i, D_i) \cdot \left(Y_i - \widehat{\gamma}^Y(V_i, D_i)\right) \\
R_i^* &= \widehat{\gamma}^Z(V_i, 1) + \frac{D_i}{\widehat{p}(V_i)} \cdot \left(Z_i - \widehat{\gamma}^Z(V_i, D_i)\right)
\end{aligned}$$

and observed characteristics X_i. Here V_i collects the confounders in OHIE, namely household size (number of adults entered on the lottery sign-up form) and survey wave. Note that while X_i can overlap with V_i, the policy $g(X_i)$ needs not vary by V_i. In the OHIE example, the policy is based on number of children and income. However, conditional on household size and survey wave, income and number of children is independent of the lottery outcome in OHIE.

Recall that $\widehat{W}_n(g)$ and $\widehat{B}_n(g)$ are supposed to approximate net benefit Γ and net excess cost R of Medicaid eligibility. I provide more precise definitions for Γ and R as the primitive assumptions are stated in terms of their components.

Let $Y(1)$ be the (potential) subjective health when one is given Medicaid eligibility, and $Y(0)$ be the (potential) subjective health when one is not given Medicaid eligibility. Recall the definition for

$$\Gamma = Y(1) - Y(0)$$

We only observe the actual subjective health Y_i.

Let $M(1)$ be the (potential) enrollment in Medicaid when one is given Medicaid eligibility, and $M(0)$ be the (potential) enrollment in Medicaid when one is not given Medicaid eligibility.

Let $C(1)$ be the (potential) cost to the government when one is given Medicaid eligibility, and $C(0)$ be the (potential) cost to the government when one is not given Medicaid eligibility. Note that $C(0) = 0$ by construction. Even when given eligibility, one might not enroll and thus incur zero cost to the government. So given the current expenditure is \$6,000 per *enrollee*, the implied expenditure *per capita under eligibility policy* $g(X)$ is actually $\$6,000 \cdot E[M(1)g(X)]$. The reason is that the expected enrollment rate is only $Pr\{M(1) = 1 \mid X \in g\}$. So the per capita *excess* cost of Medicaid eligibility policy $g(X)$ relative to the current level is

$$R = C(1) - \$6,000 \cdot M(1).$$

We only observe the actual cost ($C_i = D_i C_i(1) + (1 - D_i)C_i(0)$) and the actual Medicaid enrollment (M_i) in OHIE. We calculate $Z_i = C_i - \$6,000 \cdot M_i$.

Assumption B.4. *Suppose for all $P \in \mathcal{P}$, the following statements hold for the OHIE:*

Independent characteristics: $Pr\{D_i = 1 \mid V_i, X_i\} = Pr\{D_i = 1 \mid V_i\}$

Unconfoundedness: $(Y(1), Y(0), C(1), M(1)) \perp D_i | V_i$.

Bounded attributes: the support of variables X_i, Y_i and Z_i are bounded.

Strict overlap: There exist $\kappa \in (0, 1/2)$ such that the propensity score satisfies $p(v) \in [\kappa, 1 - \kappa]$ for all $v \in \mathcal{V}$.

B.3.1 Uniform convergence of $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$

We want to show the recentered empirical processes $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$ converge to mean-zero Gaussian processes G_P^W and G_P^B with covariance functions $\Sigma_P^W(\cdot,\cdot)$ and $\Sigma_P^B(\cdot,\cdot)$ respectively uniformly over $P \in \mathcal{P}$. The covariance functions are uniformly bounded, with diagonal entries bounded away from zero uniformly over $g \in \mathcal{G}$. Take $\widehat{W}_n(\cdot)$ for example, the recentered empirical processes is

$$\sqrt{n}\left(\frac{1}{n}\sum_i \Gamma_i^* \cdot g(X_i) - E_P[\Gamma \cdot g(X_i)]\right)$$

and can be expressed as the sum of two terms

$$\frac{1}{\sqrt{n}}\sum_i (\Gamma_i^* - \widetilde{\Gamma}_i) \cdot g(X_i) + \sqrt{n}\left(\frac{1}{n}\sum_i \widetilde{\Gamma}_i \cdot g(X_i) - E_P[\Gamma \cdot g(X_i)]\right). \qquad (28)$$

Here $\widetilde{\Gamma}_i$ are the theoretical analogs

$$\widetilde{\Gamma}_i = \gamma^Y(V_i, 1) - \gamma^Y(V_i, 0) + \alpha(V_i, D_i) \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right)$$

which is doubly-robust score with the theoretical propensity score and the CEF. A similar expansion holds for $\widehat{B}_n(\cdot)$ involving the theoretical analog

$$\widetilde{R}_i = \gamma^Z(V_i, 1) + \frac{D_i}{p(V_i)} \cdot \left(Z_i - \gamma^Z(V_i, D_i)\right).$$

The following lemmas prove the uniform convergence of $\widehat{W}_n(\cdot)$ and $\widehat{B}_n(\cdot)$.

