Disclosure of Bank-Specific Information and the Stability of Financial Systems*

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Abstract

We find that disclosing bank-specific information reallocates systemic risk, but whether it mitigates systemic bank runs depends on the information disclosed. Disclosure reveals banks' resilience to adverse shocks, and shifts systemic risk from weak to strong banks. Yet, only disclosure of banks' exposure to systemic risk can mitigate systemic bank runs because it shifts systemic risk from more vulnerable banks to those less vulnerable. Optimal disclosure thus maximally differentiates such exposure, provided that banks experience runs simultaneously, if inevitable. Disclosure of banks' idiosyncratic factors does not differentiate such exposure, rendering the resulting reallocation of systemic risk ineffective in mitigating systemic runs. In the context of disclosing stress-test results, when the quality of the banking system deteriorates, the regulator may have to face a sudden run on a huge mass of banks rather than gradually abandoning weak banks.

Keywords: information design, coordination game, financial stability

JEL Codes: G01, G21, G28

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1 Introduction

How to prevent systemic bank runs has been a central topic for policy makers and researchers since the 2007-08 financial crisis, and has focused attention on the role of individual banks[1] in initiating and amplifying systemic risk through inter-bank linkages. To improve the stability of the whole financial system, public disclosure of bank-specific information has subsequently become a regular occurrence, as exemplified by stress tests. However, the existing literature on regulatory disclosure either focuses on the disclosure of aggregate states, or ignores systemic risk and the consequent strategic complementarity between investors of different banks. Subsequently, it ignores the dependence of optimal disclosure on the nature of bank-specific information.

This paper fills this gap by studying how the disclosure of *different kinds* of *bank-specific* information affects the stability of a banking system *facing systemic risk*. We identify a novel channel: disclosure of bank-specific information differentiates banks regarding their resilience to adverse shocks, and shifts systemic risk from weak banks to strong ones. Yet, only the disclosure of banks' exposure to systemic risk ("systemic vulnerability" hereafter) can mitigate systemic bank runs, because it shifts systemic risk from banks more vulnerable to it to those less vulnerable. Optimal disclosure thus maximally differentiates banks in such exposure, provided that they experience runs simultaneously, if inevitable. Disclosure of banks' idiosyncratic factors does not differentiate banks like that, rendering the resulting reallocation of systemic risk ineffective to mitigating systemic bank runs.

Specifically, consider a model with a continuum of banks, each with a representative investor.[2] Banks hold long-term illiquid assets. If a bank survives to the maturity of its asset, its investor receives the promised payoff. Otherwise, he receives nothing. Each investor can stay with or run on his bank before the maturity of the bank's asset. If he runs, the bank definitely fails. If he stays, the probability of the bank survival is the sum of three determinants. The first is a "fundamental," which captures exogenous systematic factors that affect all banks. The second is the product of an endogenous systematic factor,

[1]While we use the label "banks," this should be interpreted broadly as financial institutions or non-financial firms who are affected by and contribute to the liquidity of the whole financial system.

[2]Our focus is on how the disclosure of bank-specific information mitigates systemic bank runs. The setup with one representative investor for each bank allows us to abstract away from the coordination problem of different investors of the same bank, which has been extensively studied in the literature. Adding the problem of within-bank coordination complicates the analysis but does not change our qualitative results.

systemic risk, captured by a decreasing function of the total mass of investors who choose to stay ("stayers"), and the bank's "factor loading;" i.e., its systemic vulnerability. The third determinant is the negative of a "cost" idiosyncratic to that bank, which captures all exogenous factors affecting only that bank. Banks differ from each other in their systemic vulnerabilities and their idiosyncratic costs.

