INFORMATION SPILLOVER IN MARKETS WITH HETEROGENEOUS TRADERS

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ABSTRACT. This paper studies the welfare impact of information spillover in divisible-good markets with heterogeneous traders and interdependent values. In a setting in which two groups of traders trade two distinct but correlated assets, one within each group, the information content in the price of one asset spillovers to the other market. Some "informed" traders who submit demand schedules may condition their demands on the prices of both assets, while others not. We prove the existence of a linear equilibrium and examine how information spillover affects trading, information efficiency, and welfare, as the fraction of the informed traders varies. In the two symmetric benchmarks, full information spillover (all traders are informed) dominates no information spillover (all traders are uninformed) in terms of trading volume and welfare. However, in markets with heterogeneous traders, information spillover can hurt overall welfare, while still improving information efficiency and liquidity; we characterize the non-monotone impact of information spillover on aggregate welfare in large finite markets. Furthermore, information spillover can account for the empirical evidence of excessive price co-movement and volatility transmission in financial markets.

KEYWORDS: information spillover, strategic trading, interdependent values

1. INTRODUCTION

It is well-known that markets are interconnected. For instance, in financial markets, trading activities of assets in one market can be informative about the values of assets in other markets. Savvy traders operating in one market will likely use information from other markets to help guide their trading decisions, amplifying underlying correlation among markets and volatility of the impact of economic fundamentals, with potential substantial welfare consequences for all traders. More concretely, during the financial

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crises in the late '90s and 2008, drastic movements in one national stock market had significant impacts on the stock markets across the world (see King et al. (1994), Forbes and Rigobon (2002), Diebold and Yilmaz (2009)). Likewise, several studies have documented the "excess comovement" of asset prices relative to the fundamentals (see Pindyck and Rotemberg (1993), Barberis et al. (2005), Veldkamp (2006)). Similar patterns and issues appear in many markets, such as real estate, petroleum, electricity, etc. Despite the significance of information spillover, there is relatively little study of its impact on strategic trading. How do strategic traders react to information spillover? For those who do not directly observe or take into account prices in other markets, how are their trading activities indirectly affected? Does information spillover enhance the information efficiency of prices? What determines traders' equilibrium surplus? Would a policy that encourages more traders to take advantage of information spillover necessarily improve welfare?

This paper develops a theoretical framework to examine these questions. In our model, there are two markets and two assets, one in each market. [1] Traders' values are correlated both within a market and across markets. Traders in each market observe noisy signals about their values and compete in demand schedules of the asset and the equilibrium asset prices are determined by market clearing conditions jointly. Following the strategic trading literature, traders' payoffs are linear in their values net off a quadratic cost, and all traders' asset values and signals are jointly normally distributed. To model information spillover, we assume that some traders in a market, who are called "informed traders," can react to the price of the asset in the other market and submit demand schedules of an asset contingent on the prices of both assets and their private signals; the remaining traders, who are called "uninformed traders," can only express their demands of an asset as functions of its own price and their private signals.[2] The fraction of informed traders in a market, which governs the extent of endogenous information spillover into this market, is the main focus of our exercise.[3]

We first solve for the unique linear Bayes Nash equilibrium in closed form in two benchmarks: either all traders are informed or they are all uninformed.[4] The differences

[1]In the online appendix, we show that our results extend to the case with more than two markets and assets.

[2]Different from insider trading models in the literature, the uninformed traders still observe private signals that are correlated with their values, so do the informed. The terms "informed" and "uninformed" in this paper refer to whether a trader has access to the price information from the other market, or more generally, some additional payoff-relevant information.

[3]When all traders in one market are uninformed, its asset price is independent of tradings in the other market, although it is still an informative signal to the informed traders in the other market. Thus, our model covers exogenous information spillover as a special case.

[4]The online appendix contains another special case with closed-form solutions where traders in one market are all informed and all traders in the other market are uninformed.

between these two equilibria illustrate the impact of information spillover, absence of heterogeneity. Notably, all traders in a market benefit from information spillover solely because the information from the asset price in the other market *lessens* competition. Intuitively, because this price is public, no trader has any informational advantage. Instead, it alleviates adverse selection faced by traders so that they are more willing to trade, which in turn lowers each trader's price impact (i.e., how much a marginal increase in her demand would change the price) and hence increases her expected surplus.

In general, trader heterogeneity, despite a realistic feature in many real-world markets, confounds the information content of prices in the presence of information spillover. In particular, equilibrium prices no longer reveal the average signals of all traders in a market, which is a key property that holds in the symmetric benchmarks. Furthermore, a traders' strategic impact depends on the composition of traders in the markets. Consequently, there are no off-the-shelf results that guarantee the existence of linear Bayes Nash equilibria for the general case. In addition, the equilibria are not in closed forms even when they exist. The main contributions of this paper are to establish the existence of an asymmetric linear equilibrium and to characterize the equilibrium and its comparative statics with respect to information spillover in large but finite markets.

For equilibrium existence, to deal with the unbounded strategy spaces and the large set of equilibrium coefficients, we first identify four bounded parameters, two in each market, that measure the weighted heterogeneity in bidders' response to their own signals and the asset price in this market, respectively. We show that these parameters uniquely pin down each trader's conditional expectation about her value as a linear function of her signal and prices; moreover, the inference coefficients are bounded under a joint restriction on the number of traders and the noise in the signal, from which we solve for traders' optimal strategies and hence the four parameters in the beginning. Applying Brouwer's fixed point theorem to this mapping of the parameters delivers existence.[5]

Because the equilibria are not in closed forms in all but the benchmark cases, we first consider the large-market limit as the market size in both markets grows to infinity and then expand the equilibrium system with respect to the total number of traders to characterize traders' inference, behavior, and welfare in large but finite markets and their comparative statics with respect to information spillover. Specifically, the limiting equilibrium is unique and in closed-form, as the numbers of traders in both markets go to infinity, keeping the fraction of informed traders in each market constant. Furthermore,

[5]The numerical exercises presented after the existence result suggest that the equilibrium is unique for a wide range of primitives, although we do not have a formal proof of uniqueness.

the limiting equilibrium is independent of the fraction of informed traders. However, both the price impact of a single trader and her inference from information spillover vanish in the large market limit in our main setting. The former follows from the fact that a single trader's demand becomes negligible in large markets and the latter is a result of the relative informativeness of signals within a market or across markets in predicting a traders' value.[6] We also provide a series of numerical examples to illustrate how the equilibrium outcomes vary with the number of traders.

To further examine analytically both the informational and strategic impacts of information spillover, we expand the equilibrium coefficients around their limits and focus on terms that are of order $1/N$ and $1/N^2$, where N is the market size. Remarkably, all these first and second order effects, which also pin down the speed of convergence, are unique and in closed form. Most importantly, with these approximations, we can compare how the informed and uninformed are differentially impacted by information spillover and characterize its comparative statics in large but finite markets.

Regarding traders' inference and behavior, we show that information spillover affects traders' marginal revenues through the inference of their values and their marginal costs through the equilibrium price impacts. For an informed trader, the information content of the asset price in the other market has a first-order effect. This direct "spillover channel" benefits more to the informed than the uninformed. Intuitively, the extra information alleviates the informed traders' adverse selection problem so that their beliefs are less responsive (first-order effect) to changes in their own asset prices and more sensitive (second-order effect) to their own signals than the uninformed. Regarding price impacts, similar to the benchmarks, because the informed are more willing to trade with less severe adverse selection, all traders' price impacts in equilibrium are lower with information spillover. Moreover, the price impacts are decreasing in the fraction of informed in a market, albeit only to a second-order effect.

Based on these characterizations, we obtain three main sets of results on traders' welfare and, at the market level, information efficiency, liquidity, price co-movement, and volatility transmission. First, the fraction of informed traders has a non-monotonic impact on aggregate welfare. Based on the large-market approximation for the expected surplus, we show that a trader's welfare increases if either (i) she relies more on her own signal ("own signal effect"), or (ii) the two types of traders react more differently to prices so as to create more trade between them ("heterogeneous beliefs effect"), or (iii) her price

[6] In the online appendix, we also consider extensions with market-specific systematic risks or supply shocks in which information spillover does not vanish in the large market limit.

impact is smaller. Among these three effects, it turns out that only the first two effects, both of which are second-order, vary with respect to the fraction of informed traders; the term containing price impact in the welfare approximation is independent of the fraction of informed traders up to the second-order. Therefore, the welfare comparative statics is solely driven by the traders' beliefs and inference in large but finite markets.

For the informed, whose expected surplus is always larger than the uninformed, more information spillover hurts their welfare. As an important step toward this result, we show that when the market sizes are large enough, informed traders almost perfectly infer the average signals in both markets from the two market prices, independent of the fraction of informed. Consequently, changes in information spillover do not affect the informativeness of prices for the informed. Thus, only the "heterogeneous beliefs effect" matters for an informed trader: the gain from trading with the uninformed shrinks and so does her expected surplus, as the fraction of the informed increases. For the uninformed, their surplus is U-shaped because of two opposing forces. Suppose there are more informed traders in a market, on the one hand, the price contains more information from the other market, which may crowd out the informativeness of her private signal (a smaller "own signal effect"), which hurts the uninformed; on the other hand, competition among the informed gets stronger (a larger "heterogeneous beliefs effect"), which benefits the uninformed. Overall, as more traders become informed, the weighted average surplus also turns out to be U-shaped as a function of information spillover. An immediate policy implication is that an increase in transparency can have negative welfare consequences in markets with adverse selection and heterogeneously informed traders.

Second, information spillover improves both information efficiency, measured by variance reduction, and liquidity. When there are more informed traders, the variance of an uninformed trader's conditional expectation of her value is always smaller; the variance of the informed, on the other hand, remains the same up to second-order approximation. That is, more informed traders improve overall learning and lower the information advantage of informed traders. In fact, we can rewrite a trader's expected surplus (up to second-order approximation) as the difference between information efficiency and the variance of her own asset price. Because information spillover leads to larger price variance, this explains the gap between information efficiency and allocation efficiency in our strategic setting. For liquidity, we show that the expectation of trading volume of a trader can be rewritten (again up to second-order approximation) as expected surplus multiplying another term related to price impact, which has a second-order effect. However, unlike for information efficiency, the price impact now plays a dominant role.

Finally, our findings are consistent with the empirical evidence of excessive price comovements and volatility transmissions mentioned in the beginning. In particular, the correlation in prices is larger than the correlation of average signals across markets, and when there is an exogenous shock to the average signals in one market, the price variance in the other market also goes up; both effects are increasing in information spillover.

This paper is related to the large literature on divisible-good double auctions with interdependent values and strategic trading in imperfectly competitive markets. Rostek and Yoon (2020) give a contemporary and comprehensive survey of the literature. Starting from the seminal work by Wilson (1979) that considers the symmetric pure common value case, subsequent papers, Vives (2011), Rostek and Weretka (2012), Ausubel et al. (2014), Vives (2014), and Du and Zhu (2017) for example, extend the analysis to general interdependent value settings with symmetric equilibria. More recent contributions by Malamud and Rostek (2017), Rostek and Yoon (2021), Rostek and Yoon (2021), Chen and Duffie (2021), Wittwer (2021), and Rostek and Wu (2021) take the market design perspective to analyze the impact of trading technologies (e.g. the arrangement of trading venues, market fragmentation, joint or independent market-clearing across venues, synthetic assets, access to information) in multi-asset markets with either private or interdependent values. An important insight is that, when traders are strategic and can trade in multiple markets, the change in traders' price impact in decentralized markets can outweigh the loss of information or market depth and hence improve overall welfare relative to centralized markets. In comparison, to isolate the cross-market information externality, our main model assumes that each trader only trades in one market so that her payoff does not directly depend on the allocations across markets and analyzes how information spillover, either exogenous or endogenous, affects her price impact and welfare.[7]

Trader heterogeneity in the presence of information spillover is another substantial departure of our paper from the literature. Different from the insider trading models initiated by Kyle (1989) in which some traders are privately informed and the rest has no private information, in our model all traders are privately informed and, in addition, a fraction of traders in each market learns some additional public signal, such as the price in the other market when information spillover is endogenous. A recent contribution by Manzano and Vives (2021) also studies asymmetric divisible good auctions with two types of traders, who also provide an extensive and insightful discussion on the relevance

[7] The online appendix contains an analysis of the case in which all traders trade in both markets.

of trader heterogeneity in many real-world markets.[8] Manzano and Vives (2021) overcome the potentially complex inference problem under heterogeneous information by studying a setting in which all traders of the same type observe a common type-specific signal so that the market price perfectly aggregates information when there are two types of traders; they also consider trader heterogeneity in other dimensions, such as the marginal cost. In contrast, prices in our model do not perfectly aggregate information and different types of traders hold distinct beliefs; we deal with this non-degenerate adverse selection by exploiting the asymptotic symmetry in large markets as explained in previous paragraphs. In addition, our focus on the comparative statics of information spillover is specific to our setting and differs from the questions addressed in Manzano and Vives (2021). Our work thus complements Manzano and Vives (2021), in both the economic insights and technical methods, and echoes their advocate for further investigations of markets with heterogeneous traders.

The rest of the paper is organized as follows. Section 2 presents the setting. Section 3 analyzes two symmetric benchmarks. Section 4 establishes equilibrium existence in the general case. Section 5 examines the equilibrium in large finite markets. Section 6 discusses the modeling assumptions and extensions. The proofs of the results are relegated to Appendices A–C. The online appendix contains further details of the extensions.

2. MODEL

Consider a setting with two risky assets, $k \in \{I, II\}$, traded in two separate markets. For each $k \in \{I, II\}$, there are $n_k \in \mathbb{N}_+$ traders who trade asset k in market k, and the set of traders in market k is denoted by $\mathcal{N}_k$. For each trader $i \in \mathcal{N}_k$, the per-unit value of asset k to her is θ_k^i. The vector of all traders' values, $(\theta_I^i, \theta_{II}^j)_{i \in \mathcal{N}_I, j \in \mathcal{N}_{II}}$, is jointly normally distributed with a zero mean vector (as a normalization). The variance of θ_k^i is $\sigma_{\theta_k}^2 > 0$. The covariances satisfy: $\text{Cov}(\theta_k^i, \theta_k^j) = \sigma_{\theta_k}^2 \rho_k > 0$, for all $i, j \in \mathcal{N}_k$, and $\text{Cov}(\theta_k^i, \theta_{-k}^j) = \sigma_{\theta_k} \sigma_{\theta_{-k}} \phi \in \mathbb{R}$, for all $i \in \mathcal{N}_k$, $j \in \mathcal{N}_{-k}$, $k, -k \in \{I, II\}$ with $k \neq -k$. We assume $1 > \rho_k \geq |\phi| > 0$.[9]

For each $k \in \{I, II\}$, trader $i \in \mathcal{N}_k$ privately observes a noisy signal $s_k^i = \theta_k^i + \varepsilon_k^i$ about her value. The noise ε_k^i is normally distributed with mean zero and variance $\sigma_{\varepsilon_k}^2$ and independent across all traders i and assets k. Let $\sigma_k^2 = \sigma_{\varepsilon_k}^2 / \sigma_{\theta_k}^2$ be the variance ratio measuring the impact of noise relative to the value in market k. The noises $(\varepsilon_I^i, \varepsilon_{II}^j)_{i \in \mathcal{N}_I, j \in \mathcal{N}_{II}}$ and the values $(\theta_I^i, \theta_{II}^j)_{i \in \mathcal{N}_I, j \in \mathcal{N}_{II}}$ are independent.

[8] Another recent paper by Andreyanov and Sadzik (2021) also studies exchanges with trader heterogeneity from a robust mechanism design perspective.

[9] The condition $\rho_k \geq |\phi|$ is necessary for the covariance matrix of all signals to be positive semi-definite.

The initial endowment of each trader $i \in \mathcal{N}_k$ is normalized to zero. The payoff of trader $i \in \mathcal{N}_k$ from trading x_k^i units of asset k at a price $p_k \in \mathbb{R}$ is

$$u_k^i(x_k^i, p_k, \theta_k^i) = (\theta_k^i - p_k) \cdot x_k^i - \frac{\gamma}{2}\left(x_k^i\right)^2,$$

which is linear in her value of the asset θ_k^i net off the asset price p_k and has a quadratic inventory cost, where $\gamma > 0$ is a commonly known constant.

Traders at each market submit net demand schedules for the corresponding asset and the equilibrium prices of the assets are simultaneously determined by the market-clearing conditions at both markets. To examine the impact of information spillover from one market to the other, we assume that there are two types of traders at each market, depending on whether their demand can be contingent on the price of the other asset. Specifically, for each $k \in \{I, II\}$, let $\mathcal{N}_k^1$ be the set of *informed* traders in market k, who can submit demand schedules depending on the prices of the assets in both markets. That is, for each $i \in \mathcal{N}_k^1$, trader i submits a demand function $x_k^i : \mathbb{R}^2 \to \mathbb{R}$ such that $x_k^i(p_k, p_{-k}) \in \mathbb{R}$ specifies the quantity of asset k trader i demands for any price vector (p_k, p_{-k}). Let $\mathcal{N}_k^0 = \mathcal{N}_k \setminus \mathcal{N}_k^1$ be the set of *uninformed* traders in market k, who submit demand schedules only as a function of the price of asset k. That is, for each $i' \in \mathcal{N}_k^0$, trader i' submits a demand function $x_k^{i'} : \mathbb{R} \to \mathbb{R}$ specifying the quantity demanded $x_k^{i'}(p_k)$ of asset k for any price $p_k \in \mathbb{R}$. A trader is a buyer if her demand is positive or a seller if her demand is negative. Let $n_k^1 = |\mathcal{N}_k^1|$ and $n_k^0 = |\mathcal{N}_k^0| = n_k - n_k^1$ be the numbers of informed and uninformed traders, respectively. Denote by $\alpha_k = n_k^1/n_k \in [0,1]$ the fraction of informed traders in market k. Given the submitted demand schedules in both markets, $(x_I^i(p_I, p_{II}), x_I^{i'}(p_I))_{i \in \mathcal{N}_I^1, i' \in \mathcal{N}_I^0}$ and $(x_{II}^j(p_{II}, p_I), x_{II}^{j'}(p_{II}))_{j \in \mathcal{N}_{II}^1, j' \in \mathcal{N}_{II}^0}$, the equilibrium price vector $(p_I^*, p_{II}^*) \in \mathbb{R}^2$ is determined by the two market-clearing conditions:

$$\sum_{i \in \mathcal{N}_I^1} x_I^i(p_I^*, p_{II}^*) + \sum_{i' \in \mathcal{N}_I^0} x_I^{i'}(p_I^*) = 0, \quad \text{and} \quad \sum_{j \in \mathcal{N}_{II}^1} x_{II}^j(p_{II}^*, p_I^*) + \sum_{j' \in \mathcal{N}_{II}^0} x_{II}^{j'}(p_{II}^*) = 0.$$

For each $k \in \{I, II\}$, an informed trader $i \in \mathcal{N}_k^1$ receives $x_k^i(p_k^*, p_{-k}^*)$ units of asset k and pays $p_k^* x_k^i(p_k^*, p_{-k}^*)$, and an uninformed trader $i' \in \mathcal{N}_k^0$ is allocated $x_k^{i'}(p_k^*)$ units of asset k and pays $p_k^* x_k^{i'}(p_k^*)$.

We adopt linear Bayes Nash equilibrium as the solution concept. A strategy of an informed trader $i \in \mathcal{N}_k^1$ is a mapping $x_k^i(p_k, p_{-k}, s_k^i)$ from her realized signal s_k^i to a demand schedule for asset k contingent on (p_k, p_{-k}). A strategy of an uninformed trader $i' \in \mathcal{N}_k^0$ is a mapping $x_k^{i'}(p_k, s_k^{i'})$ from her realized signal $s_k^{i'}$ to a demand schedule for asset k contingent on p_k. A *Bayes Nash equilibrium* is a strategy profile $(x_k^i, x_k^{i'})_{k \in \{I, II\}, i \in \mathcal{N}_k^1, i' \in \mathcal{N}_k^0}$ such that for each trader $i \in \mathcal{N}_k^1$ and signal s_k^i, the demand schedule maximizes i's expected payoff:

$$\mathbb{E}\left[u_k^i(x_k^i(p_k^*, p_{-k}^*, s_k^i), p_k^*, \theta_k^i)\middle| s_k^i, p_k^*, p_{-k}^*\right] \geq \mathbb{E}\left[u_k^i(\tilde{x}_k^i(\tilde{p}_k, \tilde{p}_{-k}, s_k^i), \tilde{p}_k, \theta_k^i)\middle| s_k^i, \tilde{p}_k, \tilde{p}_{-k}\right],$$

where (p_k^*, p_{-k}^*) is the market-clearing price vector given x_k^i and all other traders' equilibrium strategies and $(\tilde{p}_k, \tilde{p}_{-k})$ is the market-clearing price vector given any strategy $\tilde{x}_k^i$ and all other traders' equilibrium strategies, and for each $i' \in \mathcal{N}_k^0$ and $s_k^{i'}$,

$$\mathbb{E}\left[u_k^{i'}(x_k^{i'}(p_k^*, s_k^{i'}), p_k^*, \theta_k^{i'})\middle| s_k^{i'}, p_k^*\right] \geq \mathbb{E}\left[u_k^{i'}(\tilde{x}_k^{i'}(\tilde{p}_k, s_k^{i'}), \tilde{p}_k, \theta_k^{i'})\middle| s_k^{i'}, \tilde{p}_k\right],$$

where p_k^* is asset k's market-clearing price given $x_k^{i'}$ and all other traders' equilibrium strategies and $\tilde{p}_k$ is asset k's market-clearing price given any strategy $\tilde{x}_k^{i'}$ and all other traders' equilibrium strategies. A Bayes Nash equilibrium is *linear* if all traders' equilibrium strategies are linear functions. Since traders in each of the four subgroups ($\mathcal{N}_I^1$, $\mathcal{N}_I^0$, $\mathcal{N}_{II}^1$, and $\mathcal{N}_{II}^0$) are ex ante symmetric, we further restrict attention to linear Bayes Nash equilibria that are symmetric within each subgroup.