The last part of the assumption is that we have a uniformly consistent estimate for the covariance function. I argue the sample analog

$$\widehat{\Sigma}^B(g, g') = \frac{1}{n}\sum_i (\widetilde{R}_i)^2 \cdot g(X_i) \cdot g'(X_i) - \left(\frac{1}{n}\sum_i \widetilde{R}_i \cdot g(X_i)\right) \cdot \left(\frac{1}{n}\sum_i \widetilde{R}_i \cdot g'(X_i)\right)$$

is a pointwise consistent estimate for $\widehat{\Sigma}^B(g, g')$. To see this, note that the covariance function $\Sigma_P^B(\cdot, \cdot)$ maps $\mathcal{G} \times \mathcal{G}$ to $\mathbb{R}$. Under Assumption B.4 that $\mathcal{G}$ is a VC-class, the product $\mathcal{G} \times \mathcal{G}$ is also a VC-class (see e.g. Lemma 2.6.17 of van der Vaart and Wellner (1996)). By a similar argument leading to Lemma B.4 and B.6 , we conclude the uniform consistency of $\widehat{\Sigma}^B(\cdot, \cdot)$.

Lemma B.2. *Let $\mathcal{G}$ be a VC-class of subsets of $\mathcal{X}$ with VC-dimension $v < \infty$. The following sets of functions from $\mathcal{A}$ to $\mathbb{R}$*

$$\mathcal{F}^W = \{\widetilde{\Gamma}_i \cdot g(X_i) : g \in \mathcal{G}\}$$
$$\mathcal{F}^B = \{\widetilde{R}_i \cdot g(X_i) : g \in \mathcal{G}\}$$

are VC-subgraph class of functions with VC-dimension less than or equal to v for all $P \in \mathcal{P}$. For notational simplicity, we suppress the dependence of $\mathcal{F}$ on P.

Lemma B.3. *For all P in the family $\mathcal{P}$ of distributions satisfying Assumption B.4, for all $g \in \mathcal{G}$ we have*

$$E_P[\widetilde{\Gamma}_i \cdot g(X_i)] = W(g; P)$$
$$E_P[\widetilde{R}_i \cdot g(X_i)] = B(g; P)$$

Lemma B.4. *Let $\mathcal{G}$ satisfy Assumption B.4 (VC). Let U_i be a mean-zero bounded random vector of fixed dimension i.e. there exists $M < \infty$ such that i.e. $U_i \in [-M/2, M/2]$ almost surely under all $P \in \mathcal{P}$. Then the uniform deviation of the sample average of vanishes*

$$\sup_{P \in \mathcal{P}} \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_i U_i \cdot g(X_i) \right| \to_{a.s.} \mathbf{0}.$$

Lemma B.5. *Let $\mathcal{P}$ be a family of distributions satisfying Assumption B.4. Let $\mathcal{G}$ satisfy Assumption B.4 (VC). Then $\mathcal{F}^W$ and $\mathcal{F}^B$ are P-Donsker for all $P \in \mathcal{P}$. That is, the empirical process indexed by $g \in \mathcal{G}$*

$$\sqrt{n} \cdot (\frac{1}{n} \sum_i \widetilde{\Gamma}_i \cdot g(X_i) - W(g; P))$$

converge to a Gaussian process $\mathcal{GP}(0, \Sigma_P^W(\cdot, \cdot))$ uniformly in $P \in \mathcal{P}$, and the empirical process indexed by $g \in \mathcal{G}$

$$\sqrt{n} \cdot (\frac{1}{n} \sum_i \widetilde{R}_i \cdot g(X_i) - B(g; P))$$

converge to a Gaussian process $\mathcal{GP}(0, \Sigma_P^B(\cdot, \cdot))$ uniformly in $P \in \mathcal{P}$.