To disclose information about the bank's systemic vulnerability or idiosyncratic cost, the regulator assigns each bank a score. After observing the scores, each investor observes a noisy private signal of the fundamental, whose noise is independent and identically distributed (i.i.d.) across investors. He then chooses to stay with or run on his bank. Following the global games literature, we focus on the limiting case of vanishing noise to highlight the coordination issue between banks' investors instead of the (mechanical) impact of fundamental shocks. In this limit, it is well known that investors choose switching strategies in equilibrium. That is, an investor stays if and only if his signal realization is above a switching cutoff. The regulator assigns different scores based on bank-specific information, which induces the banks' investors to choose potentially different switching cutoffs. Disclosure hence partitions all banks into bank groups, with investors of each group sharing a common switching cutoff. Thus, we can regard a disclosure rule ("a disclosure" henceforth) as a collection of sub-disclosures, each of which is imposed on a different bank group.

To obtain general implications for optimal disclosure, instead of assuming a particular objective function for the regulator, we assume only that she always prefers more banks to be immune from runs, regardless of the fundamental. We find that for a disclosure to be potentially optimal, its sub-disclosures must all be robust disclosures, defined as those minimizing the maximum switching cutoffs of investors in the corresponding bank group, given the mass of stayers outside the group.[3] Subsequently, we characterize robust disclosures of different kinds of bank-specific information. We find that regardless of which kind of bank-specific information is revealed, a robust disclosure always equalizes the switching cutoffs of all investors in the bank group. However, robust disclosures of different kinds of

[3] A robust disclosure is so named for two reasons. First, the bank group as a whole is immune from runs only if its weakest constituent is immune; i.e., the bank whose investor's switching cutoff is the maximum among the whole group. By minimizing this maximum cutoff, a robust disclosure maximizes the robustness of the weakest constituent, and thus of the whole group, to adverse fundamental shocks. Second, this approach is in the spirit of maxmin expected utility theory of Gilboa and Schmeidler (1989), and Hansen and Sargent (2001) show its connection to the robust-control theory.

bank-specific information turn out to be very different qualitatively: while nondisclosure is always a robust disclosure of idiosyncratic costs, a robust disclosure of systemic vulnerabilities always maximizes the informational heterogeneity of banks in the group, provided that their investors share the same switching cutoff.

To understand these results, we can regard the disclosure of either kind of bank-specific information as effectively reallocating two fixed "budgets" across banks that are otherwise informationally homogeneous to investors. On one hand, as in a standard information-design problem, given the Bayesian plausibility constraint, disclosure of bank-specific information makes some banks informationally stronger (i.e., believed by investors to be less vulnerable to systemic risk or to have lower idiosyncratic cost by investors) than others. In other words, disclosure reallocates the constant expected systemic vulnerability or the constant expected idiosyncratic cost across banks. On the other hand, we show that, given the composition of a bank group and the mass of stayers from outside, the aggregate systemic risk born by all investors in this group when they are indifferent (i.e., their signals equal their equilibrium switching cutoffs) is constant, regardless of disclosures. But disclosure of either type of bank-specific information makes investors of informationally stronger banks bear more of such risk when they are indifferent. In other words, disclosure effectively generates an assortative matching: it reallocates more of the constant aggregate systemic risk to informationally stronger banks.

To see why investors of informationally stronger banks bear more systemic risk, first observe that when they are indifferent, they believe that investors of informationally weaker banks are less optimistic about their banks' survival and thus are less likely to stay than themselves. The opposite holds for investors of informationally weaker banks: when they are indifferent, they believe that investors of informationally stronger banks are more likely to stay than themselves. Consequently, the mass of stayers expected by investors of informationally stronger banks when they are indifferent is less than that expected by investors of informationally weaker banks when they are indifferent.

For the disclosure of different kinds of bank-specific information, the interaction of the reallocation of the two fixed "budgets" is qualitatively different and thus has different policy implications. Disclosure of systemic vulnerabilities results in a beneficial assortative matching: holding constant the expected systemic vulnerability of the whole bank group, more of the constant aggregate systemic risk is reallocated to investors of banks that are believed to be less vulnerable to systemic risk (i.e., informationally stronger in systemic vulnerabilities). This improves the average robustness of all banks to adverse shocks to the fundamental. Provided that all banks experience runs simultaneously if inevitable, disclosure of systemic vulnerabilities results in a Pareto improvement for the whole bank group. As a result, the robust disclosure of such information should maximize such improvement.