3. BENCHMARKS

To illustrate the spillover effect of asset prices, we first solve for the closed-form equilibria in two benchmark cases. To simplify notations, here we assume the two markets are symmetric in the sense that $n_I = n_{II} = n$, $\rho_I = \rho_{II} = \rho$, and $\sigma_I^2 = \sigma_{II}^2 = \sigma^2$.

3.1. **Information Spillover** ($\alpha_I = \alpha_{II} = 1$)**.** Suppose all traders can condition their demands on the prices of both assets. Consider the symmetric linear demand:

$$x_k^i(p_k, s_k^i) = a_k^1 s_k^i - B_k^1 p_k + b_k^1 p_{-k}, \tag{1}$$

where a_k^1, B_k^1, and b_k^1 are coefficients. The equilibrium is derived as follows:

(i) Trader i's first order condition is

$$\mathbb{E}\left[\theta_k^i | s_k^i, p_k, p_{-k}\right] - p_k = (\gamma + \lambda_k^1) x_k^i, \tag{2}$$

where left-hand side is the marginal revenue, and the right-hand side represents the marginal cost, which contains a term $\lambda_k^1 = dp_k/dx_k^i$, i.e., the price impact in market k, measuring how the equilibrium price reacts to a change in trader i's demand.

(ii) The price impact λ_k^1. In the two market-clearing conditions, we take derivative with respect to x_k^i to obtain

$$1 + \sum_{j \neq i, j \in \mathcal{N}_k^1} \left[\frac{\partial x_k^j}{\partial p_k}\frac{dp_k}{dx_k^i} + \frac{\partial x_k^j}{\partial p_{-k}}\frac{dp_{-k}}{dx_k^i}\right] = 0$$

and

$$\sum_{l \in \mathcal{N}^1_{-k}} \left[\frac{\partial x^l_{-k}}{\partial p_{-k}} \frac{dp_{-k}}{dx^i_k} + \frac{\partial x^l_{-k}}{\partial p_k} \frac{dp_k}{dx^i_k} \right] = 0,$$

from which we solve for the price impact:

(3)

$$\lambda^1_k = \frac{dp_k}{dx^i_k} = -\left\{ \sum_{j \neq i, j \in \mathcal{N}^1_k} \left[\frac{\partial x^j_k}{\partial p_k} - \frac{\partial x^j_k}{\partial p_{-k}} \left(\frac{\sum_{l \in \mathcal{N}^1_{-k}} \frac{\partial x^l_{-k}}{\partial p_k}}{\sum_{l \in \mathcal{N}^1_{-k}} \frac{\partial x^l_{-k}}{\partial p_{-k}}} \right) \right] \right\}^{-1} = \frac{B^1_{-k}}{(n-1)\left(B^1_k B^1_{-k} - b^1_k b^1_{-k} \right)},$$

where the second equality follows from the conjectured linear strategy profile in (1).

(iii) Trader i's inference: $\mathbb{E}\left[\theta^i_k | s^i_k, p_k, p_{-k}\right]$. The market-clearing condition for asset k implies that

$$\bar{s}^1_k = \frac{B^1_k p_k - b^1_k p_{-k}}{a^1_k}, \tag{4}$$

where $\bar{s}^1_k = (\sum_{i \in \mathcal{N}^1_k} s^i_k)/n$ is the average signal of all (informed) traders in market k. Therefore, (p_I, p_{II}) is a linear combination of $(\bar{s}_I, \bar{s}_{II})$ and hence is also jointly normal distributed. The projection theorem then implies that

$$\mathbb{E}\left[\theta^i_k | s^i_k, p_k, p_{-k}\right] = C^1_s s^i_k + \left(\frac{C^1 B^1_k}{a^1_k} - \frac{C^1_- b^1_{-k}}{a^1_{-k}} \right) p_k + \left(\frac{C^1_- B^1_{-k}}{a^1_{-k}} - \frac{C^1 b^1_k}{a^1_k} \right) p_{-k}, \tag{5}$$

where

$$C^1_s = \frac{1-\rho}{1-\rho+\sigma^2},\ C^1 = \frac{\sigma^2}{1-\rho+\sigma^2} \cdot \frac{\rho^2 + \frac{\rho(1-\rho+\sigma^2)}{n} - \phi^2}{\left(\rho + \frac{1-\rho+\sigma^2}{n}\right)^2 - \phi^2},\ C^1_- = \frac{\sigma^2 \cdot \frac{\phi}{n}}{\left(\rho + \frac{1-\rho+\sigma^2}{n}\right)^2 - \phi^2}.$$

(iv) Substituting (5) and (3) into (2) and matching coefficients, we get

(6)

$$a^1_k \equiv a^1 = \frac{(n-2)C^1_s - C^1}{\gamma(n-1)},\ B^1_k \equiv B^1 = \frac{(C^1_s + C^1)a^1}{(C^1_s + C^1)^2 - (C^1_-)^2},\ b^1_k \equiv b^1 = \frac{C^1_- a^1}{(C^1_s + C^1)^2 - (C^1_-)^2}.$$

The above strategy profile is an equilibrium if and only if $a^1 > 0$. A simple sufficient condition for $a^1 > 0$ is $(1-\rho)(n-2) > \sigma^2$. Furthermore, we have the following result regarding the equilibrium coefficients.

Proposition 3.1. *If $\alpha_I = \alpha_{II} = 1$ and $(1-\rho)(n-2) > \sigma^2$, the equilibrium is given by (6) and the parameters satisfy:*

(i) $b^1 > 0$ if and only if $\phi > 0$;
(ii) $0 < a^1 < \frac{C^1_s}{\gamma}\frac{n-2}{n-1}$ and $0 < |b^1| < B^1 < \frac{1}{\gamma}\frac{n-2}{n-1}$;
(iii) a^1, B^1, and $|b^1|$ are increasing in $|\phi|$.

3.2. **No Spillover** ($\alpha_I = \alpha_{II} = 0$)**.** Now suppose no trader can condition her demand on the price of the other asset, thus there is no information spillover and the market-clearing prices are determined separately. Following the steps in the previous benchmark, the equilibrium in market $k \in \{I, II\}$ with no spillover is given by

$$x_k^i(p_k, s_k^i) = a_k^0 s_k^i - B_k^0 p_k, \tag{7}$$

where

$$a_k^0 \equiv a^0 = \frac{(n-2)C_s^0 - C^0}{\gamma(n-1)}, \quad B_k^0 \equiv B^0 = \frac{a^0}{C_s^0 + C^0}, \tag{8}$$

$$C_s^0 = \frac{1-\rho}{1-\rho+\sigma^2}, \quad C^0 = \frac{\sigma^2}{1-\rho+\sigma^2} \cdot \frac{\rho}{\rho + \frac{1-\rho+\sigma^2}{n}}. \tag{9}$$

Moreover, trader i's price impact in market k is

$$\lambda_k^0 = \frac{dp_k}{dx_k^i} = \frac{1}{(n-1)B_k^0}. \tag{10}$$

The above strategy profile is an equilibrium if and only if $a^0 > 0$. Again, a sufficient condition is $(1-\rho)(n-2) > \sigma^2$. The result is summarized in the next proposition.

Proposition 3.2. *If $\alpha_I = \alpha_{II} = 0$ and $(1-\rho)(n-2) > \sigma^2$, the equilibrium is given by (8) and the parameters satisfy:*

(i) $0 < a^0 < \frac{C_s^0}{\gamma}\frac{n-2}{n-1}$ *and* $0 < B^0 < \frac{1}{\gamma}\frac{n-2}{n-1}$*;*

(ii) a^0 *and* B^0 *are independent of* $|\phi|$*.*

3.3. **Comparison.** This section compares traders' inference, behavior, and welfare. Corollary 3.3 shows that traders put the same weight on their own signals in both benchmarks and the uninformed place a higher weight on the price of her asset than the informed.

Corollary 3.3. *The equilibrium inference parameters satisfy $C_s^0 = C_s^1$, $C^0 > C^1$.*

Since traders' inferences are directly related to their equilibrium strategies, the next result (Corollary 3.4) compares traders' demand schedules.

Corollary 3.4. *The informed traders' demand schedules, compared to those of the uninformed, are more sensitive to their own price, i.e., $B_k^1 > B_k^0$ and more sensitive to their own signals, i.e., $a_k^1 > a_k^0$. The price impact under information spillover is lower than that under no spillover: $\lambda_k^1 < \lambda_k^0$.*

Intuitively, with a lower price p_k, an informed trader in market k is less pessimistic about the quality of the asset and thus demands more than an uninformed trader, since an informed trader's inference is less sensitive to the own price ($C^1 < C^0$). Therefore, the equilibrium market demand under information spillover is more responsive to price ($B^1_k > B^0_k$). Consequently, the equilibrium price is less sensitive to a change of trade, i.e., ($\lambda^1_k < \lambda^0_k$). As an immediate implication, informed traders are more willing to rely on their own signals ($a^1_k > a^0_k$), since they have smaller price impacts.

Finally, we compare traders' equilibrium payoffs. Denote by W^1_k (resp., W^0_k) a trader's expected equilibrium payoff in the benchmark with (resp., without) information spillover. Proposition 3.5 establishes that a trader's expected payoff is strictly decreasing in her price impact, and informed traders' payoffs are higher since information spillover lowers all traders' price impacts.

Proposition 3.5. *Traders' expected surpluses in the benchmarks are given by*

$$W^1_k = \frac{\frac{\gamma}{2} + \lambda^1_k}{(\gamma + \lambda^1_k)^2}(C^1_s)^2 \mathbb{E}(s^i_k - \bar{s}_k)^2 \quad \textit{and} \quad W^0_k = \frac{\frac{\gamma}{2} + \lambda^0_k}{(\gamma + \lambda^0_k)^2}(C^0_s)^2 \mathbb{E}(s^i_k - \bar{s}_k)^2,$$

where $W^1_k > W^0_k$.

4. The General Case

This section establishes the existence and characterization of equilibria in the general setting in which there can be both informed and uninformed traders in each market. We first characterize both types of traders' equilibrium strategy and inference, and then apply Brouwer's fixed point theorem to prove existence.

For each $k \in \{I, II\}$, we conjecture a linear demand schedule for any uninformed trader $i' \in \mathcal{N}^0_k$ in market k as

$$x^0_{k,i}(p_k, s^i_k) = a^0_k s^i_k - B^0_k p_k,$$

and another linear demand schedule for any informed trader $i \in \mathcal{N}^1_k$ in market k as

$$x^1_{k,i}(p_k, p_{-k}, s^i_k) = a^1_k s^i_k - B^1_k p_k + b^1_k p_{-k},$$

where a^0_k, B^0_k, a^1_k, B^1_k, and b^1_k are constants. Recall that $\alpha_k = n^1_k / n_k$ is the fraction of informed traders in market k. Let $B_k = \alpha_k B^1_k + (1 - \alpha_k) B^0_k$ and $b_k = \alpha_k b^1_k$ be the aggregate sensitivities of demand in market k to p_k and p_{-k}, respectively. Lemma 4.1 characterizes the equilibrium demand schedules.

Lemma 4.1. *The equilibrium demand schedule of an informed trader* $i \in \mathcal{N}^1_k$ *is*

$$x^1_{k,i}(p_k, p_{-k}, s^i_k) = \frac{1}{\gamma + \lambda^1_k}\left(\mathbb{E}\left[\theta^i_k | s^i_k, p_k, p_{-k}\right] - p_k\right), \tag{11}$$

where

$$\lambda_k^1 = \left((\alpha_k n_k - 1)(B_k^1 - b_k^1 \frac{b_{-k}}{B_{-k}}) + (1-\alpha_k) n_k B_k^0 \right)^{-1}.$$

The equilibrium demand schedule of an uninformed trader $i \in \mathcal{N}_k^0$ is

$$x_{k,i}^0(p_k, s_k^i) = \frac{1}{\gamma + \lambda_k^0} \left(\mathbb{E}\left[\theta_k^i | s_k^i, p_k \right] - p_k \right), \tag{12}$$

where

$$\lambda_k^0 = \left(\alpha_k n_k (B_k^1 - b_k^1 \frac{b_{-k}}{B_{-k}}) + ((1-\alpha_k) n_k - 1) B_k^0 \right)^{-1}.$$

Lemma 4.1 shows that a trader's demand is increasing in the conditional expectation of her value, and is decreasing in the price impact λ_k^1 (or λ_k^0). The next Lemma (Lemma 4.2) characterizes the market-clearing prices in terms of traders' signals, given the conjectured strategies of all traders. It establishes that the price p_k is a linear combination of the average signals of traders in different subgroups; as a result, the price vector (p_I, p_{II}) is also jointly normally distributed.

Lemma 4.2. *The market clearing price of asset $k \in \{I, II\}$ is*

$$p_k = D_k^1 \bar{s}_k^1 + D_k^0 \bar{s}_k^0 + d_k^1 \bar{s}_{-k}^1 + d_k^0 \bar{s}_{-k}^0, \tag{13}$$

where $\bar{s}_k^1 = \sum_{i \in \mathcal{N}_k^1} s_k^i / n_k^1$, $\bar{s}_k^0 = \sum_{i' \in \mathcal{N}_k^0} s_k^{i'} / n_k^0$, $k, -k \in \{I, II\}, k \neq -k$, and

$$D_k^1 = \frac{B_{-k} \alpha_k a_k^1}{B_k B_{-k} - b_k b_{-k}},\ d_k^1 = \frac{b_k \alpha_{-k} a_{-k}^1}{B_k B_{-k} - b_k b_{-k}}, D_k^0 = \frac{B_{-k}(1-\alpha_k) a_k^0}{B_k B_{-k} - b_k b_{-k}},\ d_k^0 = \frac{b_k (1-\alpha_{-k}) a_{-k}^0}{B_k B_{-k} - b_k b_{-k}}.$$

Next we examine traders' conditional expectations of their values. Because of trader heterogeneity, there are two types of learning: "cross assets" and "cross subgroups." Unlike the benchmark cases, traders' conditional expectations of their values and equilibrium strategies do not have closed-form solutions due to the asymmetry. To facilitate the economic interpretations of the characterization and to establish equilibrium existence, we introduce the following parameters that capture the impact of signals on market-clearing prices:

- The price impact of own-asset signals, ζ_k:

$$\zeta_k \equiv D_k^1 + D_k^0. \tag{14}$$

- The price impact of cross-asset signals relative to own-asset signals, δ_k:

$$\delta_k \equiv \frac{d_k^1 + d_k^0}{D_k^1 + D_k^0}. \tag{15}$$

- The price impact of cross-subgroup signals within a market, π_k:

$$\pi_k \equiv \frac{D_k^1}{D_k^1 + D_k^0} = \frac{\alpha_k a_k^1}{\alpha_k a_k^1 + (1-\alpha_k) a_k^0}. \tag{16}$$

The next result (Lemma 4.3) expresses traders' conditional expectations as functions of the parameters $(\zeta_k,\ \delta_k, \pi_k)_{k\in\{I,II\}}$. Besides the associated economic interpretations, these parameters are technically convenient to handle, allowing us to establish tight bounds for the fixed point mapping in the existence proof.

Lemma 4.3. *Given the conjectured equilibrium strategies, the conditional expectation of the value of an informed trader $i \in \mathcal{N}_k^1$ is*

$$\mathbb{E}\left[\theta_k^i | s_k^i, p_k, p_{-k}\right] = C_{ks}^1 s_k^i + C_k^1 p_k + c_k^1 p_{-k}, \tag{17}$$

and the conditionally expected value of an uninformed trader $i \in \mathcal{N}_k^0$ is

$$\mathbb{E}\left[\theta_k^i | s_k^i, p_k\right] = C_{ks}^0 s_k^i + C_k^0 p_k, \tag{18}$$

where $(C_{ks}^1, C_k^1, c_k^1, C_{ks}^0, C_k^0)_{k\in\{I,II\}}$ are coefficients that depend on $(\delta_k, \zeta_k, \pi_k, \sigma_k^2, \rho_k, n_k, \phi)_{k\in\{I,II\}}$. The exact form of is given in equation (25) and (27) in Appendix B.

Now we state the first main results of the paper (Theorem 4.4) that establishes the existence of a linear equilibrium and demonstrate its properties.

Theorem 4.4. *There exists $\bar{\sigma}_k^2 \in (0, (1-\rho_k)(n_k-2))$ such that if $\sigma_k^2 < \bar{\sigma}_k^2$ for $k = I, II$,*[10] *there exists a linear Bayes Nash equilibrium such that*

(1) $a_k^1 > 0$, $B_k^1 > 0$, $|b_k^1| > 0$, $a_k^0 > 0$ and $B_k^0 > 0$.
(2) C_{ks}^1, $C_{ks}^0 \in (0,1)$, $C_k^1, C_k^0 \in [0, \frac{n_k-2}{n_k-1}]$.
(3) $b_k^1 > 0$ and $c_k^1 > 0$ if and only if $\phi > 0$.

Here we provide a sketch of proof for Theorem 4.4. First, we define two parameters to capture the heterogeneity between informed and uninformed traders:

$$\pi_k = \frac{\alpha_k a_k^1}{\alpha_k a_k^1 + (1-\alpha_k) a_k^0},\ \Pi_k = \frac{\alpha_k B_k^1}{\alpha_k B_k^1 + (1-\alpha_k) B_k^0}. \tag{19}$$

Then we construct a fixed point mapping as follows:

Step 1: The input of the mapping is $\pi_k \in [0,1]$ and $\Pi_k \in [0,1]$ for $k \in \{I, II\}$.

Step 2: Fix any $(\pi_k, \Pi_k)_{k\in\{I,II\}}$, we solve for the unique $(C_{ks}^1, C_k^1, c_k^1, C_{ks}^0, C_k^0)_{k\in\{I,II\}}$, as well as $(\delta_k, \zeta_k)_{k\in\{I,II\}}$.

[10]The condition $\sigma_k^2 < \bar{\sigma}_k^2$ holds if n_k is large enough or σ_k^2 is small enough.

Step 3: Given $(C^1_{ks}, C^1_k, c^1_k, C^0_{ks}, C^0_k)_{k\in\{I,II\}}$, we get the unique $(a^1_k, B^1_k, a^0_k, B^0_k)_{k\in\{I,II\}}$, which are all positive numbers, and also (b^1_I, b^1_{II}).

Step 4: Given $(a^1_k, B^1_k, a^0_k, B^0_k)_{k\in\{I,II\}}$, we obtain the output of the mapping $\pi_k \in [0,1]$ and $\Pi_k \in [0,1]$ for $k \in \{I, II\}$.

Along the proof of Theorem 4.4, we also identify the following properties regarding the inference parameters and equilibrium strategies. First, both types of traders respond positively to private signals ($a^1_k > 0$ and $a^0_k > 0$) and negatively to the own price ($B^1_k > 0$ and $B^0_k > 0$). This is guaranteed by the assumption that the noise of the private signal σ^2_k is small enough or there is a large number of trader n_k. Intuitively, if a trader's signal is very informative about her value, her demand would mostly rely on her signal as opposed to the prices. If instead it relies more on the information content of prices, then a low price, for example, would imply that the asset is less valuable and thus the trader would lower her demand, violating optimality. Likewise, if there are more traders, the price would be less sensitive to each individual trader's demand, thus the trader is more willing to submit a larger demand when the price is lower. Second, informed traders react to the price from the other market positively ($b^1_k > 0$ and $c^1_k > 0$) if and only if the two assets are positively correlated ($\phi > 0$). Under positive (negative) correlation, a higher price in other market serves a good (bad) news about the value of the asset in a trader's own market. Finally, the sensitivity of the own price for both types of traders is less than $\frac{n_k-2}{n_k-1}$ to prevent the own price from being so precise to crowd out the informativeness of private signals and make traders completely abandon their private signals.