Lemma B.6. *Let $\mathcal{P}$ be a family of distributions satisfying Assumption B.4. Let $\mathcal{G}$*

satisfy Assumption B.4 (VC). Then the estimation errors vanishes

$$\sup_{P\in\mathcal{P}}\sup_{g\in\mathcal{G}}\left|\frac{1}{\sqrt{n}}\sum_i(\Gamma_i^* - \widetilde{\Gamma}_i)\cdot g(X_i)\right| \to_p 0$$
$$\sup_{P\in\mathcal{P}}\sup_{g\in\mathcal{G}}\left|\frac{1}{\sqrt{n}}\sum_i(R_i^* - \widetilde{R}_i)\cdot g(X_i)\right| \to_p 0$$

B.3.2 Proofs of auxiliary lemmas

Proof. **Proof of Lemma B.2.** This lemma follows directly from Lemma A.1 of Kitagawa and Tetenov (2018). □

Proof. **Proof of Lemma B.3.** Under Assumption B.4, we prove each $\widetilde{\Gamma}_i$ is an conditionally unbiased estimate for Γ:

$$E_P[\widetilde{\Gamma}_i g(X_i)] = E_P[E_P[\widetilde{\Gamma}_i \mid X_i]g(X_i)] = E_P[E_P[\Gamma \mid X_i]g(X_i)] = W(g; P).$$

We focus on $E_P[\widetilde{\Gamma}_i \mid X_i] = E_P[E_P[\widetilde{\Gamma}_i \mid V_i, X_i] \mid X_i]$. However, conditional on V_i, by unconfoundedness and strict overlap we have

$$\begin{aligned} E_P[\widetilde{\Gamma}_i \mid V_i, X_i] &= E[Y_i \mid V_i, X_i, D_i = 1] - E[Y_i \mid V_i, X_i, D_i = 0] \\ &= E[Y_i(1) - Y_i(0) \mid V_i, X_i] = E[\Gamma_i \mid V_i, X_i] \end{aligned}$$

Specifically

$$\begin{aligned} E[\widetilde{\Gamma}_i \mid V_i, X_i] &= E\left[\gamma^Y(V_i, 1) - \gamma^Y(V_i, 0) \mid V_i, X_i\right] + E\left[\alpha(V_i, D_i)\cdot\left(Y_i - \gamma^Y(V_i, D_i)\right) \mid V_i, X_i\right] \\ &= E\left[Y_i \mid V_i, D_i = 1\right] - E\left[Y_i \mid V_i, D_i = 0\right] + \\ &E\left[Y_i(1) - Y_i(0) \mid V_i, X_i\right] - \left(E\left[Y_i \mid V_i, D_i = 1\right] - E\left[Y_i \mid V_i, D_i = 0\right]\right) \\ &= E[\Gamma \mid V_i, X_i] \end{aligned}$$

Specifically, we expand $E[\alpha(V_i, D_i)Y_i \mid V_i, X_i]$

$$
\begin{aligned}
&= E\left[\frac{Y_i}{p(V_i)} \mid D_i = 1, V_i, X_i\right] \cdot Pr\{D_i = 1 \mid V_i, X_i\} - E\left[\frac{Y_i}{1 - p(V_i)} \mid D_i = 0, X_i\right] \cdot Pr\{D_i = 0 \mid V_i, X_i\} \\
&= E\left[D_iY_i(1) + (1 - D_i)Y_i(0) \mid D_i = 1, V_i, X_i\right] - E\left[D_iY_i(1) + (1 - D_i)Y_i(0) \mid D_i = 0, V_i, X_i\right] \\
&= E\left[Y_i(1) \mid V_i, X_i\right] - E\left[Y_i(0) \mid V_i, X_i\right]
\end{aligned}
$$

where the second line holds by independent characteristic such that

$$Pr\{D_i = 1 \mid V_i, X_i\} = Pr\{D_i = 1 \mid V_i\} =: p(V_i)$$

and the last line follows from unconfoundedness. Similarly, we show

$$E\left[\alpha(V_i, D_i) \cdot \gamma^Y(V_i, D_i) \mid V_i, X_i\right] = E\left[Y_i \mid V_i, D_i = 1\right] - E\left[Y_i \mid V_i, D_i = 0\right]$$

Similar argument holds for $E_P[\widetilde{R}_i \cdot g(X_i)]$, with the only modification:

$$
\begin{aligned}
&E[\frac{D_i}{p(V_i)} C_i \mid V_i, X_i] = E\left[\frac{C_i}{p(V_i)} \mid D_i = 1, V_i, X_i\right] \cdot Pr\{D_i = 1 \mid V_i, X_i\} \\
&= E[C_i(1) \mid V_i, X_i]
\end{aligned}
$$