4.1. **Numerical Exercises.** For fixed market sizes, the inference problems depend on the composition of traders in both markets, which complicates the characterizations of traders' best replies and thus the equilibrium demand schedules. Here we provide several numerical examples on the impact of how information spillover α_k (see Figures 1–3). From the mapping constructed in Theorem 4.4, we apply the fixed point iteration to get a unique numerical solution of (π_k, Π_k) and hence all the equilibrium parameters.[11]

First, the informed and the uninformed react differently to their own prices ($B^1_k - B^0_k \neq 0$), and their reactions to their own signals also differ but only mildly ($a^1_k - a^0_k \neq 0$). In addition, different from the benchmarks, it is *not* always the case that informed traders trade more aggressively, i.e., $a^1_k > a^0_k$ (see Figure 1), while it is still true that informed traders are more sensitive to prices ($B^1_k > B^0_k$). When n_k is small enough, an uninformed

[11] For simplicity, we focus on the symmetric case where $\alpha_I = \alpha_{II}$ and $n_I = n_{II} \equiv n$, where $n = 10, 50$. The inputs of the fixed point iteration are $\pi = \pi_I = \pi_{II}$ and $\Pi \equiv \Pi_I = \Pi_{II}$. It turns out that Π is the unique positive root of a cubic polynomial, hence we can further simplify the input to be π, which is one-dimensional and significantly simplifies the numerical analysis. See the details in the online appendix.

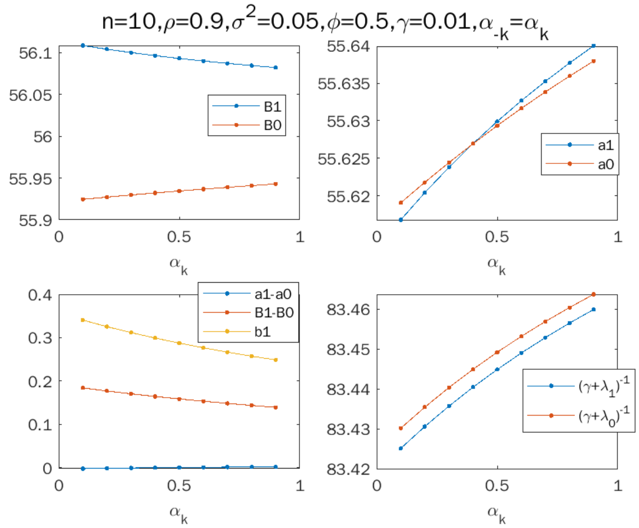

FIGURE 1. Equilibrium Outcome: $n = 10$

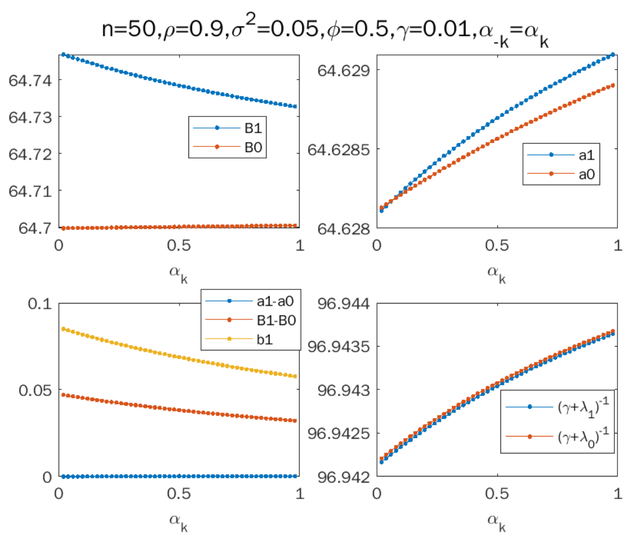

FIGURE 2. Equilibrium Outcome: $n = 50$

trader may trade more aggressively due to a free-riding effect. Since informed traders are more sensitive to prices, then a larger fraction of informed traders lowers the price impact, which benefits all traders in the market. Compared to the informed trader, an uninformed trader faces one more informed in the market, so she could benefit more from the smaller price impact of her residual demand. However, when the market size is large, this free-riding effect vanishes, since the impact of a single trader becomes insignificant, consequently, the informational advantage of an informed trader dominates: $a_k^1 > a_k^0$ (see Figure 2 and Proposition 5.2).

Second, information spillover (α_k) has monotonic impacts on the equilibrium beliefs and strategies. If there are more informed traders (a higher α_k), then market k becomes more competitive (lower price impacts λ_k^1 and λ_k^0); both types of traders trade more aggressively (higher a_k^1 and a_k^0); and informed traders are less responsive to both prices, relative to uninformed traders (lower $B_k^1 - B_k^0$ and b_k^1). Based on these observations, we characterize the equilibrium in large markets in the next section.

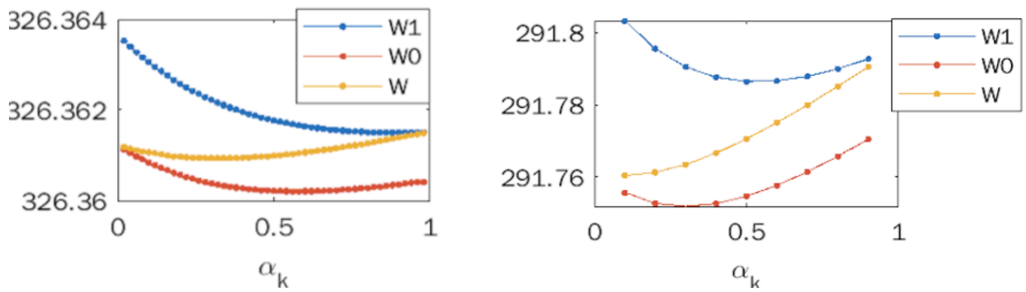

FIGURE 3. Welfare: $n = 50$ and $n = 10$

Finally, the impacts of information spillover (α_k) on welfare are non-monotonic (See Figure 3). To be specific, when the number of traders are large enough (see the left figure $n = 50$ in Figure 3), the welfare of informed traders (W_k^1) is decreasing in α_k, while the welfare of uninformed traders (W_k^0) and the aggregate welfare $W_k \equiv \alpha_k W_k^1 + (1 - \alpha_k) W_k^0$ are U-shaped in α_k. Note that the welfare in the benchmarks corresponds to the two extreme points in the interval $\alpha_k \in [0,1]$; thus, the asymmetry between the two types of traders plays has important welfare implications in the general (i.e., interior) case.

5. Analysis of Large Markets

To further examine the equilibrium properties, this section considers settings in which the numbers of traders at both markets grow large, which allow us to disentangle the information channel of price spillover from its strategic impact on traders' inference, behavior, and welfare. We focus on the generic case in which $\rho_k > |\phi|$ for each k, that is, traders' values are more correlated within a market than across markets. Let $N = n_I + n_{II}$ be the total number of traders. Denote by $\chi_k = n_k / N \in (0,1)$ the proportion of traders in market $k \in \{I, II\}$. Recall that $\alpha_k \in (0,1)$ is the fraction of informed traders in market k. We will take N to infinity while holding both (χ_k) and (α_k) fixed. That is, we study equilibria in large markets keeping the relative sizes of different subgroups the same. In particular, we examine the inference and demand parameters in the orders of $1/N$ and $1/N^2$, which capture most of the direct and indirect spillover effects when N is large.

5.1. Comparison between two types of traders.

We first compare traders' inferences and strategies in large markets. In particular, the unique limiting equilibrium is symmetric and independent of the fraction of the informed. This "asymptotic symmetry" lays out the foundation toward the analysis of how information spillover and the composition of traders influence the equilibrium in later sections.

Proposition 5.1 characterizes and compares traders' inferences. Notably, the infinite-market limit as well as the first and second order effects are all unique.

Proposition 5.1. *Traders' inference parameters satisfy the following:*

(1) $|c_k^1| = \frac{c_k^*}{n_k} + o(\frac{1}{N})$, $|\delta_k| = \frac{\delta_k^*}{n_k} + o(\frac{1}{N})$, $C_k^0 - C_k^1 = \frac{\Delta_k}{n_k} + o(\frac{1}{N})$, *and* $C_{ks}^1 - C_{ks}^0 = \frac{\Delta_{ks}}{n_k^2} + o(\frac{1}{N^2})$, *where* $c_k^* = \frac{\sigma_{\theta_k}(1-\rho_k)|\phi|}{\sigma_{\theta_{-k}}(\rho_k\rho_{-k}-\phi^2)} \frac{1}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}}$.

(2) $c_k^* > 0$, $\delta_k^* > 0$, $\Delta_k > 0$, *and* $\Delta_{ks} > 0$.[12]

(3) $\lim_{N\to\infty} c_k^1 = \lim_{N\to\infty} \delta_k = 0$, $\lim_{N\to\infty} C_k^1 = \lim_{N\to\infty} C_k^0 = 1 - C_k^*$, *and* $\lim_{N\to\infty} C_{ks}^1 = \lim_{N\to\infty} C_{ks}^0 = C_k^*$, *where* $C_k^* = \frac{1-\rho_k}{1-\rho_k+\sigma_k^2}$.

First, an informed trader's inference depends on the price in the other market, $|c_k^1| > 0$ when the two assets are correlated ($\phi \neq 0$), as she directly takes into account this information externality. Since p_k is predominantly determined by the average signal $\bar{s}_k$ in market k, as the number of traders n_k grows large, $\bar{s}_k$ is almost a perfect signal of her value, i.e., $\frac{\text{Cov}(\theta_k^i, \bar{s}_k)}{\text{Var}(\bar{s}_k)}$ is close to 1. Therefore, the residual explanatory power of $\bar{s}_{-k}$ converges to zero and is proportional to $\frac{1}{n_k}$, i.e., $1 - \frac{\text{Cov}(\theta_k^i, \bar{s}_k)}{\text{Var}(\bar{s}_k)} = \frac{\sigma_{\varepsilon_k}^2}{\text{Var}(\bar{s}_k)} \frac{1}{n_k}$.[13] Consequently, the cross-asset effect is non-zero $|\delta_k| > 0$. Because of information spillover, the demand of the informed in market k depends on the price p_{-k}, which is correlated with $\bar{s}_{-k}$, thus the equilibrium price p_k depends on $\bar{s}_{-k}$ (i.e., $|\delta_k| > 0$). Since this cross-asset effect is a direct consequence of information externality, $|\delta_k|$ is also proportional to $\frac{1}{n_k}$.

Second, the uninformed are more sensitive to the own price than the informed: $C_k^0 > C_k^1$. Intuitively, the information disadvantage of the uninformed makes them put more weight on the own price than the informed. Since this is also a direct implication of information externalities, $C_k^0 - C_k^1$ is proportional to $\frac{1}{n_k}$.

Finally, the uninformed are less sensitive to their own signals than the informed ($C_{ks}^0 < C_{ks}^1$). Since the uninformed put more weight on the own price ($C_k^0 > C_k^1$), which contains information about the other market ($|\delta_k| > 0$), it crowds out their reliance on their own signals than the informed. Since the difference $C_{ks}^1 - C_{ks}^0$ is jointly determined by both $C_k^0 - C_k^1$ and $|\delta_k|$, it is proportional to $\frac{1}{n_k^2}$.

Based on Proposition 5.1, we can now characterize and compare traders' equilibrium strategies in Proposition 5.2. In particular, the price impacts for both types of traders

[12] All these parameters are solved in closed-form in the proof.

[13] We have $\text{Cov}(\theta_k^i, \bar{s}_k) = (\rho_k + \frac{1-\rho_k}{n_k})\sigma_{\theta_k}^2$ and $\text{Var}(\bar{s}_k) = (\rho_k + \frac{1-\rho_k+\sigma_k^2}{n_k})\sigma_{\theta_k}^2$.

are asymptotically the same up to second-order approximations. This is because price impacts are determined by the price elasticity of demand $B_k \equiv \alpha_k B_k^1 + (1-\alpha_k)B_k^0$, so if the market size is large, each trader faces with approximately the same set of market participants. Consequently, the behavior differences between the two types of traders are completely due to their information differences. Formally, an informed trader's demand is more sensitive to the own signal ($a_k^1 > a_k^0$), since her belief is more sensitive to the own signal than the uninformed ($C_{ks}^1 > C_{ks}^0$); an informed trader's demand is also more sensitive to the own price ($B_k^1 > B_k^0$), since her conditional expectation about her value is less sensitive to the own price ($C_k^1 < C_k^0$), that is, with a lower price, she is less pessimistic about the asset value and is willing to buy more assets than the uninformed.

Proposition 5.2. *Traders' equilibrium strategies satisfy the following:*

(1) $\lambda_k^1 = \frac{1}{(n_k-1)B_k} + o(\frac{1}{N^2})$ *and* $\lambda_k^0 = \frac{1}{(n_k-1)B_k} + o(\frac{1}{N^2})$.

(2) $a_k^1 - a_k^0 = \frac{a_k^*}{n_k^2} + o(\frac{1}{N^2})$, $B_k^1 - B_k^0 = \frac{B_k^*}{n_k} + o(\frac{1}{N})$, *and* $|b_k^1| = \frac{b_k^*}{n_k} + o(\frac{1}{N})$.

(3) $a_k^* = \frac{1}{\gamma}\Delta_{ks} > 0$, $B_k^* = \frac{1}{\gamma}\Delta_k > 0$, *and* $b_k^* = \frac{1}{\gamma}c_k^* > 0$.

(4) $\lim_{N\to\infty} a_k^1 = \lim_{N\to\infty} a_k^0 = \frac{C_k^*}{\gamma}$, $\lim_{N\to\infty} B_k^1 = \lim_{N\to\infty} B_k^0 = \frac{C_k^*}{\gamma}$, $\lim_{N\to\infty} b_k^1 = 0$, *and* $\lim_{N\to\infty} \lambda_k^1 = \lim_{N\to\infty} \lambda_k^0 = 0$.

5.2. **Impacts of information spillover.** Next, we study how information spillover, parameterized by the fraction α_k, affects traders' inference and behavior in market k.

For inference, the first step is to analyze the key parameters δ_k and ζ_k, representing cross-asset and own-asset effects, respectively. Let $\bar{s}_k$ be the average signal of all traders in market $k \in \{I, II\}$. From Lemma 4.2, we get

$$p_k = \zeta_k(\bar{s}_k + \delta_k \bar{s}_{-k}) + o(\frac{1}{N}), \tag{20}$$

where ζ_k is the impact of $\bar{s}_k$ on p_k and δ_k is the relative impact of $\bar{s}_{-k}$ on p_k. From Proposition 5.1, a larger α_k implies that the total demand in market k is more sensitive to p_{-k}, since there are more informed traders reacting to p_{-k}. Consequently, the equilibrium price p_k is more sensitive to $\bar{s}_{-k}$ (larger $|\delta_k|$). In other words, p_k is less informative of $\bar{s}_{-k}$. Furthermore, since the inference of the informed is less sensitive to p_k than that of the uninformed, a larger α_k implies that the total demand in market k is less sensitive to p_k, which is predominantly determined by $\bar{s}_k$. As a result, the equilibrium price p_k is less sensitive to $\bar{s}_k$ (lower ζ_k). That is, p_k is more informative of $\bar{s}_k$. Lemma 5.3 below formally establishes these relationships.

Lemma 5.3. *If we ignore the terms of order* $o(\frac{1}{N})$*, then*

- $|\delta_k|$ *and* $|\delta_k \zeta_k|$ *are increasing in* α_k.

- *ζ_k is decreasing in α_k.*

Applying Lemma 5.3, we establish the comparative statics of the inference parameters in Proposition 5.4.

Proposition 5.4. *The impacts of information spillover on inference are the following:*

(1) First order effect: if we ignore the terms of order $o(\frac{1}{N})$, then
  - *C_k^1 is increasing in α_k, and $|c_k^1|$ is decreasing in α_k.*
  - *C_k^0 is independent of α_k;*

(2) Second order effect: if we ignore the terms of order $o(\frac{1}{N^2})$, then
  - *C_{ks}^1 is independent of α_k;*
  - *C_{ks}^0 is decreasing in α_k if and only if $\alpha_k < \frac{1-\rho_k}{\sigma_k^2}$.*

According to Proposition 5.4, information spillover has a first-order effect on the informed traders' inference from prices, but has no first-order effect on that of the uninformed. For the informed, conditioning on observing p_{-k}, the predicting power of p_k on $\bar{s}_{-k}$ is of second-order effect, thus we only need to focus on the direct inference of $\bar{s}_k$ from p_k. A larger α_k has two effects: (i) a smaller ζ_k, implying that p_k is more informative about $\bar{s}_k$ (a larger C_k^1); (ii) a larger $|\delta_k|$, implying that p_k contains more information about of $\bar{s}_{-k}$, which crowds out the informativeness of p_{-k} in predicting $\bar{s}_{-k}$ (a smaller $|c_k^1|$). For the uninformed, the price p_k contains two sources of information: $\bar{s}_k$ and $\bar{s}_{-k}$. Recall that Lemma 5.3 shows that with a larger fraction of informed traders, p_k is more informative about $\bar{s}_k$ and less informative about $\bar{s}_{-k}$. It turns out that their first-order effects exactly cancel out so that the informativeness of p_k remains unchanged.[14]

Proposition 5.4 also establishes that information spillover has a second-order effect on traders' inference from their own signals. For the informed, C_{ks}^1 is independent of α_k. This is because they can almost perfectly infer the average signals from both markets ($\bar{s}_k$ and $\bar{s}_{-k}$), independent of α_k (see (20)). For the uninformed, C_{ks}^0 is decreasing in α_k if and only if $\alpha_k < \frac{1-\rho_k}{\sigma_k^2}$.[15] This non-monotonicity follows from two opposing effects. On the one hand, the correlation between a trader's value and the price is stronger when there are more informed traders in the market. As a result, the uninformed rely more on the price instead of their signals to predict their values (smaller C_{ks}^0). On the other hand, the market price becomes noisier with more informed traders and thus the correlation between a trader's signal and the price is weaker, which implies that the signal contains more information (larger C_{ks}^0). In total, these two effects generate the non-monotonity in Proposition 5.4.

[14] It is possible that α_k has a second-order effect on C_k^0.

[15] If $\frac{1-\rho_k}{\sigma_k^2} > 1$, then C_{ks}^0 is decreasing in α_k for all $\alpha_k \in [0,1]$.

Moreover, when the fraction of the informed is small, the former dominates the latter, thus C_{ks}^0 is decreasing in α_k for small α_k.[16]

Next, we study how information spillover affects traders' equilibrium strategies. The first step (Lemma 5.5) is to analyze the price impact: λ_k^1 and λ_k^0. Recall that $B_k \equiv \alpha_k B_k^1 + (1-\alpha_k)B_k^0$ is the price elasticity of the market demand.

Lemma 5.5. *The following results hold:*

- *First order effect: if we ignore the terms of order $o(\frac{1}{N})$, then B_k is increasing in α_k; λ_k^1 and λ_k^0 are independent of α_k.*
- *Second order effect: if we ignore the terms of order $o(\frac{1}{N^2})$, then $\lambda_k^1 = \lambda_k^0 = \frac{1}{(n_k-1)B_k}$ is decreasing in α_k.*

Lemma 5.5 states that the price elasticity of the market demand (B_k) is increasing in α_k. Intuitively, the demand of an informed trader is more responsive to price, relative to an uninformed trader: $B_k^1 > B_k^0$ (see Proposition 5.2). Thus, if there are more informed traders, the market demand becomes more elastic. Consequently, the price impacts λ_k^1 and λ_k^0 are smaller, where the changes are in the second order.

Proposition 5.6 then establishes the comparative statics of the equilibrium.

Proposition 5.6. *The equilibrium impacts of information spillover are the following:*

(1) First order effect: if we ignore the terms of order $o(\frac{1}{N})$, then
- *B_k^1 and $|b_k^1|$ are decreasing in α_k.*
- *B_k^0 is independent of α_k.*

(2) Second order effect: if we ignore the terms of order $o(\frac{1}{N^2})$, then
- *a_k^1 and a_k^0 are increasing in α_k.*

The first part of Proposition 5.6 follows from the fact that information spillover has a second-order effect on price impacts (Lemma 5.5) and a first-order effect on inference (Proposition 5.4). For the second part, note that even though the inference from signals C_{ks}^1 is independent of α_k, the price impact of the informed is smaller when there are more informed traders, thus they trade more aggressively with a higher signal, i.e., a_k^1 is increasing in α_k. For the uninformed, their price impact is also smaller with more informed traders, yet their inference from signals C_{ks}^0 is non-monotone in α_k. Overall, the former dominates the latter and hence the uninformed also trade more aggressively with a higher signal, i.e., a_k^0 is increasing in α_k.

[16] See the online appendix for more detailed analysis of this non-monotonicity.

5.3. **Welfare Analysis.** This section presents the main results on traders' welfare. Denote the ex ante expected surplus of an informed (resp., uninformed) trader in market k by W_k^1 (resp., W_k^0). Recall that $\bar{s}_k$ is the average signal of all traders in market k.

Lemma 5.7 characterizes W_k^1 and W_k^0. Importantly, it decomposes an informed (resp., uninformed) trader's welfare into three parts: (1) an own signal effect: $\mathbb{E}[C_{ks}^1(s_k^i - \bar{s}_k)]^2$ (resp., $\mathbb{E}[C_{ks}^0(s_k^i - \bar{s}_k)]^2$), respectively; (2) a heterogeneous beliefs effect: $(1-\alpha_k)^2\mathbb{E}[(C_k^1 - C_k^0)\bar{s}_k + c_k^1\bar{s}_{-k}]^2$ (resp., $\alpha_k^2\mathbb{E}[(C_k^1 - C_k^0)\bar{s}_k + c_k^1\bar{s}_{-k}]^2$); and (3) a strategic effect related to the price impacts.

Lemma 5.7. *Ignoring the terms of order $o(\frac{1}{N^2})$, we have*

(1) $W_k^1 = \frac{\frac{\gamma}{2}+\lambda_k^1}{(\gamma+\lambda_k^1)^2}\left[\mathbb{E}[C_{ks}^1(s_k^i - \bar{s}_k)]^2 + (1-\alpha_k)^2\mathbb{E}[(C_k^1 - C_k^0)\bar{s}_k + c_k^1\bar{s}_{-k}]^2\right].$

(2) $W_k^0 = \frac{\frac{\gamma}{2}+\lambda_k^0}{(\gamma+\lambda_k^0)^2}\left[\mathbb{E}[C_{ks}^0(s_k^i - \bar{s}_k)]^2 + \alpha_k^2\mathbb{E}[(C_k^1 - C_k^0)\bar{s}_k + c_k^1\bar{s}_{-k}]^2\right].$

Furthermore, the strategic effects satisfy $\frac{\frac{\gamma}{2}+\lambda_k^1}{(\gamma+\lambda_k^1)^2} = \frac{\frac{\gamma}{2}+\lambda_k^0}{(\gamma+\lambda_k^0)^2} = \frac{1}{2\gamma}(1 - (\frac{1}{(n_k-1)C_k^*})^2)$, *which is independent of* α_k.

Notably, information spillover does not affect welfare through the strategic effect up to the second order. That is, the welfare impact of information spillover can be almost completely attributed to the information effects. To see this, let us consider the informed. The same reasoning applies to the uninformed. Recall that a larger α_k lowers the price impact λ_k^1, so the informed trade more aggressively. However, the marginal benefit of trading equals its marginal cost in equilibrium so that there is no first or second order welfare change when α_k increases. Proposition 5.8 goes on to characterize how the own signal and heterogeneous beliefs effects and thus traders' welfare react to changes in α_k.

Proposition 5.8. *If we ignore the terms of order $o(\frac{1}{N^2})$, then*

(1) W_k^1 *is decreasing in* α_k.

(2) W_k^0 *is decreasing* α_k *if and only if* $\alpha_k < \alpha_k^* \equiv \frac{1-\rho_k}{1-\rho_k+\sigma_k^2}$.

(3) $W_k \equiv \alpha_k W_k^1 + (1-\alpha_k)W_k^0$ *is decreasing in* α_k *if and only if* $\alpha_k < \hat{\alpha}_k \equiv \frac{1-\rho_k}{2(1-\rho_k)+\sigma_k^2}$.

(4) W_k *evaluated at* $\alpha_k = 1$ *and* $\alpha_k = 0$ *are the same.*

(5) $W_k^1 - W_k^0$ *is positive and decreasing in* α_k.

Proposition 5.8 shows that for any given market composition, the informed are always better than the uninformed ($W_k^1 > W_k^0$), but the welfare advantage $W_k^1 - W_k^0$ shrinks as α_k increases. For the informed, since C_{ks}^1 is independent of α_k, the own signal effect is also independent of α_k. The heterogeneous beliefs effect is decreasing in α_k: a larger α_k means that there are fewer uninformed from whom the informed can take advantage. Therefore, the welfare of the informed is decreasing in α_k. For the uninformed, the own signal effect is non-monotonic (Proposition 5.4) and the heterogeneous belief effect is increasing in α_k. When the fraction of the informed is small ($\alpha_k < \alpha_k^*$), the information disadvantage of the uninformed dominates, so that the welfare of the uniformed is U-shaped in α_k.

Perhaps more surprisingly, the aggregate (i.e., weighted average) welfare $W_k \equiv \alpha_k W_k^1 + (1-\alpha_k)W_k^0$ is also U-shaped. That is, more information spillover does not always improve the aggregate welfare. To understand this, we consider the aggregate own signal and heterogeneous beliefs effects, respectively. The aggregate own signal effect displays a U-shape, since a larger α_k implies that both there are more informed traders who are better than the uninformed and the uninformed may get worse, and the latter dominates when the fraction of informed is small, i.e., $\alpha_k < \hat{\alpha}_k$. The aggregate heterogeneous beliefs effect displays a reversed U-shape, as intuitively the market becomes more heterogeneous when α_k is in the middle. Overall, the aggregate own asset effect dominates the aggregate heterogeneous belief effect, so W_k is U-shaped in α_k.

5.4. **Information Efficiency and Liquidity.** In this section, we first examine the impact of information spillover on information efficiency, captured by an individual trader's uncertainty reduction, formally defined as:

$$\tau_k^1 = \mathrm{Var}(\theta_k^i) - \mathrm{Var}(\theta_k^i | s_k^i, p_k, p_{-k}),\ \tau_k^0 = \mathrm{Var}(\theta_k^i) - \mathrm{Var}(\theta_k^i | s_k^i, p_k).$$

Since a larger fraction of informed traders α_k improves the informativeness of the price p_k, i.e., a higher correlation between the value and the price (see the online appendix for details of this statement), uninformed traders in market k get more precise estimation of their values by observing only p_k; whereas informed traders, who observe both prices, can almost perfectly estimate the average signals from both markets, independent of α_k. Thus, we have the following result (Proposition 5.9).

Proposition 5.9. *If we ignore the terms of order $o(\frac{1}{N^2})$, then*

(1) τ_k^1 is independent of α_k.
(2) τ_k^0 is increasing in α_k.
(3) $\tau_k^1 - \tau_k^0$ is positive and decreasing in α_k.

The next result (Proposition 5.10) characterizes the relationship between welfare and information efficiency. The welfare of any trader can be decomposed into two parts: information efficiency and price volatility. Hence, the gap between allocation efficiency and information efficiency comes from the volatility of prices. This gives an alternative explanation about the welfare impact of information spillover: as α_k increases, the welfare deterioration for both types of traders is due to the fact p_k becomes more volatile (higher

$\mathrm{Var}(p_k)$) and all traders are risk averse. Another immediate implication is that the advantage of the informed $W_k^1 - W_k^0$ is completely determined by the information gain $\tau_k^1 - \tau_k^0$, since price variance has the same effect on all traders.

Proposition 5.10. *If we ignore the terms of order $o(\frac{1}{N^2})$, then*

(1) $W_k^1 = \frac{\frac{\gamma}{2}+\lambda_k^1}{(\gamma+\lambda_k^1)^2}\left(\tau_k^1 - \mathrm{Var}(p_k)\right)$ *and* $W_k^0 = \frac{\frac{\gamma}{2}+\lambda_k^0}{(\gamma+\lambda_k^0)^2}\left(\tau_k^0 - \mathrm{Var}(p_k)\right)$.
(2) $W_k^1 - W_k^0 = \frac{1}{2\gamma}(\tau_k^1 - \tau_k^0)$.
(3) $\mathrm{Var}(p_k)$ *is increasing in* α_k.

Next, we study liquidity (i.e., trading volumes) in both markets. Define the liquidity indices of the informed, the uninformed, and an average trader respectively as:

$$L_k^1 \equiv \mathbb{E}(x_{k,i}^1)^2,\ L_k^0 \equiv \mathbb{E}(x_{k,i}^0)^2,\ L_k \equiv \alpha_k L_k^1 + (1-\alpha_k)L_k^0.$$

Proposition 5.11 extablishes that liquidity are determined by the price impact λ_k^1 (or λ_k^0) and welfare. With more informed traders, the uninformed trade more aggressively since the price impact plays a bigger role than information disadvantage, while the informed may trade less aggressively since the loss of information advantage may dominate the gain from a lower price impact if α_k is small enough. Moreover, the total liquidity is increasing in α_k.

Proposition 5.11. *If we ignore the terms of order $o(\frac{1}{N^2})$, then*

(1) $L_k^1 = (\frac{\gamma}{2} + \lambda_k^1)^{-1} W_k^1$, $L_k^0 = (\frac{\gamma}{2} + \lambda_k^0)^{-1} W_k^0$, *and* $L_k = (\frac{\gamma}{2} + \lambda_k^1)^{-1} W_k$.
(2) L_k^1 *is decreasing in* α_k *if and only if* $\alpha_k < \alpha_k^L \equiv \frac{1}{2}(1 - \frac{1-\rho_k}{\sigma_k^2})$.
(3) L_k^0 *and* L_k *are increasing in* α_k.
(4) $L_k^1 - L_k^0 = \frac{2}{\gamma}(W_k^1 - W_k^0)$ *is positive and decreasing in* α_k.

5.5. **Price co-movement and volatility transmission.** Here we show that information spillover can account for the empirical patterns of market prices mentioned in the introduction. First, it exacerbates the price comovement across markets beyond the correlation in the fundamentals (see Proposition 5.12). Define r_p as the correlation between p_k and p_{-k}, and define r_s as the correlation between $\bar{s}_k$ and $\bar{s}_{-k}$, which captures the correlation between fundamentals in two markets. Intuitively, a larger fraction of informed traders in each market makes the market price contain more information about the other market, thereby increasing the correlation between the two prices.

Proposition 5.12. *Information spillover amplifies price co-movement:*

(1) $|r_p|$ *is increasing in both* α_k *and* α_{-k}.
(2) $|r_p| \geq |r_s|$.

(3) $|r_p| = |r_s|$ *if and only if* $\alpha_k = \alpha_{-k} = 0$.

(4) $r_p > 0$ *and* $r_s > 0$ *if and only if* $\phi > 0$.

Furthermore, information spillover amplifies the transmission of price volatility between the two markets (Proposition 5.13). Suppose that the variance of the average signal $\bar{s}_{-k}$ increases, due to higher volatility of true value θ^i_{-k} or noises ε^i_{-k}. Define $\Delta_k \equiv \Delta\text{Var}(p_k)/\Delta\text{Var}(p_{-k})$ as the change in price volatility in market k, relative to market $-k$, which captures the transmission of price volatility. Intuitively, because prices provide information about the average signals of both markets: $p_k = \zeta_k(\bar{s}_k + \delta_k \bar{s}_{-k}) + o(\frac{1}{N})$ and $p_{-k} = \zeta_{-k}(\bar{s}_{-k} + \delta_{-k}\bar{s}_k) + o(\frac{1}{N})$, a larger α_k (and thus a higher δ_k) implies a higher degree of linkage between two prices.

Proposition 5.13. *The change in price volatility* Δ_k *is increasing in* α_k.

6. Discussion and Extensions

We conclude with a brief discussion of the modeling assumptions and several extensions studied in more detail in the online appendix.

Endogenous or exogenous information spillover. As mentioned in the introduction, our results cover both endogenous and exogenous information spillover. When there are informed traders in both markets, the equilibrium prices and their information content are endogenously determined through joint market clearing. When only one market has informed traders, the price in the other market, which is determined by its own market-clearing condition, can be viewed as exogenous and "semi-public" information that is only available to the informed in the former market. Therefore, in addition to prices from other markets, our analysis applies to broader settings in which a fraction of traders (i.e., insiders) commonly observe some informative signals about their values before trading.

Trading in both markets. Our model assumes that each trader only participates in one of the two markets in order to isolate the impact of information externality from prices. The online appendix contains an analysis of the case in which all traders trade in both markets and have multidimensional private information. Assuming a trader's demand for each asset can be contingent on both asset prices, we derive the unique symmetric equilibrium. Compared with the full information spillover benchmark in Section 3.1, the equilibrium features two extra incentives, in addition to the information externality from prices: cross-market (and within-market) price impacts and cost linkage of holding different assets; furthermore, traders trade more aggressively based on their signals and prices when they participate in both markets. In the more general case with both traders who trade both assets and those who only trade one asset, the above three types of incentives would confound each other due to this new trader heterogeneity, which is an interesting question that goes beyond the scope of our current analysis.

Additional comparative statics. The analysis in Section 5 focuses on the comparative statics of the equilibrium in market k of information spillover, parameterized by the fraction α_k of informed traders in a market. The online appendix contains further comparative statics results, such as the cross-market impact of information spillover (i.e., how α_{-k} affects equilibrium in market k), which we show are negligible up to second-order approximations, and the correlation $|\phi|$ between the two assets, which we show amplifies the impact of information spillover.

More than two markets. We focus on the two-market setting for notational simplicity. We show in the online appendix that, when there are more than two markets, information spillover has stronger impacts on the equilibrium, as the prices from other markets provide more information about traders' values. Furthermore, an increase in the number of markets is qualitatively similar to an increase in the asset correlation $|\phi|$.

Positive information spillover in large markets. In our main setting, the impact of information spillover vanishes as the market size increases to infinity, which is a consequence of the assumption that a trader's residual uncertainty diminishes when the market size increases, conditional on the average signal of traders in her market (in particular, $s_k^i = \theta_k^i + \varepsilon_k$ and $\rho_k > |\phi|$). There are several ways to enrich the model to reinstall information spillover in the large market limit. We pursue two such extensions in detail in the online appendix: systematic risks and supply shocks. Since the underlying logic of these two extensions is similar, here we only briefly discuss the former.

We model systematic risk by introducing an extra market-specific normally-distributed noise e_k in the signals of all traders in market k: $s_k^i = \theta_k^i + e_k + \varepsilon_k$, for all $i \in \mathcal{N}_k$. Even when these noises e_I and e_{II} are independent, in the large market limit, the average signal of traders in market k will not filter out all the noise in s_k^i. As in the main setting, we first solve for the unique symmetric equilibrium in the benchmark where all traders are informed and examine how the systematic noise e_k affects traders' inference. In particular, it muffles the informativeness of the average signal $\bar{s}_k$, but this also implies that traders may rely more on information spillover (i.e., $\bar{s}_{-k}$) if e_k becomes noisier. Furthermore, in the large market limit, information spillover matters for traders' inference as long as the variance of the systematic risk is positive.

In the general case with heterogeneously informed traders, the presence of systematic risk implies that the limiting equilibrium remains asymmetric, which further complicates the equilibrium characterization even in large but finite markets. To make progress, extend the approximation method in the main setting to investigate equilibria around a double limit, first taking the market size to infinity and then taking the variance of the system risk to zero. This again allows us to characterize the comparative statics of information spillover up to second-order approximations. Furthermore, we show that the results in our main setting are robust to systematic risks.

Appendix A. Proofs of the Results in Section 3

A.1. **Proof of Proposition 3.1.** We first solve for $\mathbb{E}\left[\theta_k^i | s_k^i, p_k, p_{-k}\right]$. It follows from the market clearing condition (4) that $p_k = a_k^1(B_{-k}^1 \bar{s}_k^1 + b_k^1 \bar{s}_{-k}^1)/(B_k^1 B_{-k}^1 - b_k^1 b_{-k}^1)$, so it suffices to solve for $\mathbb{E}[\theta_k^i | s_k^i, \bar{s}_k^1, \bar{s}_{-k}^1]$. We have $\mathrm{Cov}(\bar{s}_k^1, \theta_k^i) = \frac{(1+(n-1)\rho)\sigma_\theta^2}{n}$, $\mathrm{Cov}(\bar{s}_k^1, s_k^i) = \mathrm{Var}(\bar{s}_k^1) = \frac{(1+(n-1)\rho+\sigma^2)\sigma_\theta^2}{n}$, and $\mathrm{Cov}(\bar{s}_{-k}^1, \theta_k^i) = \mathrm{Cov}(\bar{s}_{-k}^1, s_k^i) = \mathrm{Cov}(\bar{s}_k^1, \bar{s}_{-k}^1) = \phi\sigma_\theta^2$. Thus,

$$
\begin{aligned}
&\mathbb{E}\left[\theta_k^i | s_k^i, \bar{s}_k^1, \bar{s}_{-k}^1\right] \\
=& C_s^1 s_k^i + C^1 \bar{s}_k^1 + C_-^1 \bar{s}_{-k}^1 = C_s^1 s_k^i + \left(\frac{C^1 B_k^1}{a_k^1} - \frac{C_-^1 b_{-k}^1}{a_{-k}^1}\right) p_k + \left(\frac{C_-^1 B_{-k}^1}{a_{-k}^1} - \frac{C^1 b_k^1}{a_k^1}\right) p_{-k},
\end{aligned}
$$

where

$$
C_s^1 = \frac{1-\rho}{1-\rho+\sigma^2},\ C^1 = \frac{\sigma^2}{1-\rho+\sigma^2} \cdot \frac{\rho^2 + \frac{\rho(1-\rho+\sigma^2)}{n} - \phi^2}{\left(\rho + \frac{1-\rho+\sigma^2}{n}\right)^2 - \phi^2},\ C_-^1 = \frac{\sigma^2 \cdot \frac{\phi}{n}}{\left(\rho + \frac{1-\rho+\sigma^2}{n}\right)^2 - \phi^2}.
$$

Then we prove the properties of the equilibrium parameters. First, since $C_-^1 > 0$ if and only if $\phi > 0$ and b_k^1 has the same sign as C_-^1, we have $b^1 > 0$ if and only if $\phi > 0$. Second, since $\rho^2 + \rho(1-\rho+\sigma^2)/n - \phi^2 > |\phi|(1-\rho+\sigma^2)/n$, we have $C^1 \geq |C_-^1|$ and thus $\frac{|b^1|}{B^1} = \frac{|C_-^1|}{C_s^1 + C^1} < \frac{|C_-^1|}{C^1} < 1$. Since $C^1 > 0$, we have $a^1 = \frac{(n-2)C_s^1 - C^1}{\gamma(n-1)} < \frac{C_s^1}{\gamma}\frac{n-2}{n-1}$. Since $C^1 > |C_-^1|$, we also have $B^1 = \frac{a^1}{C_s^1 + C^1 - |C_-^1|} < \frac{a^1}{C_s^1} < \frac{1}{\gamma}\frac{n-2}{n-1}$. Finally, since $\rho^2 + \rho(1-\rho+\sigma^2)/n < (\rho + (1-\rho+\sigma^2)/n)^2 - \phi^2$, we have $\frac{\partial C^1}{\partial |\phi|} < 0$. Together with $\frac{\partial a^1}{\partial C^1} < 0$, we have $\frac{\partial a^1}{\partial |\phi|} > 0$. In addition, since $\frac{\partial C_-^1}{\partial |\phi|} > 0$, both $\frac{B^1}{a^1}$ and $\frac{|b^1|}{a^1}$ are increasing in C_-^1 and decreasing in C^1, and thus both are increasing in $|\phi|$. Together with $\frac{\partial a^1}{\partial |\phi|} > 0$, we have that B^1 and $|b^1|$ are increasing in $|\phi|$.

A.2. **Proof of Proposition 3.2.** From the market-clearing condition, $p_k = a_k^0 \bar{s}_k^0 / B_k^0$, where $\bar{s}_k^0 = \sum_{i' \in \mathcal{N}_k^0} s_k^{i'}/n$, we have that $\mathrm{Cov}(p_k, \theta_k^{i'}) = a_k^0(1+(n-1)\rho)\sigma_\theta^2/(nB_k^0)$, $\mathrm{Cov}(p_k, s_k^{i'}) =$

$a_k^0\left(1+(n-1)\rho+\sigma^2\right)\sigma_\theta^2/(nB_k^0)$, $\mathrm{Var}(p_k)=(a_k^0)^2\left(1+(n-1)\rho+\sigma^2\right)\sigma_\theta^2/(n(B_k^0)^2)$. Thus,

$$\mathbb{E}\left[\theta_k^{i'}|s_k^{i'},p_k\right]=C_k^0 s_k^{i'}+C^0\bar{s}_k^0=C_k^0 s_k^{i'}+C^0\frac{B_k^0 p_k}{a_k^0},$$

where

$$C_s^0=\frac{(1-\rho)}{1-\rho+\sigma^2},\ C^0=\frac{\sigma^2}{1-\rho+\sigma^2}\frac{\rho}{\rho+\frac{1-\rho+\sigma^2}{n}}.$$

Finally, it is clear that $a^0>0$ and $B^0>0$ are independent of ϕ. Since $C^0>0$, we have $a^0=\frac{(n-2)C_s^0-C^0}{\gamma(n-1)}<\frac{C_s^0}{\gamma}\frac{n-2}{n-1}$ and thus $B^0=\frac{a^0}{C_s^0+C^0}<\frac{a^0}{C_s^0}<\frac{1}{\gamma}\frac{n-2}{n-1}$.