□

Proof. **Proof of Lemma B.4.** Denote the following set of functions from $\mathcal{U}$ to $\mathbb{R}$

$$\mathcal{F}^U = \{U_i \cdot g(X_i) : g \in \mathcal{G}\}$$

and it has uniform envelope $\bar{F} = M/2$ since U_i is bounded. This envelop function is bounded uniformly over $\mathcal{P}$. Also, by Assumption B.4 (VC) and Lemma B.2, $\mathcal{F}^U$ is VC-subgraph class of functions with VC-dimension at most v. By Lemma 4.14 and Proposition 4.18 of Wainwright (2019), we conclude that $\mathcal{F}^U$ has Rademacher complexity $2\sqrt{M^2 \frac{v}{n}}$. Then by Proposition 4.12 of Wainwright (2019) we conclude that $\mathcal{F}^U$ are P-Glivenko–Cantelli for each $P \in \mathcal{P}$, with an $O(\sqrt{\frac{v}{n}})$ rate of convergence. Note that this argument does not use any constants that depend on P but only M

and v, so we can actually get uniform convergence over $\mathcal{P}$. $\square$

Proof. **Proof of Lemma B.5.** Note that Assumption B.4 imply that $\mathcal{F}^W$ and $\mathcal{F}^B$ have uniform envelope $\bar{F} = M/(2\kappa)$. $\mathcal{F}^W$ and $\mathcal{F}^B$ thus have square integrable envelop functions uniformly over $\mathcal{P}$. Also, by Assumption B.4 (VC) and Lemma B.2, $\mathcal{F}^W$ and $\mathcal{F}^B$ are VC-subgraph class of functions with VC-dimension at most v. Even though both $\mathcal{F}^W$ and $\mathcal{F}^B$ depend on P, a similar argument for Theorem 1 in Rai (2019) show that $\mathcal{F}^W$ and $\mathcal{F}^B$ are P-Donsker uniformly in $P \in \mathcal{P}$. $\square$

Proof. **Proof of Lemma B.6.** We focus on the deviation in $\Gamma_i^* - \widetilde{\Gamma}_i$. The deviation in $R_i^* - \widetilde{R}_i$ can be proven to vanish in a similar manner. Denote $\Delta\gamma^Y(V_i) = \gamma^Y(V_i, 1) - \gamma^Y(V_i, 0)$. For any fixed policy g, we expand the deviation $\frac{1}{\sqrt{n}}\sum_i(\Gamma_i^* - \widetilde{\Gamma}_i)g(X_i)$ into three terms

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right) \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) \qquad (29)\\
&+ \frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\Delta\widehat{\gamma}^Y(V_i) - \Delta\gamma^Y(V_i) - \alpha(V_i, D_i) \cdot \left(\widehat{\gamma}^Y(V_i, D_i) - \gamma^Y(V_i, D_i)\right)\right)\\
&- \frac{1}{\sqrt{n}}\sum_i g(X_i)\left(\widehat{\gamma}^Y(V_i, D_i) - \gamma^Y(V_i, D_i)\right) \cdot \left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right) \cdot
\end{aligned}
$$

Denote these three summands by $D_1(g)$, $D_2(g)$ and $D_3(g)$. We will bound all three summands separately. Recall we use the full sample to estimate the propensity score and the CEF with a saturated model. The purpose of the above expansion is to separately bound the estimation error from the estimated CEF and propensity score, and the deviation from taking sample averages. For cross-fitted estimators for the propensity score and the CEF, a similar bound can be found in Athey and Wager (2021).

Uniform consistency of the estimated CEF and propensity score Denote with $b(V_i, D_i)$ the dictionary that spans (V_i, D_i), and $b(V_i,)$ the dictionary that spans V_i. The saturated models are therefore parameterized as $\gamma^Z(V_i, D_i) = \gamma' b(V_i, D_i)$ and $p(V_i) = \beta' b(V_i)$. Under standard argument, the OLS estimators $\widehat{\gamma}$ and $\widehat{\beta}$ are asymptotically normal uniformly over $P \in \mathcal{P}$:

$$\sqrt{n} \cdot (\widehat{\gamma} - \gamma) = O_P(1), \ \sqrt{n} \cdot \left(\widehat{\beta} - \beta\right) = O_P(1).$$

Furthermore, the in-sample L_2 errors from the estimated CEF and propensity score vanish. Consider

$$\frac{1}{n} \sum_i \left(\widehat{\gamma}^Z(V_i, D_i) - \gamma^Z(V_i, D_i)\right)^2 = \frac{1}{n} \sum_i \left((\widehat{\gamma} - \gamma)' \, b(V_i, D_i)\right)^2 = (\widehat{\gamma} - \gamma)' \, \widehat{M} \, (\widehat{\gamma} - \gamma)$$

where $\widehat{M} = \frac{1}{n} \sum_i b(V_i, D_i) b(V_i, D_i)'$. It converges in probability to a fixed matrix $M = E[b(V_i, D_i) b(V_i, D_i)']$. So the in-sample L_2 error from the estimated CEF vanishes at the rate of $n^{-1/2}$.