A.3. **Proof of Corollary 3.3.** First, $C_s^0=C_s^1=(1-\rho)/(1-\rho+\sigma^2)$. Next, we have

$$C^0-C^1=\frac{\sigma^2}{1-\rho+\sigma^2}\cdot\frac{\frac{(1-\rho+\sigma^2)\phi^2}{n}}{\left(\rho+\frac{1-\rho+\sigma^2}{n}\right)\left(\left(\rho+\frac{1-\rho+\sigma^2}{n}\right)^2-\phi^2\right)}>0.$$

A.4. **Proof of Corollary 3.4.** Part (i) follows directly from the closed-form solutions. By Corollary 3.3, we have $a_k^1-a_k^0=\frac{C^0-C^1}{\gamma(n-1)}>0$. Since $C^0>C^1$, then we have

$$\frac{B_k^1}{a_k^1}=\frac{C_s^1+C^1}{(C_s^1+C^1)^2-(C_-^1)^2}>\frac{1}{C_s^1+C^1}>\frac{1}{C_s^0+C^0}=\frac{B_k^0}{a_k^0}.$$

Together with $a_k^1>a_k^0$, we have $B_k^1>B_k^0$. Since $a_k^1=\frac{C_s^1}{\gamma+\lambda_k^1}$, $a_k^0=\frac{C_s^0}{\gamma+\lambda_k^0}$, and $C_s^1=C_s^0$, then $a_k^1>a_k^0$ implies that $\lambda_k^1<\lambda_k^0$.

A.5. **Proof of Proposition 3.5.** With full information spillover, for any trader $i\in\mathcal{N}_k^1$, $x_k^i=\frac{1}{\gamma+\lambda_k^1}(\mathbb{E}[\theta_k^i|s_k^i,p_k,p_{-k}]-p_k)$, then her expected payoff is

$$W_k^1=\mathbb{E}((\mathbb{E}[\theta_k^i|s_k^i,p_k,p_{-k}]-p_k)x_k^i-\frac{\gamma(x_k^i)^2}{2})=(\frac{\gamma}{2}+\lambda_k^1)\mathbb{E}(x_k^i)^2.$$

By the market-clearing condition, the equilibrium demand is $x_k^i=a^1 s_k^i-B^1 p_k+b^1 p_k=a^1(s_k^i-\bar{s}_k)$. Together with the fact that $a^1=\frac{1}{\gamma+\lambda_k^1}C_s^1$, we have

$$W_k^1=\frac{\frac{\gamma}{2}+\lambda_k^1}{(\gamma+\lambda_k^1)^2}(C_s^1)^2\mathbb{E}(s_k^i-\bar{s}_k)^2.$$

Similarly, $W_k^0=\frac{\frac{\gamma}{2}+\lambda_k^0}{(\gamma+\lambda_k^0)^2}(C_s^0)^2\mathbb{E}(s_k^i-\bar{s}_k)^2$. Since $\lambda_k^1<\lambda_k^0$ and $C_s^1=C_s^0$, we have $W_k^1>W_k^0$.

APPENDIX B. PROOFS OF THE RESULTS IN SECTION 4

B.1. **Proof of Lemma 4.1.** The first-order condition of an uninformed trader i' is

$$\mathbb{E}\left[\theta_k^{i'}|s_k^{i'}, p_k\right] - p_k = (\gamma + \frac{dp_k}{dx_k^{i'}})x_k^{i'}. \tag{21}$$

In addition, from the market-clearing conditions, we have

$$1 + \sum_{j' \in \mathcal{N}_k^0, j' \neq i'} \frac{\partial x_k^{j'}}{\partial p_k} \frac{dp_k}{dx_k^{i'}} + \sum_{i \in \mathcal{N}_k^1} \left[\frac{\partial x_k^i}{\partial p_k} \frac{dp_k}{dx_k^{i'}} + \frac{\partial x_k^i}{\partial p_{-k}} \frac{dp_{-k}}{dx_k^{i'}}\right] = 0$$

and

$$\sum_{l' \in \mathcal{N}_{-k}^0} \frac{\partial x_{-k}^{l'}}{\partial p_{-k}} \frac{dp_{-k}}{dx_k^{i'}} + \sum_{l \in \mathcal{N}_{-k}^1} \left[\frac{\partial x_{-k}^l}{\partial p_{-k}} \frac{dp_{-k}}{dx_k^{i'}} + \frac{\partial x_{-k}^l}{\partial p_k} \frac{dp_k}{dx_k^{i'}}\right] = 0.$$

Thus, the price impact of an uninformed trader i' is

$$\frac{dp_k}{dx_k^{i'}} = -\left(\sum_{j' \in \mathcal{N}_k^0, j' \neq i'} \frac{\partial x_k^{j'}}{\partial p_k} + \sum_{i \in \mathcal{N}_k^1} \frac{\partial x_k^i}{\partial p_k} - \sum_{i \in \mathcal{N}_k^1} \frac{\partial x_k^i}{\partial p_{-k}} \frac{\sum_{l \in \mathcal{N}_{-k}^1} \frac{\partial x_{-k}^l}{\partial p_k}}{\sum_{l' \in \mathcal{N}_{-k}^0} \frac{\partial x_{-k}^{l'}}{\partial p_{-k}} + \sum_{l \in \mathcal{N}_{-k}^1} \frac{\partial x_{-k}^l}{\partial p_{-k}}}\right)^{-1}. \tag{22}$$

Substituting (22) into (21) and using the conjectured linear strategies, we obtain

$$x_k^{i'}(p_k, s_k^{i'}) = \frac{1}{\gamma + \frac{dp_k}{dx_k^{i'}}}\left(\mathbb{E}\left[\theta_k^{i'}|s_k^{i'}, p_k\right] - p_k\right) = \frac{1}{\gamma + \lambda_k^0}\left(\mathbb{E}\left[\theta_k^{i'}|s_k^{i'}, p_k\right] - p_k\right), \tag{23}$$

where λ_k^0 is given by (12). Similarly we obtain the strategy of the informed as in (11).

B.2. **Proof of Lemma 4.2.** The market-clearing conditions can be rewritten as

$$\alpha_k a_k^1 \bar{s}_k^1 + (1-\alpha_k) a_k^0 \bar{s}_k^0 = \left(\alpha_k B_k^1 + (1-\alpha_k) B_k^0\right) p_k - (1-\alpha_k)\, b_k^1 p_{-k} = B_k p_k - b_k p_{-k},$$

for $k, -k \in \{I, II\}$ and $k \neq -k$. Thus, we have $p_k = D_k^1 \bar{s}_k^1 + D_k^0 \bar{s}_k^0 + d_k^1 \bar{s}_{-k}^1 + d_k^0 \bar{s}_{-k}^0$, where the parameters are given in the statement of this lemma.

B.3. **Proof of Lemma 4.3.** We first define some new parameters:

$$C_k^* = \frac{1-\rho_k}{1-\rho_k+\sigma_k^2},\ \kappa_k = 1-\rho_k+\sigma_k^2,\ \eta_k = \frac{\sigma_{\theta_{-k}}}{\sigma_{\theta_k}}, \tag{24}$$

$$e_{k1} = \frac{\pi_k}{\alpha_k}\frac{\kappa_k}{n_k},\ e_{k2} = (\frac{(1-\pi_k)^2}{1-\alpha_k} + \frac{\pi_k^2}{\alpha_k})\frac{\kappa_k}{n_k},\ e_{k3} = \frac{1-\pi_k}{1-\alpha_k}\frac{\kappa_k}{n_k},$$

$$y_{k1} = \frac{\kappa_k e_{k2} - e_{k1}^2}{\kappa_k - e_{k1}}\frac{\phi}{\eta_k},\ \ y_{k2} = \rho_k(\rho_{-k} + e_{-k2}) - \phi^2,$$

$$y_{k3} = (\rho_k + e_{k2})(\rho_{-k} + e_{-k2}) - \phi^2 - \frac{e_{k1} - e_{k2}}{\kappa_k - e_{k1}}((\rho_k + e_{k1})(\rho_{-k} + e_{-k2}) - \phi^2).$$

Informed traders: For an informed trader $i \in \mathcal{N}_k^1$, since $(\theta_k^i, s_k^i, p_k, p_{-k})$ is jointly normal, we have $E(\theta_k^i | s_k^i, p_k, p_{-k}) = C_{ks}^1 s_k^i + C_k^1 p_k + c_k^1 p_{-k}$. Define $X = \theta_k^i$, $Y = (s_k^i, p_k, p_{-k})$. By the projection theorem, $E[X|Y] = E(X) + \Sigma_{X,Y}\Sigma_{Y,Y}^{-1}(Y - E(Y))$, where

$$\Sigma_{X,Y} = (\mathrm{Cov}(\theta_k^i, s_k^i), \mathrm{Cov}(\theta_k^i, p_k), \mathrm{Cov}(\theta_k^i, p_{-k})),$$

$$\Sigma_{Y,Y} = \begin{bmatrix} \mathrm{Cov}(s_k^i, s_k^i) & \mathrm{Cov}(s_k^i, p_k) & \mathrm{Cov}(s_k^i, p_{-k}), \\ \mathrm{Cov}(p_k, s_k^i) & \mathrm{Cov}(p_k, p_k) & \mathrm{Cov}(p_k, p_{-k}) \\ \mathrm{Cov}(p_{-k}, s_k^i) & \mathrm{Cov}(p_{-k}, p_k) & \mathrm{Cov}(p_{-k}, p_{-k}) \end{bmatrix}.$$

Define $\delta_k = \frac{d_k^1 + d_k^0}{D_k^1 + D_k^0}$, $\zeta_k = D_k^1 + D_k^0$, $\pi_k = \frac{D_k^1}{D_k^1 + D_k^0}$. Since $\frac{D_k^1}{D_k^1 + D_k^0} = \frac{d_{-k}^1}{d_{-k}^1 + d_{-k}^0}$ and by $p_k = D_k^1 \bar{s}_k^1 + D_k^0 \bar{s}_k^0 + d_k^1 \bar{s}_{-k}^1 + d_k^0 \bar{s}_{-k}^0$ (see Lemma 4.2), we have

$$p_k = \zeta_k[(\pi_k \bar{s}_k^1 + (1 - \pi_k)\bar{s}_k^0) + \delta_k(\pi_{-k}\bar{s}_{-k}^1 + (1 - \pi_{-k})\bar{s}_{-k}^0)].$$

In addition, we have $\Sigma_{X,Y}\Sigma_{Y,Y}^{-1} = (C_{ks}^1, C_k^1, c_k^1)$, where

(25)

$$C_{ks}^1 = C_k^* - \frac{(1 - C_k^*)(e_{k1} - e_{k2})y_{k2}}{(\kappa_k - e_{k1})y_{k3}},\ C_k^1 = \frac{(1 - C_k^*)(y_{k2} - y_{k1}\delta_{-k})}{(1 - \delta_k\delta_{-k})y_{k3}\zeta_k},\ c_k^1 = \frac{(1 - C_k^*)(y_{k1} - y_{k2}\delta_k)}{(1 - \delta_k\delta_{-k})y_{k3}\zeta_{-k}}.$$

Define $\beta_k = \frac{\alpha_k a_k + (1-\alpha_k)a_{k0}}{\alpha_k B_k + (1-\alpha_k)B_{k0}}$. Then, $\beta_k = (1 - \delta_k\delta_{-k})\zeta_k$. Substituting this into (25), we get

$$C_k^1 = \frac{(1 - C_k^*)(y_{k2} - y_{k1}\delta_{-k})}{y_{k3}\beta_k},\ c_k^1 = \frac{(1 - C_k^*)(y_{k1} - y_{k2}\delta_k)}{y_{k3}\beta_{-k}}. \tag{26}$$

Uninformed traders: For an uninformed trader $i \in \mathcal{N}_k^0$, since (θ_k^i, s_k^i, p_k) is jointly normal, we have

$$E(\theta_k^i | s_k^i, p_k) = C_{ks}^0 s_k^i + C_k^0 p_k.$$

Define $X = \theta_k^i$, $Y_0 = (s_k^i, p_k)$. By projection theorem, we can solve for

$$E[X|Y_0] = E(X) + \Sigma_{X,Y_0}\Sigma_{Y_0,Y_0}^{-1}(Y_0 - E(Y_0)),$$

where $\Sigma_{X,Y_0} = (\mathrm{Cov}(\theta_k^i, s_k^i)$, $\Sigma_{Y_0,Y_0} = \begin{bmatrix} \mathrm{Cov}(s_k^i, s_k^i) & \mathrm{Cov}(s_k^i, p_k) \\ \mathrm{Cov}(p_k, s_k^i) & \mathrm{Cov}(p_k, p_k) \end{bmatrix}$.

Therefore, we have $\Sigma_{X,Y_0}\Sigma_{Y_0,Y_0}^{-1} = (C_{ks}^0, C_k^0)$, where

(27)

$$C_{ks}^0 = C_k^* + \frac{(1 - C_k^*)(y_{k2}\eta_k^2\delta_k^2 + \rho_k(e_{k2} - e_{k3}) - e_{k3}\phi\eta_k\delta_k)}{H_k^0},\ C_k^0 = \frac{(1 - C_k^*)(\rho_k(\kappa_k - e_{k3}) + \eta_k\delta_k\kappa_k\phi)}{H_k^0\zeta_k},$$

$$H_k^0 = \eta_k^2\delta_k^2((\kappa_k + \rho_k)(\rho_{-k} + e_{-k2}) - \phi^2) + (\kappa_k + \rho_k)(\rho_k + e_{k2}) - (\rho_k + e_{k3})^2 + 2\phi(\kappa_k - e_{k3})\eta_k\delta_k.$$

B.4. **Proof of Theorem 4.4.** Define $\beta_k \equiv \frac{\alpha_k a_k^1+(1-\alpha_k)a_k^0}{\alpha_k B_k^1+(1-\alpha_k)B_k^0}$, $\bar{C}_k^1 = C_k^1 + \frac{\delta_{-k}\zeta_{-k}}{\zeta_k}c_k^1$, $\pi_k = \frac{\alpha_k a_k^1}{\alpha_k a_k^1+(1-\alpha_k)a_k^0}$ and $\Pi_k = \frac{\alpha_k B_k^1}{\alpha_k B_k^1+(1-\alpha_k)B_k^0}$.

We construct a fixed point mapping in Steps 1–4 for the case where $\phi > 0$, which satisfies the conditions in the Brouwer's fixed point theorem (Step 5). Step 6 discusses the case where $\phi < 0$.

Step 1: Given $\pi_k \in [0,1]$ and $\Pi_k \in [0,1]$, there is a unique δ_k, ζ_k, β_k, C_{ks}^1, C_{ks}^0, C_k^1, C_k^0, c_k^1, $\bar{C}_k^1$.

Given π_k, we get a unique solution e_{k1}, e_{k2}, e_{k3}, y_{k1}, y_{k2}, y_{k3}. An immediate result is that we get $C_{ks}^1 = C_k^* - \frac{(1-C_k^*)(e_{k1}-e_{k2})y_{k2}}{(\kappa_k-e_{k1})y_{k3}}$.

Next, we solve for δ_k. By Lemma 4.2, we get

$$\delta_k = \beta_{-k}\frac{\alpha_k b_k^1}{\alpha_k a_k^1 + (1-\alpha_k)a_k^0} = \frac{\beta_{-k}\pi_k b_k^1}{a_k^1} = \frac{\pi_k\beta_{-k}c_k^1}{C_{ks}^1}. \tag{28}$$

From(26) and (28), we get $\delta_k = (1-C_k^*)(\frac{y_{k1}}{y_{k3}} - \frac{y_{k2}}{y_{k3}}\delta_k)\frac{\pi_k}{C_{ks}^1}$. Consequently,

$$\delta_k = \frac{(1-C_k^*)y_{k1}}{(1-C_k^*)y_{k2} + \frac{C_{ks}^1}{\pi_k}y_{k3}}. \tag{29}$$

Since we have solved for y_{k1},y_{k2},y_{k3},π_k,C_{ks}^1, then we get the solution of δ_k.

By the definition of β_k, we have $\frac{\pi_k}{\Pi_k} = \frac{a_k^1}{B_k^1}\frac{1}{\beta_k} = \frac{C_{ks}^1}{(1-C_k^1)\beta_k}$. Together with (26), we get

$$\beta_k = (1-C_k^*)(\frac{y_{k2}}{y_{k3}} - \frac{y_{k1}}{y_{k3}}\delta_{-k}) + \frac{\Pi_k C_{ks}^1}{\pi_k}. \tag{30}$$

By definition, we have $\zeta_k = \frac{\beta_k}{1-\delta_k\delta_{-k}}$. Since we know β_k, δ_k and δ_{-k}, then there is a unique solution ζ_k. By Lemma 4.3, we solve for C_{ks}^0, C_k^0, c_k^1 and C_k^1, which are functions of $(\delta_k, \zeta_k, e_{k1}, e_{k2}, e_{k3}, y_{k1}, y_{k2}, y_{k3})$. Moreover, we solve for $\bar{C}_k^1 \equiv C_k^1 + \frac{\delta_{-k}\zeta_{-k}}{\zeta_k}c_k^1$.

Step 2: Given $C_{ks}^1, C_k^1, C_{ks}^0 \in [0,1]$, $\bar{C}_k^1, C_k^0 \in [0, \frac{n_k-2}{n_k-1}]$, there is a unique solution to $a_k^1 > 0$, $B_k^1 > 0$, $a_k^0 > 0$, $B_k^0 > 0$, $b_k^1 > 0$. Consequently, there is a unique $\pi_k \in (0,1)$, $\Pi_k \in (0,1)$.

By the definition of δ_k, we have $\delta_k = \frac{\zeta_{-k}}{\zeta_k}\frac{\alpha_k b_k^1}{\alpha_k B_k^1+(1-\alpha_k)B_k^0}$. Therefore, $\frac{\alpha_{-k}b_{-k}^1}{\alpha_{-k}B_{-k}^1+(1-\alpha_{-k})B_{-k}^0} = \frac{\delta_{-k}\zeta_{-k}}{\zeta_k}$. Together with $\frac{b_k^1}{B_k^1} = \frac{c_k^1}{1-C_k^1}$ and $\bar{C}_k^1 = C_k^1 + \frac{\delta_{-k}\zeta_{-k}}{\zeta_k}c_k^1$, we get

$$B_k^1 - \frac{\alpha_{-k}b_{-k}^1}{\alpha_{-k}B_{-k}^1 + (1-\alpha_{-k})B_{-k}^0}b_k^1 = B_k^1(1 - \frac{b_k^1}{B_k^1}\frac{\delta_{-k}\zeta_{-k}}{\zeta_k}) = B_k^1\frac{1-\bar{C}_k^1}{1-C_k^1}.$$

Substitute the above expression into the definition of λ_k^1 and λ_k^0, we get

$$\lambda_k^1 = \left((n\alpha_k - 1)B_k^1\frac{1-\bar{C}_k^1}{1-C_k^1} + (1-\alpha_k)n_k B_k^0\right)^{-1}, \lambda_k^0 = \left(n_k\alpha_k B_k^1\frac{1-\bar{C}_k^1}{1-C_k^1} + ((1-\alpha_k)n_k - 1)B_k^0\right)^{-1}.$$

Together with $B_k^1 = \frac{1-C_k^1}{\gamma+\lambda_k^1}$ and $B_k^0 = \frac{1-C_k^0}{\gamma+\lambda_k^0}$, we get

$$B_k^0 = \frac{1-C_k^0}{\gamma} - \frac{x}{\gamma[n_k\alpha_k\frac{1-\bar{C}_k^1}{1-C_k^1} + ((1-\alpha_k)n_k - 1)x]} \equiv H_0(x),$$

$$B_k^1 = \frac{1-C_k^1}{\gamma} - \frac{1}{\gamma[(n_k\alpha_k - 1)\frac{1-\bar{C}_k^1}{1-C_k^1} + (1-\alpha_k)n_k x]} \equiv H_1(x),$$

where $x \equiv \frac{B_k^0}{B_k^1}$. We divide $H_0(x)$ by $H_1(x)$ and get $x = \frac{H_0(x)}{H_1(x)}$. Define

$$G(x) \equiv xH_1(x) - H_0(x).$$

We need to solve $G(x) = 0$ to get the solution $x = \frac{B_k^0}{B_k^1}$.

Notice that if $\bar{C}_k^1 < 1$ and $C_k^1 < 1$, then $G(x)$ is continuous for $x \geq 0$. Moreover, $H_1(x)$ is increasing in x and $H_0(x)$ is decreasing in x for any $x \geq 0$. Define $y = \frac{1-\bar{C}_k^1}{1-C_k^1}$. It is apparent that $y > 0$.

First, there is a solution $x^* > 0$ such that $G(x^*) = 0$. This holds since $G(0) = -\frac{1-C_k^0}{\gamma} < 0$ and $G(+\infty) = +\infty$.

Second, we prove that the solution $x^* \in (0,\infty)$ is unique. Define $\underline{x}$ such that $H_1(x) > 0$ if and only if $x > \underline{x}$ (it is possible that $\underline{x} = 0$). We prove that $G(x) < 0$ for $x < \underline{x}$. If $x \geq y$, then $H_1(x) \geq H_1(y) = \frac{1-C_k^1}{\gamma} - \frac{1}{\gamma(n_k-1)y} = \frac{1-C_k^1}{\gamma(1-\bar{C}_k^1)}(\frac{n_k-2}{n_k-1} - \bar{C}_k^1) > 0$, where the last inequality holds since $\bar{C}_k^1 < \frac{n_k-2}{n_k-1}$. By the definition of $\underline{x}$, we have $x > \underline{x}$. Therefore, $x < \underline{x}$ implies that $x < y$ and hence $H_0(x) > H_0(y) = \frac{1}{\gamma}(\frac{n_k-2}{n_k-1} - C_k^0) > 0$, which holds since $C_k^0 < \frac{n_k-2}{n_k-1}$. Since $xH_1(x) < 0$ and $H_0(x) > 0$ for any $x < \underline{x}$, then we reach the conclusion that $G(x) = xH_1(x) - H_0(x) < 0$ for $x < \underline{x}$. Consequently, any solution x^* of $G(x^*) = 0$ satisfies $x^* > \underline{x}$.

We then prove that $G(x)$ is increasing in x for $x \geq \underline{x}$. Since $H_1(x) > 0$ and $H_1(x)$ is increasing in x for $x \geq \underline{x}$, then $xH_1(x)$ is increasing for $x \geq \underline{x}$, Together with the fact that $H_0(x)$ is decreasing in x, we have $G(x) = xH_1(x) - H_0(x)$ is increasing in $x \geq \underline{x}$. Therefore, the solution x^* such that $G(x^*) = 0$ is unique.

Next, we solve for the unique $B_k^1 = H_1(x^*) > 0$ and $B_k^0 = H_0(x^*) > 0$. Moreover, $x^* \in (0,y)$ if and only if $\bar{C}_k^1 < C_k^0$. We know that $G(y) = yH_1(y) - H_0(y) = y\frac{1-C_k^1}{\gamma} - \frac{1-C_k^0}{\gamma} = \frac{1}{\gamma}(C_k^0 - \bar{C}_k^1)$. There are two cases: (1) If $\bar{C}_k^1 > C_k^0$, then $G(y) < 0$, which means that $G(x^*) = 0 > G(y)$. Since $x^* > \underline{x}$ and $y > \underline{x}$ and $G(x)$ is increasing for $x > \underline{x}$, then $x^* > y$. Moreover, $B_k^1 = H(x^*) > H_1(y) = \frac{1-C_k^1}{\gamma(1-C_k)}(\frac{n_k-2}{n_k-1} - \bar{C}_k^1) > 0$. Furthermore, $B_k^0 = H_0(x^*) = x^*H(x^*) > 0$. (2) If $\bar{C}_k^1 \leq C_k^0$, then $G(y) \geq 0$, which means that $G(x^*) = 0 \leq G(y)$. Consequently, $x^* \leq$

y. Moreover, $B_k^0 = H_0(x^*) \geq H_0(y) = \frac{1}{\gamma}(\frac{n_k-2}{n_k-1} - C_k^0) > 0$. Furthermore, $B_k^1 = H_1(x^*) = \frac{1}{x^*}H_0(x^*) > 0$.

Since we have solved $B_k^1 > 0$ and $B_k^0 > 0$, then together with the fact that $C_{ks}^1, C_k^1, C_{ks}^0 \in [0,1]$, $\bar{C}_k^1, C_k^0 \in [0, \frac{n_k-2}{n_k-1}]$, we get the unique $a_k^1 = \frac{C_{ks}^1}{1-C_k^1}B_k^1 > 0$, $a_k^0 = \frac{C_{ks}^0}{1-C_k^0}B_k^0 > 0$ and $b_k^1 = \frac{c_k^1}{1-C_k^1}B_k^1 > 0$. Finally, by definition, we get the unique $\pi_k \in (0,1)$ and $\Pi_k \in (0,1)$.

Step 3: We show that (i) $\delta_k \in (0, \frac{y_{k1}}{y_{k2}})$, $\delta_k\delta_{-k} \in (0,1)$; (ii) $C_{ks}^1 < 1$, $C_{ks}^0 < 1$; (iii) if $C_{ks}^1 > 0$, then $\beta_k > 0$, $\zeta_k > 0$, $C_k^1 \in (0,1)$, $C_k^0 > 0$, and $c_k^1 > 0$.

First, we prove that $\delta_k \in (0, \frac{y_{k1}}{y_{k2}})$, $\delta_k\delta_{-k} \in (0,1)$ and $\delta_{-k} < \frac{y_{k2}}{y_{k1}}$. By equation (29) and $\frac{C_{ks}^1}{\pi_k} > 0$, we get $\delta_k < \frac{y_{k1}}{y_{k2}}$. Moreover, by $C_k^* < 1$, $y_{k1} > 0$, $y_{k2} > 0$, $y_{k3} > 0$, we get $\delta_k > 0$. Consequently, $\delta_k\delta_{-k} < \frac{y_{k1}}{y_{k2}}\frac{y_{-k1}}{y_{-k2}}$. Since $\eta_k y_{k1} < y_{-k2}$, $\eta_{-k}y_{-k1} < y_{k2}$ and $\eta_k\eta_{-k} = 1$, then we get $\frac{y_{k1}}{y_{k2}}\frac{y_{-k1}}{y_{-k2}} < 1$. Therefore, $\delta_k\delta_{-k} < \frac{y_{k1}}{y_{k2}}\frac{y_{-k1}}{y_{-k2}} < 1$. Moreover, $\delta_{-k} < \frac{y_{-k1}}{y_{-k2}} < \frac{y_{k2}}{y_{k1}}$.

Second, we prove that $C_{ks}^1 < 1$ and $C_{ks}^0 < 1$. By (25), we get $1 - C_{ks}^1 = (1-C_k^*)(1 + \frac{(e_{k1}-e_{k2})y_{k2}}{(\kappa_k - e_{k1})y_{k3}}) = \frac{1-C_k^*}{y_{k3}}\Big(\rho_k(\rho_{-k} + e_{-k2}) - \phi^2 + (\rho_{-k} + e_{-k2})(e_{k2} - \frac{e_{k1}-e_{k2}}{\kappa_k - e_{k1}}e_{k1})\Big) > 0$, due to the fact that $\rho_k(\rho_{-k}+e_{-k2}) - \phi^2 > 0$ and $e_{k2} - \frac{e_{k1}-e_{k2}}{\kappa_k - e_{k1}}e_{k1} > 0$. In all, $C_{ks}^1 < 1$. By (27), $1 - C_{ks}^0 = \frac{1-C_k^*}{H_k^0}[\eta_k^2\delta_k^2\kappa_k(\rho_{-k}+e_{-k2}) + \kappa_k e_{k2} - e_{k3}^2 + (\rho_k + 2\phi\eta_k\delta_k)(\kappa_k - e_{k3}) + e_{k3}\phi\eta_k\delta_k] > 0$, due to the fact that $\kappa_k e_{k2} - e_{k3}^2 > 0$ and $\kappa_k - e_{k3} > 0$. Therefore, $C_{ks}^0 < 1$.

Third, we prove that $\beta_k > 0$ and $\zeta_k > 0$. By (30), $\delta_{-k} < \frac{y_{k2}}{y_{k1}}$ and $C_{ks}^1 > 0$, $\pi_k > 0$ and $\Pi_k > 0$, then we have $\beta_k > 0$. Since $\delta_k\delta_{-k} < 1$, then $\zeta_k = \frac{\beta_k}{1-\delta_k\delta_{-k}} > 0$.

Next, we prove that $C_k^1 \in (0,1)$. By equation (26) and (30), we get

$$C_k^1 = \frac{(1-C_k^*)(y_{k2} - y_{k1}\delta_{-k})}{y_{k3}\beta_k} = \frac{(1-C_k^*)(\frac{y_{k2}}{y_{k3}} - \frac{y_{k1}}{y_{k3}}\delta_{-k})}{(1-C_k^*)(\frac{y_{k2}}{y_{k3}} - \frac{y_{k1}}{y_{k3}}\delta_{-k}) + C_{ks}^1\frac{\Pi_k}{\pi_k}}.$$

Since $\delta_{-k} < \frac{y_{k2}}{y_{k1}}$ and $\beta_k > 0$, then $C_k^1 > 0$. Furthermore, $C_{ks}^1\frac{\Pi_k}{\pi_k} > 0$ guarantees that $C_k^1 < 1$.

Finally, we prove that $C_k^0 > 0$ and $c_k^1 > 0$. Since $H_k^0 > 0$ and $\kappa_k - e_{k3} > 0$, then $C_k^0 = \frac{(1-C_k^*)(\rho_k(\kappa_k - e_{k3}) + \eta_k\delta_k\kappa_k\phi)}{H_k^0\zeta_k} > 0$. By equation (26), $c_k^1 > 0$ is equivalent to $\delta_k < \frac{y_{k1}}{y_{k2}}$, which holds.

Step 4: If n_k is large or σ_k^2 is small, then $\bar{C}_k^1, C_k^0 \in [0, \frac{n_k-2}{n_k-1}]$ and $C_{ks}^1, C_{ks}^0, C_k^1 \in [0,1]$.

First, we prove that $\bar{C}_k^1 \in [0, \frac{n_k-2}{n_k-1}]$. From (26), we get $\bar{C}_k^1 = C_k^1 + \frac{\delta_{-k}\zeta_{-k}}{\zeta_k}c_k^1 = \frac{1-C_k^*}{\zeta_k}\frac{y_{k2}}{y_{k3}}$. Hence, we need to prove that $\zeta_k > \frac{n_k-1}{n_k-2}\frac{(1-C_k^*)y_{k2}}{y_{k3}}$. By (30) and $\zeta_k = \frac{\beta}{1-\delta_k\delta_{-k}}$, we need to prove that

$$\frac{(1-C_k^*)(\frac{y_{k2}}{y_{k3}} - \frac{y_{k1}}{y_{k3}}\delta_{-k}) + \frac{\Pi_k C_{ks}^1}{\pi_k}}{1-\delta_k\delta_{-k}} > \frac{n_k-1}{n_k-2}\frac{(1-C_k^*)y_{k2}}{y_{k3}}. \tag{31}$$

We check the case $\delta_{-k}=0$, then (31) is equivalent to

$$n_k - 2 > \frac{1-C_k^*}{C_{ks}^1}\frac{y_{k2}}{y_{k3}}\frac{\pi_k}{\Pi_k}. \tag{32}$$

As $n_k \to +\infty$ or $\sigma_k^2 \to 0$, we get $\lim \frac{y_{k2}}{y_{k3}} = 1$, $\lim \frac{\pi_k}{\Pi_k} = 1$ and $\lim C_{ks}^1 = C_k^*$. The right hand side of (32) converges to $\frac{1-C_k^*}{C_k^*} = \frac{\sigma_k^2}{1-\rho_k}$. Moreover, if we ignore the first order term $o(1)$, then (32) is equivalent to and $\sigma_k^2 < (1-\rho_k)(n_k-2)$. Since $\lim \delta_{-k} = 0$, then (31) also holds.

Second, we prove that $C_k^0 < \frac{n_k-2}{n_k-1}$. Since $C_k^0 < \frac{1-C_k^*}{\zeta_k}$, then we need to prove that $\zeta_k > \frac{n_k-1}{n_k-2}(1-C_k^*)$. Since $\lim \frac{y_{k2}}{y_{k3}} = 1$, then the condition to guarantee $C_k^0 < \frac{n_k-2}{n_k-1}$ also guarantees that $\bar{C}_k^1 < \frac{n_k-2}{n_k-1}$.

Finally, we prove that $C_{ks}^1, C_{ks}^0, C_k^1 \in [0,1]$. By Step 3, we only need to prove that $C_{ks}^1 > 0$ and $C_{ks}^0 > 0$, which holds since $C_{ks}^1 \to C_k^* > 0$, $C_{ks}^0 \to C_k^* > 0$, as $n_k \to +\infty$ or $\sigma_k^2 \to 0$.

Step 5: Since the mapping constructed above is continuous and maps from a convex compact set to a convex compact set. By Brouwer's fixed point theorem, there exists a fixed point. By construction, we obtain the equilibrium parameters, a_k^1, B_k^1, a_k^0, B_k^0, b_k^1, $C_{ks}^1, C_k^1, c_k^1, C_{ks}^0$, and C_k^0 that satisfy the properties stated in Theorem 4.4.

Step 6: The $\phi < 0$ case. If $\phi < 0$, then y_{k1}, δ_k, ϕ, b_k^1, and c_k^1 are all negative. Therefore, when we replace δ_k, ϕ, b_k^1, c_k^1 with $|\delta_k|$, $|\phi|$, $|b_k^1|$, $|c_k^1|$, all the analysis above remains valid.

APPENDIX C. PROOFS OF THE RESULTS IN SECTION 5

C.1. **Proof of Propositions 5.1 and 5.2.** We prove these two results in five steps.

Step 1: The limit as $N \to +\infty$.

Theorem 4.4 shows that $\delta_k \le \frac{y_{k1}}{y_{k2}}$. As $N \to +\infty$, $y_{k1} \to 0$, and hence $\delta_k \to 0$. Consequently, $\lim c_k^1 = 0$, $\lim C_{ks}^1 = C_{ks}^0 = C_k^*$, $\lim C_k^1 = \lim C_k^0 = 1 - C_k^*$, $\lim \lambda_k^1 = \lim \lambda_k^0 = 0$, $\lim a_k^1 = \lim a_k^0 = \lim B_k^1 = \lim B_k^0 = \frac{C_k^*}{\gamma}$. Moreover, $\lim \beta_k = \lim \zeta_k = 1$, $\lim \pi_k = \alpha_k$.

Step 2: The coefficients C_{ks}^1, C_k^1, c_k^1, C_{ks}^0, and C_k^0, ignoring the term $o(\frac{1}{N^2})$.

In equilibrium, $a_k^1 - a_k^0 = o(\frac{1}{N})$ (which is verified in Step 7). Consequently,

$$\pi_k - \alpha_k = \frac{\alpha_k(1-\alpha_k)(a_k^1 - a_k^0)}{\alpha_k a_k^1 + (1-\alpha_k)a_k^0} = o(\frac{1}{N}).$$

Substituting $\pi_k = \alpha_k + o(\frac{1}{N})$ into (24), we get (by ignoring $o(\frac{1}{N^2})$)

$$e_{k1} = e_{k2} = e_{k3} = \frac{\kappa_k}{n_k},\ y_{k1} = \frac{\kappa_k \phi}{\eta_k n_k},$$
$$y_{k2} = \rho_k(\rho_{-k} + \frac{\kappa_{-k}}{n_{-k}}) - \phi^2,\ y_{k3} = (\rho_k + \frac{\kappa_k}{n_k})(\rho_{-k} + \frac{\kappa_{-k}}{n_{-k}}) - \phi^2. \tag{33}$$

Substituting (33) into (25) and (27) and ignoring $o(\frac{1}{N^2})$, we get

$$C^1_{ks} = C^*_k,\ C^1_k = \frac{(1-C^*_k)(y_{k2} - y_{k1}\delta_{-k})}{y_{k3}\beta_k},\ c^1_k = \frac{(1-C^*_k)(y_{k1} - y_{k2}\delta_k)}{y_{k3}\beta_{-k}},$$
$$C^0_{ks} = C^*_k - \frac{(1-C^*_k)\eta_k^2\delta_k(y_{k1} - y_{k2}\delta_k)}{H^0_k},\ C^0_k = \frac{(1-C^*_k)\kappa_k(\frac{n_k-1}{n_k}\rho_k + \phi\eta_k\delta_k)}{H^0_k\zeta_k},$$
$$H^0_k = \eta_k^2\delta_k^2((\rho_k+\kappa_k)(\rho_{-k} + \frac{\kappa_{-k}}{n_{-k}}) - \phi^2) + \frac{n_k-1}{n_k}\kappa_k(\rho_k + \frac{\kappa_k}{n_k}) + 2\frac{n_k-1}{n_k}\kappa_k\phi\eta_k\delta_k. \tag{34}$$

Step 3: The parameter $|\delta_k|$.

Since $C^1_{ks} = C^*_k + o(\frac{1}{N^2})$ and $\pi_k = \alpha_k + o(\frac{1}{N})$, then by (29), we get $\delta_k = \frac{(1-C^*_k)y_{k1}}{(1-C^*_k)y_{k2} + \frac{C^1_{ks}}{\pi_k}y_{k3}} + o(\frac{1}{N^2}) = \frac{y_{k1}}{y_{k2} + \frac{C^*_k}{\alpha_k(1-C^*_k)}y_{k3}} + o(\frac{1}{N^2}) = \frac{y_{k1}}{y_{k2}}\frac{\alpha_k}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}} + o(\frac{1}{N})$, which holds by $y_{k3} = y_{k2} + o(1)$ and $\frac{C^*_k}{1-C^*_k} = \frac{1-\rho_k}{\sigma_k^2}$. Since $\frac{y_{k1}}{y_{k2}} = \frac{\kappa_k\phi}{\eta_k(\rho_k\rho_{-k} - \phi^2)}\frac{1}{n_k}$, we have

$$|\delta_k| = \frac{\delta^*_k}{n_k} + o(\frac{1}{N}), \quad \text{and} \quad \delta^*_k = \frac{\kappa_k|\phi|}{\eta_k(\rho_k\rho_{-k} - \phi^2)}\frac{\alpha_k}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}} > 0. \tag{35}$$

Step 4: The coefficients $|c^1_k|$, $C^1_{ks} - C^0_{ks}$, and $C^0_k - C^1_k$.

By $\beta_{-k} = 1 + o(1)$ and $y_{k3} = y_{k2} + o(1)$, we have $c^1_k = \frac{(1-C^*_k)(y_{k1} - y_{k2}\delta_k)}{y_{k3}\beta_{-k}} = (1-C^*_k)(\frac{y_{k1}}{y_{k2}} - \delta_k) + o(\frac{1}{N})$. By (24) and (35) and $y_{k2} = \rho_k\rho_{-k} - \phi^2 + o(1)$,

$$|c^1_k| = \frac{c^*_k}{n_k} + o(\frac{1}{N}) \quad \text{and} \quad c^*_k = \frac{(1-\rho_k)|\phi|}{\eta_k(\rho_k\rho_{-k} - \phi^2)}\frac{1}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}} > 0. \tag{36}$$

Since $H^0_k = \kappa_k\rho_k + o(1)$, then $C^1_{ks} - C^0_{ks} = \frac{(1-C^*_k)\eta_k^2\delta_k(y_{k1} - y_{k2}\delta_k)}{H^0_k} = \frac{(1-C^*_k)\eta_k^2\delta_k(y_{k1} - y_{k2}\delta_k)}{\kappa_k\rho_k} + o(\frac{1}{N^2})$. By (24) and (35),

$$C^1_{ks} - C^0_{ks} = \frac{\Delta_{ks}}{n_k^2} + o(\frac{1}{N^2}) \quad \text{and} \quad \Delta_{ks} = \frac{(1-\rho_k)\phi^2}{\rho_k(\rho_k\rho_{-k} - \phi^2)}\frac{\alpha_k}{(\alpha_k + \frac{1-\rho_k}{\sigma_k^2})^2} > 0. \tag{37}$$

Since $H^0_k = \frac{n_k-1}{n_k}\kappa_k(\rho_k + \frac{\kappa_k}{n_k}) + 2\frac{n_k-1}{n_k}\kappa_k\phi\eta_k\delta_k + o(\frac{1}{N})$, then

$$C^0_k = (1-C^*_k)(1 - \frac{\frac{\kappa_k}{n_k} + \phi\eta_k\delta_k}{\rho_k})\frac{1}{\beta_k} + o(\frac{1}{N}) \tag{38}$$
$$C^1_k = \frac{(1-C^*_k)y_{k2}}{y_{k3}}\frac{1}{\beta_k} + o(\frac{1}{N}) = (1-C^*_k)(1 - \frac{\frac{\kappa_k}{n_k}(\rho_{-k} + \frac{\kappa_{-k}}{n_{-k}})}{\rho_k(\rho_{-k} + \frac{\kappa_{-k}}{n_{-k}}) - \phi^2})\frac{1}{\beta_k} + o(\frac{1}{N}).$$
$$C^0_k - C^1_k = \frac{(1-C^*_k)\eta_k\phi}{\rho_k\beta_k}(\frac{y_{k1}}{y_{k2}} - \delta_k) + o(\frac{1}{N}).$$

By (38), (24) and (35), we obtain

$$C_k^0 - C_k^1 = \frac{\Delta_k}{n_k} + o(\frac{1}{N}) \quad \text{and} \quad \Delta_k = \frac{(1-\rho_k)\phi^2}{\rho_k(\rho_k\rho_{-k} - \phi^2)} \frac{1}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}} > 0. \tag{39}$$

Step 5: The parameters λ_k^1 and λ_k^0.