Similarly, consider expanding $\frac{1}{n} \sum_i \left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right)^2$ as

$$\frac{1}{n} \sum_i \left(\frac{1}{\widehat{\beta}' b(V_i)} - \frac{1}{\beta' b(V_i)} \right)^2 D_i^2 + \left(\frac{1}{1 - \widehat{\beta}' b(V_i)} - \frac{1}{1 - \beta' b(V_i)} \right)^2 (1 - D_i)^2$$

With a first-order Taylor approximation, for each term in the summand, the dominating term would be

$$\frac{1}{n} \sum_i \left(\left(\widehat{\beta} - \beta\right)' \frac{-b(V_i)}{(\beta' b(V_i))^2} \right)^2 D_i^2 = \left(\widehat{\beta} - \beta\right)' \left(\frac{1}{n} \sum_i \frac{b(V_i) b(V_i)'}{(\beta' b(V_i))^2} D_i^2 \right) \left(\widehat{\beta} - \beta\right)$$

where the middle term converges to a fixed matrix as implied by $\beta' b(V_i)$ being bounded away from zero and one. So the in-sample L_2 error from the estimated propensity score also vanishes at the rate of $n^{-1/2}$.

Bounding the deviation We now bound each term in (29). Plugging in the first-order Taylor approximation with a remainder term to the estimated propensity score, we have

$$
\begin{aligned}
D_1(g) =& \frac{1}{\sqrt{n}} \sum_i g(X_i) \left(Y_i - \gamma^Y(V_i, D_i)\right) \cdot \\
& \left(\left(\frac{1}{\widehat{\beta}' b(V_i)} - \frac{1}{\beta' b(V_i)} \right) D_i + \left(\frac{1}{1 - \widehat{\beta}' b(V_i)} - \frac{1}{1 - \beta' b(V_i)} \right) (1 - D_i) \right) \\
=& \frac{1}{\sqrt{n}} \sum_i g(X_i) \left(\widehat{\beta} - \beta\right)' \frac{-b(V_i)}{(\beta' b(V_i))^2} \cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) + \\
& \frac{1}{\sqrt{n}} \sum_i g(X_i) \left(\widehat{\beta} - \beta\right)' \frac{b(V_i) b(V_i)'}{(\widetilde{\beta}' V_i)^3} \left(\widehat{\beta} - \beta\right) \cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right) \\
=& \underbrace{\sqrt{n} \left(\widehat{\beta} - \beta\right)'}_{O_P(1)} \underbrace{\frac{1}{n} \sum_i g(X_i) \frac{-b(V_i)}{(\beta' b(V_i))^2} \cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right)}_{o_P(1)} + o_P(1)
\end{aligned}
$$

where $\widetilde{\beta}$ is a sequence between $\widehat{\beta}$ and β. This remainder term therefore converges to zero. Uniform convergence of the sample average follows from Lemma B.4: the random vector $\frac{-b(V_i)}{(\beta' b(V_i))^2} \cdot D_i \cdot \left(Y_i - \gamma^Y(V_i, D_i)\right)$ is mean-zero, has fixed dimension, and bounded.

We can decompose $D_2(g)$ into the product of two terms

$$
D_2(g) = \underbrace{\sqrt{n} \left(\widehat{\gamma} - \gamma\right)'}_{O_p(1)} \frac{1}{n} \sum_i g(X_i) \left(\Delta b(V_i, D_i) - \alpha(V_i, D_i) \cdot b(V_i, D_i)\right)
$$

Uniform convergence of the sample average again follows from Lemma B.4. We thus conclude $D_1(g)$ and $D_2(g)$ vanish uniformly over $g \in \mathcal{G}$ and over $P \in \mathcal{P}$.

For $D_3(g)$, we apply the Cauchy-Schwarz inequality to note that

$$
D_3(g) \leq \sqrt{n} \cdot \sqrt{\frac{1}{n} \sum_i \left(\widehat{\gamma}^Y(V_i, D_i) - \gamma^Y(V_i, D_i)\right)^2} \cdot \sqrt{\frac{1}{n} \sum_i \left(\widehat{\alpha}(V_i, D_i) - \alpha(V_i, D_i)\right)^2}
$$

The terms in the square root are the in-sample L_2 errors from the estimated CEF and propensity score, which vanish at the rate of $n^{-1/2}$ uniformly over $P \in \mathcal{P}$ as shown in the paragraph above. We thus conclude $D_3(g)$ vanishes uniformly over $g \in \mathcal{G}$ and over $P \in \mathcal{P}$. □