Since $b_k^1 = o(1)$ and $b_{-k} = o(1)$ and $B_k - B_k^1 = o(1)$, then $(\lambda_k^1)^{-1} = (\alpha_k n_k - 1)(B_k^1 - b_k^1 \frac{b_{-k}}{B_{-k}}) + (1-\alpha_k)n_k B_k^0 = n_k B_k - B_k^1 + o(1) = (n_k - 1)B_k + o(1)$, which implies that

$$\lambda_k^1 = \frac{1}{(n_k-1)B_k} + o(\frac{1}{N^2}), \;\; \lambda_k^0 = \frac{1}{(n_k-1)B_k} + o(\frac{1}{N^2}) \tag{40}$$

The result for λ_k^0 follows by the same logic. It is clear that $\lambda_k^1 = o(1)$ and $\lambda_k^0 = o(1)$.

Step 6: The coefficients $a_k^1 - a_k^0$, $B_k^1 - B_k^0$, and $|b_k^1|$.

Since $\lambda_k^1 = o(1)$ and $c_k^1 = o(1)$, then $|b_k^1| = \frac{1}{\gamma + \lambda_k^1}|c_k^1| = \frac{1}{\gamma}|c_k^1| + o(\frac{1}{N})$. Similarly, $B_k^1 - B_k^0 = \frac{1}{\gamma}(C_k^0 - C_k^1) + o(\frac{1}{N})$, $a_k^1 - a_k^0 = \frac{1}{\gamma}(C_{ks}^1 - C_{ks}^0) + o(\frac{1}{N^2})$. Consequently, $a_k^1 - a_k^0 = \frac{a_k^*}{n_k} + o(\frac{1}{N^2})$, $B_k^1 - B_k^0 = \frac{B_k^*}{n_k} + o(\frac{1}{N})$, $|b_k^1| = \frac{b_k^*}{n_k} + o(\frac{1}{N})$, where $a_k^* = \frac{1}{\gamma}\Delta_{ks} > 0$, $B_k^* = \frac{1}{\gamma}\Delta_k > 0$, $b_k^* = \frac{1}{\gamma}c_k^* > 0$.

Step 7: We verify that $a_k^1 - a_k^0 = o(\frac{1}{N})$.

By Step 6, $a_k^1 - a_k^0 = \frac{1}{\gamma}(C_{ks}^1 - C_{ks}^0) + o(\frac{1}{N^2}) = \frac{1}{\gamma}\frac{\Delta_{ks}}{n_k^2} + o(\frac{1}{N^2}) = o(\frac{1}{N})$.

C.2. **Proof of Lemma 5.3.** First, (35) implies that $|\delta_k|$ is increasing in α_k.

Next, since $C_k^0 - C_k^1 = \frac{\Delta_k}{n_k} + o(\frac{1}{N})$ and $C_k^1 = 1 - C_k^* + o(1)$, then $\frac{C_{k0}^1 - C_k^1}{1 - C_k^1} = \frac{\Delta_k}{C_k^* n_k} + o(\frac{1}{N})$. Since $B_k^1 - B_k^0 = \frac{1}{\gamma}(C_k^0 - C_k^1) + o(\frac{1}{N})$ and $B_k^1 = \frac{1}{\gamma}(1 - C_k^0) + o(1)$, then $\frac{B_k^1 - B_k^0}{B_k^1} = \frac{C_k^0 - C_k^1}{1 - C_k^1} + o(\frac{1}{N})$. By (39), $\frac{B_k}{B_k^1} = 1 - (1-\alpha_k)\frac{B_k^1 - B_k^0}{B_k^1} = 1 - (1-\alpha_k)\frac{\Delta_k}{C_k^* n_k} + o(\frac{1}{N})$. Since $\Delta_k > 0$ is decreasing in α_k, we have that $\frac{B_k}{B_k^1}$ is increasing in α_k.

Next, from (30), we have

$$\beta_k = (1 - C_k^*)(\frac{y_{k2}}{y_{k3}} - \frac{y_{k1}}{y_{k3}}\delta_{-k}) + C_{ks}^1 \frac{\alpha_k}{\pi_k}\frac{B_k^1}{B_k} = \frac{(1-C_k^*)y_{k2}}{y_{k3}} + C_k^* \frac{B_k^1}{B_k} + o(\frac{1}{N}).$$

Since $\frac{B_k}{B_k^1}$ is increasing in α_k, then β_k is decreasing in α_k. Since $\zeta_k = \frac{\beta_k}{1 - \delta_k \delta_{-k}} = \beta_k + o(\frac{1}{N})$, then ζ_k is also decreasing in α_k.

Finally, since $\frac{B_k}{B_k^1} = 1 + o(1)$, then $\zeta_k = \beta_k + o(\frac{1}{N}) = (1 - C_k^*)\frac{y_{k2}}{y_{k3}} + C_k^* + o(1)$. Therefore, $|\delta_k \zeta_k| = |\frac{y_{k1}}{y_{k2}}| \frac{\alpha_k}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}}((1 - C_k^*)\frac{y_{k2}}{y_{k3}} + C_k^*) + o(\frac{1}{N})$, which is increasing in α_k.

C.3. **Proof of Proposition 5.4.** Since $\beta_k = \frac{\alpha_k a_k^1 + (1-\alpha_k)a_k^0}{\alpha_k B_k^1 + (1-\alpha_k)B_k^0} = \frac{C_k^*}{1 - (\alpha_k C_k^1 + (1-\alpha_k)C_k^0)} + o(\frac{1}{N^2})$, then $\beta_k = C_k^* + \beta_k(C_k^0 - \alpha_k(C_k^0 - C_k^1))$.

Substituting (38) into this expression, we get

$$\beta_k = 1-(1-C_k^*)\frac{\frac{\kappa_k}{n_k}+\phi\eta_k\delta_k}{\rho_k} - \alpha_k\frac{(1-C_k^*)\eta_k\phi}{\rho_k}\left(\frac{y_{k1}}{y_{k2}}-\delta_k\right) = 1-\frac{\frac{\kappa_k}{n_k}+\phi\eta_k\delta_k}{\rho_k}+\frac{\kappa_k C_k^*}{\rho_k n_k}+o\left(\frac{1}{N}\right).$$

Substituting the above expression of β_k into (38), we obtain

$$C_k^0 = (1-C_k^*)\left(1-\frac{\kappa_k C_k^*}{\rho_k n_k}\right)+o\left(\frac{1}{N}\right). \tag{41}$$

Therefore, C_k^0 is independent of α_k.

By (39), $C_k^0 - C_k^1$ is decreasing in α_k. Since C_k^0 is independent of α_k, then C_k^1 is increasing in α_k. By (36), $|c_k^1|$ is decreasing in α_k. Since $C_{ks}^1 = C_k^* + o(\frac{1}{N^2})$, then $\frac{\partial C_{ks}^1}{\partial \alpha_k} = 0$. By (37), $C_{ks}^0 - C_{ks}^1 = -\frac{\Delta_{ks}}{n_k^2} + o(\frac{1}{N^2})$. Thus, $\frac{\partial \Delta_{ks}}{\partial \alpha_k} > 0$ if and only if $\alpha_k < \frac{1-\rho_k}{\sigma_k^2}$. Therefore, $\frac{\partial C_{ks}^0}{\partial \alpha_k} < 0$ if and only if $\alpha_k < \frac{1-\rho_k}{\sigma_k^2}$.

C.4. **Proof of Lemma 5.5.** By (39), $\alpha_k(B_k^1 - B_k^0) = \frac{\alpha_k}{\gamma}(C_k^0 - C_k^1) + o(\frac{1}{N}) = \frac{1}{\gamma}\frac{\Delta_k\alpha_k}{n_k} + o(\frac{1}{N})$, which is increasing in α_k. Together with $\frac{\partial B_k^0}{\partial \alpha_k} = 0$, we have $B_k = \alpha_k B_k^1 + (1-\alpha_k)B_k^0 = B_k^0 + \alpha_k(B_k^1 - B_k^0)$ is increasing in α_k. By (40), we have $\lambda_k^1 = \frac{1}{(n_k-1)B_k} + o(\frac{1}{N^2})$. Since $\frac{\partial B_k}{\partial \alpha_k} > 0$, then $\frac{\partial \lambda_k^1}{\partial \alpha_k} < 0$. Similarly, $\frac{\partial \lambda_k^0}{\partial \alpha_k} < 0$.

C.5. **Proof of Proposition 5.6.** We first prove that a_k^1 and a_k^0 are increasing in α_k. We know that $a_k^1 = \frac{1}{\gamma+\lambda_k^1}C_{ks}^1$. Since λ_k^1 is decreasing in α_k and C_{ks}^1 is independent of α_k, then a_k^1 is increasing in α_k. Since $a_k^0 = \frac{C_{ks}^0}{\gamma+\lambda_k^0}$, then

$$a_k^0 - \frac{C_k^*}{\gamma} = \frac{C_{ks}^0 - C_k^*}{\gamma} + \left(\frac{1}{\gamma+\lambda_k^0} - \frac{1}{\gamma}\right)C_{ks}^0 = \frac{1}{\gamma}\left(C_{ks}^0 - C_k^* - \frac{C_k^*}{1+\gamma(\lambda_k^0)^{-1}}\right) + o\left(\frac{1}{N^2}\right) \tag{42}$$

Since $(\lambda_k^0)^{-1} = (n_k-1)B_k + o(\frac{1}{N^2})$, $B_k = \alpha_k B_k^1 + (1-\alpha_k)B_k^0$, $\gamma(B_k^1 - B_k^0) = \frac{\Delta_k}{n_k} + o(\frac{1}{N})$, $\gamma B_k^0 = C_k^* + o(1)$, and $\gamma B_k = C_k^* + o(1)$, then

$$\frac{1}{1+\gamma(\lambda_k^0)^{-1}} = \frac{1}{1+\gamma(n_k-1)B_k} = \frac{1}{1+\gamma(n_k-1)B_k^0} - \frac{\gamma(n_k-1)\alpha_k(B_k^1-B_k^0)}{(1+\gamma(n_k-1)B_k)(1+\gamma(n_k-1)B_k^0)}$$
$$= \frac{1}{1+\gamma(n_k-1)B_k^0} - \frac{1}{(C_k^*)^2}\frac{\alpha_k\Delta_k}{n_k^2} + o\left(\frac{1}{N^2}\right).$$

Substituting the above expression and (37) into (42), we obtain

$$a_k^0 = \frac{C_k^*}{\gamma} + \frac{1}{\gamma}\left(-\frac{C_k^*}{1+\gamma(n_k-1)B_k^0} + \frac{1}{n_k^2}\left(\frac{\alpha_k\Delta_k}{C_k^*} - \Delta_{ks}\right)\right) + o\left(\frac{1}{N^2}\right) \tag{43}$$

where

$$\frac{\alpha_k \Delta_k}{C_k^*} - \Delta_{ks} = \frac{(1-\rho_k)\phi^2}{\rho_k(\rho_k\rho_{-k}-\phi^2)} \frac{\alpha_k}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}} (\frac{1}{C_k^*} - \frac{1}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}}).$$

Since $C_k^* < \frac{1-\rho_k}{\sigma_k^2}$, then $\frac{1}{C_k^*} - \frac{1}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}} > 0$ is increasing in α_k. Since $\frac{\alpha_k}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}}$ is increasing in α_k and B_k^0 is independent of α_k, then a_k^0 is increasing in α_k.

Then we prove that B_k^1 and $|b_k^1|$ are decreasing in α_k and B_k^0 is independent of α_k. Since $B_k = \frac{C_k^*}{\gamma} + o(1)$, then $\lambda_k^1 = \frac{1}{(n_k-1)B_k} + o(\frac{1}{N^2}) = \frac{\gamma}{(n_k-1)C_k^*} + o(\frac{1}{N})$. Therefore, λ_k^1 is independent of α_k if we ignore the term $o(\frac{1}{N})$. By the same logic, λ_k^0 is independent of α_k if we ignore the term $o(\frac{1}{N})$. Consequently, B_k^1, $|b_k^1|$ and B_k^0 are completely determined by $1 - C_k^1$, $|c_k^1|$ and $1 - C_k^0$, in term of comparative statics of α_k. Since $1 - C_k^1$ and $|c_k^1|$ are decreasing in α_k, so are B_k^1 and $|b_k^1|$. Since C_k^0 is independent of α_k, so is B_k^0.

C.6. **Proof of Lemma 5.7.** Define $a_k = \alpha_k a_k^1 + (1-\alpha_k)a_k^0$, $B_k = \alpha_k B_k^1 + (1-\alpha_k)B_k^0$, and $b_k = \alpha_k b_k^1$. We first solve for $E(x_{k,i}^1)^2$ and $E(x_{k,i}^0)^2$. Since $a_k^1 - a_k = o(\frac{1}{N})$ and $a_k\bar{s}_k - B_k p_k + b_k p_{-k} = 0$, then $x_{k,i}^1 = a_k^1(s_k^i - \bar{s}_k) + (B_k - B_k^1)p_k - (b_k - b_k^1)p_{-k} + o(\frac{1}{N})$.

Since $p_k = \zeta_k(\bar{s}_k + \delta_k \bar{s}_{-k}) + o(\frac{1}{N}) = \bar{s}_k + o(1)$, $B_k - B_k^1 = -(1-\alpha_k)(B_k^1 - B_k^0) = o(\frac{1}{N})$, and $b_k - b_k^1 = -(1-\alpha_k)b_k^1 = o(1)$, we have

$$x_{k,i}^1 = a_k^1(s_k^i - \bar{s}_k) - (1-\alpha_k)(B_k^1 - B_k^0)\bar{s}_k + (1-\alpha_k)b_k^1\bar{s}_{-k} + o(\frac{1}{N}). \tag{44}$$

Since $E(s_k^i - \bar{s}_k) = E(s_k^i - \bar{s}_k)\bar{s}_k = E(s_k^i - \bar{s}_k)\bar{s}_{-k} = 0$, then (44) implies that

$$E(x_{k,i}^1)^2 = (a_k^1)^2\mathbb{E}(s_k^i - \bar{s}_k)^2 + (1-\alpha_k^2)\mathbb{E}((B_k^1 - B_k^0)\bar{s}_k - b_k^1\bar{s}_{-k})^2 + o(\frac{1}{N^2}).$$

By $(a_k^1, B_k^1 - B_k^0, b_k^1) = (\gamma + \lambda_k^1)^{-1}(C_{ks}^1, C_k^0 - C_k^1, c_k^1) + o(\frac{1}{N^2})$, we have

$$E(x_{k,i}^1)^2 = \frac{1}{(\gamma+\lambda_k^1)^2}\Big((C_{ks}^1)^2\mathbb{E}(s_k^i - \bar{s}_k)^2 + (1-\alpha_k)^2 H_k\Big) + o(\frac{1}{N^2}). \tag{45}$$

where $H_k = \mathbb{E}((C_k^1 - C_k^0)\bar{s}_k + c_k^1\bar{s}_{-k})^2$. Similarly, we get

$$E(x_{k,i}^0)^2 = \frac{1}{(\gamma+\lambda_k^0)^2}\Big((C_{ks}^0)^2\mathbb{E}(s_k^i - \bar{s}_k)^2 + \alpha_k^2 H_k\Big) + o(\frac{1}{N^2}). \tag{46}$$

Next, we solve for W_k^1 and W_k^0. Since $E(\theta_k^i|s_k^i, p_k, p_{-k}) - p_k = (\gamma + \lambda_k^1)x_{k,i}^1$, then

$$W_k^1 = \mathbb{E}[x_{k,i}^1(E(\theta_k^i|s_k^i, p_k, p_{-k}) - p_k) - \frac{\gamma}{2}(x_{k,i}^1)^2] = (\frac{\gamma}{2} + \lambda_k^1)E(x_{k,i}^1)^2.$$

Substituting (45) into the above expression, we get

$$W_k^1 = \frac{\frac{\gamma}{2}+\lambda_k^1}{(\gamma+\lambda_k^1)^2}\Big((C_{ks}^1)^2\mathbb{E}(s_k^i - \bar{s}_k)^2 + (1-\alpha_k)^2\mathbb{E}((C_k^1 - C_k^0)\bar{s}_k + c_k^1\bar{s}_{-k})^2\Big) + o(\frac{1}{N^2}). \tag{47}$$

Similarly,

$$W_k^0 = \frac{\frac{\gamma}{2} + \lambda_k^0}{(\gamma + \lambda_k^0)^2} \Big((C_{ks}^0)^2 \mathbb{E}(s_k^i - \bar{s}_k)^2 + \alpha_k^2 \mathbb{E}((C_k^1 - C_k^0)\bar{s}_k + c_k^1 \bar{s}_{-k})^2 \Big) + o(\frac{1}{N^2}). \tag{48}$$

Finally, we estimate $\frac{\frac{\gamma}{2}+\lambda_k^1}{(\gamma+\lambda_k^1)^2}$ and $\frac{\frac{\gamma}{2}+\lambda_k^0}{(\gamma+\lambda_k^0)^2}$. By $\lambda_k^1 = \frac{1}{(n_k-1)B_k} + o(\frac{1}{N^2})$, $\lambda_k^1 = o(1)$ and $(\gamma+\lambda_k^1)B_k = C_k^* + o(1)$, we get $\frac{\frac{\gamma}{2}+\lambda_k^1}{(\gamma+\lambda_k^1)^2} = \frac{1}{2\gamma}(1 - (\frac{\lambda_k^1}{\gamma+\lambda_k^1})^2) = \frac{1}{2\gamma}(1 - (\frac{1}{(n_k-1)B_k(\gamma+\lambda_k^1)})^2) + o(\frac{1}{N^2}) = \frac{1}{2\gamma}(1 - (\frac{1}{(n_k-1)C_k^*})^2) + o(\frac{1}{N^2})$. Similarly, $\frac{\frac{\gamma}{2}+\lambda_k^0}{(\gamma+\lambda_k^0)^2} = \frac{1}{2\gamma}(1 - (\frac{1}{(n_k-1)C_k^*})^2) + o(\frac{1}{N^2})$.

C.7. **Proof of Proposition 5.8.** We first compute two preliminary estimations.

Since $\frac{C_k^1 - C_k^0}{c_k^1} = \frac{\phi\eta_k}{\rho_k} + o(\frac{1}{N^2})$, $\lim E(\bar{s}_k)^2 = \rho_k \sigma_{\theta_k}^2$, and $\lim E(\bar{s}_{-k})^2 = \rho_{-k}\sigma_{\theta_{-k}}^2$, then

$$\begin{aligned} &\mathbb{E}(s_k^i - \bar{s}_k)^2 = E(s_k^i)^2 - E(\bar{s}_k)^2 = X_k, \\ &E[(C_k^1 - C_k^0)\bar{s}_k + c_k^1\bar{s}_{-k}]^2 = (c_k^1)^2 E(\frac{\phi\eta_k}{\rho_k}\bar{s}_k - \bar{s}_{-k})^2 = (c_k^1)^2 Y_k + o(\frac{1}{N^2}), \end{aligned} \tag{49}$$

where $X_k \equiv \frac{n_k-1}{n_k}(1 - \rho_k + \sigma_k^2)\sigma_{\theta_k}^2$ and $Y_k \equiv \frac{\rho_k\rho_{-k} - \phi^2}{\rho_k}\sigma_{\theta_{-k}}^2$. By (47) and (49), we have

$$W_k^1 = \frac{\frac{\gamma}{2} + \lambda_k^1}{(\gamma + \lambda_k^1)^2} \left((C_k^*)^2 X_k + \frac{1}{n_k^2} \frac{(1-\rho_k)^2\phi^2\sigma_{\theta_k}^2}{\rho_k(\rho_k\rho_{-k} - \phi^2)} \frac{(1-\alpha_k)^2}{(\alpha_k + \frac{1-\rho_k}{\sigma_k^2})^2} \right) + o(\frac{1}{N^2}), \tag{50}$$

which holds since $C_{ks}^1 = C_k^* + o(\frac{1}{N^2})$ and (36). Thus, we have $\frac{dW_k^1}{d\alpha_k} < 0$.

By (48) and (49), we have

$$W_k^0 = \frac{\frac{\gamma}{2} + \lambda_k^0}{(\gamma + \lambda_k^0)^2} \left((C_k^*)^2 X_k - \frac{1}{n_k^2} \frac{(1-\rho_k)^2\phi^2\sigma_{\theta_k}^2}{\rho_k(\rho_k\rho_{-k} - \phi^2)} \frac{2\alpha_k - \alpha_k^2}{(\alpha_k + \frac{1-\rho_k}{\sigma_k^2})^2} \right) + o(\frac{1}{N^2}), \tag{51}$$

which holds because of (36) and (37). Consequently, $\frac{dW_k^0}{d\alpha_k} < 0$ if and only if $\alpha_k < \frac{1-\rho_k}{1-\rho_k+\sigma_k^2}$.

Then by (50) and (51), we obtain

$$W_k^1 - W_k^0 = \frac{1}{2\gamma n_k^2} \frac{(1-\rho_k)^2\phi^2\sigma_{\theta_k}^2}{\rho_k(\rho_k\rho_{-k} - \phi^2)} \frac{1}{(\alpha_k + \frac{1-\rho_k}{\sigma_k^2})^2} + o(\frac{1}{N^2}). \tag{52}$$

Thus, $W_k^1 - W_k^0$ is decreasing in α_k and $W_k^1 - W_k^0 > 0$.

By (50), (51), and $\frac{\frac{\gamma}{2}+\lambda_k^1}{(\gamma+\lambda_k^1)^2} = \frac{\frac{\gamma}{2}+\lambda_k^0}{(\gamma+\lambda_k^0)^2} + o(\frac{1}{N^2})$, we have

$$W_k = \frac{\frac{\gamma}{2} + \lambda_k^1}{(\gamma + \lambda_k^1)^2} \left((C_k^*)^2 X_k - \frac{1}{n_k^2} \frac{(1-\rho_k)^2\phi^2\sigma_{\theta_k}^2}{\rho_k(\rho_k\rho_{-k} - \phi^2)} \frac{\alpha_k(1-\alpha_k)}{(\alpha_k + \frac{1-\rho_k}{\sigma_k^2})^2} \right) + o(\frac{1}{N^2}). \tag{53}$$

Hence, W_k is decreasing in α_k if and only if $\alpha_k < \hat{\alpha}_k = \frac{1-\rho_k}{2(1-\rho_k)+\sigma_k^2}$. Finally, by (53), W_k is the same when $\alpha_k = 1$ or $\alpha_k = 0$.

C.8. **Proof of Proposition 5.9.** Let $\mathrm{Var}_k^1 = \mathrm{Var}(\theta_k^i | s_k^i, p_k, p_{-k})$, $\mathrm{Var}_k^0 = \mathrm{Var}(\theta_k^i | s_k^i, p_k)$. Let $X = \theta_k^i$, $Y = (s_k^i, \bar{s}_k, \bar{s}_{-k})$ and $Y_1 = (s_k^i, p_k, p_{-k})$ and $Y_0 = (s_k^i, p_k)$.

Since $\Sigma_{X,Y}\Sigma_{Y,Y}^{-1} = (C_k^*, (1-C_k^*)\frac{y_{k2}}{y_{k3}}, (1-C_k^*)\frac{y_{k1}}{y_{k3}})$ and $\Sigma_{Y,X} = (1, \rho_k + \frac{1-\rho_k}{n_k}, \phi)\sigma_{\theta_k}^2$, then by the projection theorem,

$$\mathrm{Var}_k^1 = \mathrm{Var}[X|Y] = \mathrm{Var}(X) - \Sigma_{X,Y}\Sigma_{Y,Y}^{-1}\Sigma_{Y,X} = \frac{\sigma_k^2}{\kappa_k}(1-\rho_k^1)\sigma_{\theta_k}^2, \tag{54}$$

where $\rho_k^1 = \rho_k - \frac{\sigma_k^2}{n_k}\frac{y_{k2}}{y_{k3}} = \rho_k - \frac{\sigma_k^2}{n_k} + \frac{\sigma_k^2}{n_k^2}\frac{\rho_{-k}\kappa_k}{\rho_k\rho_{-k}-\phi^2} + o(\frac{1}{N^2})$ is independent of α_k. Therefore, $\tau_k^1 = \mathrm{Var}(\theta_k^i) - \mathrm{Var}_k^1$ is independent of α_k.

Since $\Sigma_{X,Y_0}\Sigma_{Y_0,Y_0}^{-1} = (C_{ks}^0, C_k^0)$ and $\Sigma_{Y_0,X} = (\mathrm{Cov}(\theta_k^i, s_k^i), \mathrm{Cov}(\theta_k^i, p_k))$ (see Lemma 4.3), then by the projection theorem,

$$\mathrm{Var}_k^0 = \mathrm{Var}[X|Y_0] = \mathrm{Var}(X) - \Sigma_{X,Y_0}\Sigma_{Y_0,Y_0}^{-1}\Sigma_{Y_0,X} = \frac{\sigma_k^2}{\kappa_k}(1-\rho_k^0)\sigma_{\theta_k}^2, \tag{55}$$

where $\rho_k^0 = \rho_k - \frac{\sigma_k^2}{n_k} + \frac{\sigma_k^2}{n_k^2}(\frac{\kappa_k}{\rho_k} + \frac{\kappa_k}{\rho_k}\frac{\phi^2}{\rho_k\rho_{-k}-\phi^2}\frac{\alpha_k}{\alpha_k+\frac{1-\rho_k}{\sigma_k^2}}(2 - \frac{\alpha_k}{\alpha_k+\frac{1-\rho_k}{\sigma_k^2}})) + o(\frac{1}{N^2})$ is increasing in α_k. Since $\tau_k^0 = \mathrm{Var}(\theta_k^i) - \mathrm{Var}_k^0$ is increasing in ρ_k^0, then τ_k^0 is increasing in α_k.

C.9. **Proof of Proposition 5.10.** Let $E_k^1 = \mathbb{E}(\theta_k^i | s_k^i, p_k, p_{-k})$ and $E_k^0 = \mathbb{E}(\theta_k^i | s_k^i, p_k)$. By (54) and (55), we have

$$\tau_k^1 - \tau_k^0 = \mathrm{Var}_k^0 - \mathrm{Var}_k^1 = \frac{\sigma_k^2}{\kappa_k}(\rho_k^1 - \rho_k^0)\sigma_{\theta_k}^2 = \frac{1}{n_k^2}\frac{(1-\rho_k)\phi^2}{\rho_k(\rho_k\rho_{-k}-\phi^2)}\frac{1}{(\alpha_k + \frac{1-\rho_k}{\sigma_k^2})^2}. \tag{56}$$

Comparing (52) with (56), we get

$$W_k^1 - W_k^0 = \frac{1}{2\gamma}\left(\tau_k^1 - \tau_k^0\right). \tag{57}$$

Next, we deduce $W_k^1 - W_k^0$ in a different way. By Lemma 5.7, we have

$$W_k^1 - W_k^0 = \frac{1}{2\gamma}\left(\mathbb{E}((E_k^1 - p_k)^2 - \mathbb{E}(E_k^0 - p_k)^2\right) + o(\frac{1}{N^2}). \tag{58}$$

By the law of total variance, $\mathrm{Var}(\theta_k^i) = \mathbb{E}(\mathrm{Var}_k^1) + \mathrm{Var}(E_k^1) = \mathrm{Var}_k^1 + \mathbb{E}(E_k^1)^2$, which holds since Var_k^1 is a constant by normality and $\mathbb{E}(E_k^1) = 0$. Therefore,

$$\mathbb{E}(E_k^1)^2 = \mathrm{Var}(\theta_k^i) - \mathrm{Var}_k^1 = \tau_k^1, \quad \mathbb{E}(E_k^0)^2 = \mathrm{Var}(\theta_k^i) - \mathrm{Var}_k^0 = \tau_k^0. \tag{59}$$

Thus, we have $\mathbb{E}((E_k^1 - p_k)^2 - \mathbb{E}(E_k^0 - p_k)^2 = \tau_k^1 - \tau_k^0 + 2\mathbb{E}(p_k(E_k^0 - E_k^1))$.

We then substitute the above equation to (58) and get

$$W_k^1 - W_k^0 = \frac{1}{2\gamma}\left(\tau_k^1 - \tau_k^0 + 2\mathbb{E}(p_k(E_k^0 - E_k^1))\right) + o(\frac{1}{N^2}). \tag{60}$$

From (57) and (60), we get $\mathbb{E}(p_k(E_k^0 - E_k^1)) = o(\frac{1}{N^2})$. Since $(1-\alpha_k)x_{k,i}^0 + \alpha_k x_{k,i}^1 = 0$, then $(1-\alpha_k)E_k^0 + \alpha_k E_k^1 - p_k = 0$. Therefore, we have

$$\mathbb{E}(p_k E_k^0) = \mathbb{E}(p_k E_k^1) = \mathrm{Var}(p_k). \tag{61}$$

Hence, by (59) and (61), $\mathbb{E}(E_k^1 - p_k)^2 = \mathbb{E}(E_k^1)^2 - \mathbb{E}(p_k)^2 = \tau_k^1 - \mathrm{Var}(p_k)$.

Similarly, $\mathbb{E}(E_k^0 - p_k)^2 = \tau_k^0 - \mathrm{Var}(p_k)$. Therefore, we get the decomposition of W_k^1 and W_k^0 in Proposition 5.10. In addition, since τ_k^1 is independent of α_k and W_k^1 is decreasing in α_k, $\mathrm{Var}(p_k)$ is increasing in α_k.

C.10. **Proof of Proposition 5.11.** By Step 2 of the proof of Lemma 5.7, we have

$$L_k^1 = (\frac{\gamma}{2} + \lambda_k^1)^{-1} W_k^1,\ L_k^0 = (\frac{\gamma}{2} + \lambda_k^0)^{-1} W_k^0,\ L_k = (\frac{\gamma}{2} + \lambda_k^1)^{-1} W_k.$$

Since $\lambda_k^1 = \frac{1}{(n_k-1)B_k} + o(\frac{1}{N^2})$ and $B_k^1 - B_k^0 = \frac{1}{\gamma}\frac{\Delta_k}{n_k} + o(\frac{1}{N})$, we have

$$\frac{1}{(\gamma + \lambda_k^1)^2} = \frac{1}{(\gamma + ((n_k-1)B_k^0)^{-1})^2} + \frac{2}{\gamma^2}\frac{\alpha_k \Delta_k}{(C_k^*)^2 n_k^2} + o(\frac{1}{N^2}). \tag{62}$$

Substituting $\mathbb{E}(s_k^i - \bar{s}_k)^2$ and $E[(C_k^1 - C_k^0)\bar{s}_k + c_k^1 \bar{s}_{-k}]^2$ into (45), we get

$$L_k^1 = E(x_{k,i}^1)^2 = \frac{1}{(\gamma + \lambda_k^1)^2}\left((C_k^*)^2 X_k + \frac{1}{n_k^2}(c_k^*(1-\alpha_k))^2 Y_k\right) + o(\frac{1}{N^2}). \tag{63}$$

We substitute Δ_k into (62), and then substitute c_k^* and (62) into (63), where we take out the term in (63) that is related to α_k:

$$\frac{1}{\gamma^2}\frac{(1-\rho_k)^2\phi^2\sigma_{\theta_k}^2}{\rho_k(\rho_k\rho_{-k} - \phi^2)n_k^2}\left(\frac{(1-\alpha_k)^2}{(\alpha_k + \frac{1-\rho_k}{\sigma_k^2})^2} + \frac{2(1+\frac{\sigma_k^2}{1-\rho_k})\alpha_k}{\alpha_k + \frac{1-\rho_k}{\sigma_k^2}}\right).$$

Taking derivative to α_k, we get that L_k^1 is decreasing in α_k if and only if $\alpha_k < \frac{1}{2}(1 - \frac{1-\rho_k}{\sigma_k^2})$.

Next, we study $L_k^0 = E(x_{k,i}^0)^2$. Substituting $\mathbb{E}(s_k^i - \bar{s}_k)^2$ and $E[(C_k^1 - C_k^0)\bar{s}_k + c_k^1\bar{s}_{-k}]^2$ into (46), we get

$$L_k^0 = E(x_{k,i}^0)^2 = (a_k^0)^2 X_k + \frac{1}{(\gamma + \lambda_k^0)^2}\frac{1}{n_k^2}(c_k^*\alpha_k)^2 Y_k + o(\frac{1}{N^2}). \tag{64}$$

Since a_k^0, $\frac{1}{(\gamma+\lambda_k^0)^2}$ and $c_k^*\alpha_k$ are increasing in α_k, then L_k^0 is increasing in α_k.

Then, we estimate $L_k \equiv \alpha_k E(x^1_{k,i})^2 + (1-\alpha_k)E(x^0_{k,i})^2$. From (53) and $W_k = (\frac{\gamma}{2} + \lambda^1_k)L_k$, we get

$$L_k = \frac{1}{(\gamma+\lambda^1_k)^2}\left((C^*_k)^2 X_k - \frac{1}{n_k^2}\frac{(1-\rho_k)^2\phi^2\sigma^2_{\theta_k}}{\rho_k(\rho_k\rho_{-k}-\phi^2)}\frac{\alpha_k(1-\alpha_k)}{(\alpha_k+\frac{1-\rho_k}{\sigma^2_k})^2}\right) + o(\frac{1}{N^2}). \tag{65}$$

We substitute (62) into (65), from which we take out term that is related to α_k:

$$\frac{1}{\gamma^2}\frac{(1-\rho_k)^2\phi^2\sigma^2_{\theta_k}}{\rho_k(\rho_k\rho_{-k}-\phi^2)n_k^2}\left(\frac{-\alpha_k(1-\alpha_k)}{(\alpha_k+\frac{1-\rho_k}{\sigma^2_k})^2} + \frac{2(1+\frac{\sigma^2_k}{1-\rho_k})\alpha_k}{\alpha_k+\frac{1-\rho_k}{\sigma^2_k}}\right).$$

Taking derivative to α_k, we get that L_k is increasing in α_k.

Finally, since $L^1_k - L^0_k = (\frac{\gamma}{2}+\lambda^1_k)^{-1}(W^1_k - W^0_k)$, $W^1_k - W^0_k = o(\frac{1}{N})$ and $\lambda^1_k = o(1)$, then

$$L^1_k - L^0_k = \frac{2}{\gamma}(W^1_k - W^0_k) + o(\frac{1}{N}).$$

C.11. **Proof of Proposition 5.12.** First, we calculate r_p and r_s. Since $p_k = \zeta_k(\bar{s}_k + \delta_k\bar{s}_{-k})$ and $p_{-k} = \zeta_{-k}(\bar{s}_{-k} + \delta_{-k}\bar{s}_k)$, then

$$\mathrm{Cov}(p_k, p_{-k}) = \zeta_k\zeta_{-k}[(1+\delta_k\delta_{-k})\mathrm{Cov}(\bar{s}_k,\bar{s}_{-k}) + \delta_k\mathrm{Var}(\bar{s}_{-k}) + \delta_{-k}\mathrm{Var}(\bar{s}_k)],$$

$$\mathrm{Var}(p_k) = \zeta_k^2[\mathrm{Var}(\bar{s}_k) + \delta_k^2\mathrm{Var}(\bar{s}_{-k}) + 2\delta_k\mathrm{Cov}(\bar{s}_k,\bar{s}_{-k})],$$

$$\mathrm{Var}(p_{-k}) = \zeta_{-k}^2[\mathrm{Var}(\bar{s}_{-k}) + \delta_{-k}^2\mathrm{Var}(\bar{s}_k) + 2\delta_{-k}\mathrm{Cov}(\bar{s}_k,\bar{s}_{-k})].$$

Define $\omega_k = \sqrt{\frac{\mathrm{Var}(\bar{s}_{-k})}{\mathrm{Var}(\bar{s}_k)}}$ and $\omega_{-k} = \sqrt{\frac{\mathrm{Var}(\bar{s}_k)}{\mathrm{Var}(\bar{s}_{-k})}}$. We get

$$r_p = \frac{\mathrm{Cov}(p_k,p_{-k})}{\sqrt{\mathrm{Var}(p_k)\mathrm{Var}(p_{-k})}} = \frac{(1+\delta_k\delta_{-k})r_s + \delta_k\omega_k + \delta_{-k}\omega_{-k}}{\sqrt{1+\delta_k^2\omega_k^2+2r_s\omega_k\delta_k}\sqrt{1+\delta_{-k}^2\omega_{-k}^2+2r_s\omega_{-k}\delta_{-k}}}.$$

Since $\mathrm{Cov}(\bar{s}_k,\bar{s}_{-k}) = \phi\sigma_{\theta_k}\sigma_{\theta_{-k}}$, $\mathrm{Var}(\bar{s}_k) = (\rho_k + \frac{\kappa_k}{n_k})\sigma^2_{\theta_k}$ and $\mathrm{Var}(\bar{s}_{-k}) = (\rho_{-k} + \frac{\kappa_{-k}}{n_{-k}})\sigma^2_{\theta_{-k}}$, then $r_s = \frac{\phi}{\sqrt{(\rho_k+\frac{\kappa_k}{n_k})(\rho_{-k}+\frac{\kappa_{-k}}{n_{-k}})}}$.

Second, we prove that $r_p > 0$ and $r_s > 0$ if and only if $\phi > 0$. If $\phi > 0$, then $r_s > 0$, $\delta_k > 0$ and $\delta_{-k} > 0$, thus $r_p > 0$. If $\phi < 0$, then $r_s < 0$, $\delta_k < 0$ and $\delta_{-k} < 0$, thus $r_p < 0$.

Next, we prove that $|r_p|$ is increasing in α_k and α_{-k}. We first check the case where $\phi > 0$. We have

$$\frac{\partial r_p}{\partial \delta_k} = \frac{\omega_k(1-r^2)(1-\delta_k\delta_{-k})}{(1+\delta_k^2\omega_k^2+2r_s\omega_k\delta_k)^{\frac{3}{2}}\sqrt{1+\delta_{-k}^2\omega_{-k}^2+2r_s\omega_{-k}\delta_{-k}}} > 0,$$

using the fact that $r_s < 1$, $1-\delta_k\delta_{-k} > 0$, and $\omega_k\omega_{-k} = 1$. If $\phi < 0$, we replace ϕ, r_p, r_s, δ_k and δ_{-k} with $|\phi|$, $|r_p|$, $|r_s|$, $|\delta_k|$ and $|\delta_{-k}|$. By definition, $|r_p|$ is increasing in $|\delta_k|$. Together with the fact that $|\delta_k|$ is increasing in α_k, we have $|r_p|$ is increasing in α_k. By symmetry, $|r_p|$ is also increasing in α_{-k}.

Finally, we prove that $|r_p| > |r_s|$. Notice that $|r_p| = |r_s|$ if $\alpha_k = \alpha_{-k} = 0$. Since $|r_p|$ is increasing in α_k and α_{-k}, then $|r_p| > |r_s|$.

C.12. **Proof of Proposition 5.13.** Since

$$\mathrm{Var}(p'_k) - \mathrm{Var}(p_k) = \zeta_k^2 \delta_k^2 (\mathrm{Var}(\bar{s}'_{-k}) - \mathrm{Var}(\bar{s}_{-k}))$$

$$\mathrm{Var}(p'_{-k}) - \mathrm{Var}(p_{-k}) = \zeta_{-k}^2 (\mathrm{Var}(\bar{s}'_{-k}) - \mathrm{Var}(\bar{s}_{-k}))$$

then

$$\Delta_k \equiv \frac{\mathrm{Var}(p'_k) - \mathrm{Var}(p_k)}{\mathrm{Var}(p'_{-k}) - \mathrm{Var}(p_{-k})} = \frac{(\delta_k \zeta_k)^2}{\zeta_{-k}^2}.$$

Since $|\delta_k \zeta_k|$ is increasing in α_k and ζ_{-k} is independent of α_k, then Δ_k is increasing in α_k.

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