Specifically, we show that if the physical systemic vulnerabilities of the banks in a group do not differ very much from each other, such that investors in the group share the same switching cutoff even if full disclosure of this group is applied, then full disclosure is the desired robust disclosure of systemic vulnerability. Otherwise, the robust disclosure assigns as many scores as possible, and maximizes informational heterogeneity between any two scores, provided that all investors in the bank group still share the same switching cutoff. We further show that as the number of scores allowed approaches infinity, the robust disclosures converge to a limiting disclosure. We characterize the limiting disclosure, and show that the common switching cutoff that it induces is the infimum of those induced by all robust disclosures.

In contrast, disclosure of idiosyncratic costs does not generate informational heterogeneity in systemic vulnerability. Thus, the resulting assortative matching, which instead reallocates more of the constant aggregate systemic risk to investors of banks that are believed to have lower idiosyncratic costs, is ineffective in mitigating systemic bank runs. Consequently, nondisclosure is a robust disclosure of idiosyncratic costs for any bank group.[4]

Our results shed light on the public disclosure of stress-test results with the presence of systemic risk. Suppose the regulator's objective is to maximize the mass of banks immune from runs in a hypothetical adverse state of the economy. Then the sub-disclosure of her optimal disclosure for the bank group immune from runs must be its corresponding robust disclosure, which maximizes its joint resilience. To maximize the informational strength of this group, banks subject to runs consist exclusively of physically weak banks, with full disclosure applied, unless the state is so adverse that no bank is immune from runs, regardless of disclosures. Consequently, two novel implications of systemic risk are underscored. First, un-

[4]This result is due to our assumption that each bank has a representative investor. This precludes the role of disclosure of bank-specific information in mitigating miscoordination between investors of the same bank, which has been well studied in the literature. Our results concerning how such disclosures affect the stability of the banking system by reallocating systemic risk across banks persist even if this assumption is relaxed.

like that of idiosyncratic factors, optimal disclosure of systemic vulnerabilities entails further differentiation of banks immune from runs, due to the aforementioned beneficial assortative matching. Second, when the quality of the banking system deteriorates, the regulator may have to face a sudden run on a huge mass of banks rather than gradually abandoning weak banks. This is because, while the sacrifice of physically weak banks enhances the informational strength of the rest, it also reduces the mass of stayers and thus increases the systemic risk faced by the investors of unsacrificed banks. When the second effect dominates, an infinitesimal sacrifice of weak banks would worsen the resilience of the others, calling for further sacrifice, until the first effect dominates.

The rest of this paper is organized as follows. Section 1.1 reviews the related literature. Section 2 sets up the model. Section 3 illustrates the main insight of our results with a simplified example of binary scores. Section 4 presents our main results. Section 5 demonstrates the insight from our main results on the optimal disclosure of stress-test results. Section 6 shows that our main results are robust to informative priors and correlation of bank-specific information in different dimensions. Section 7 concludes. All proofs are relegated to the appendix unless otherwise specified.

1.1 Literature Review

Our paper is mainly related to two strands of the literature. The first strand is the discussion of bank transparency and disclosures. A particularly prevalent question is how to design bank stress tests. Goldstein and Sapra (2014) comprehensively review the nature and cost-benefit analysis of disclosing the results of stress tests. Our paper centers around the two effects of stress tests that they highlight: market discipline and coordination failure. Although we share an important point with Bouvard et al. (2015) that there are multiple banks in the system and that stress tests help distinguish between them, in their model, banks are independent of each other, so each faces a separate coordination game with their own homogeneous investors. Goldstein and Huang (2016) and Inostroza and Pavan (2020) incorporate Bayesian persuasion into coordination games, where a regime is subject to a run by homogeneous investors. They study disclosure of the aggregate state but not of the bank-specific information. Note that Inostroza and Pavan (2020) also discuss discriminatory disclosures which release different information to different investors. Goldstein and Leitner

(2018), Williams (2017) and Orlov et al. (2018) also model stress tests in Bayesian persuasion but do not focus on coordination issues. They augment their models with other elements including asset markets, capital requirements, and investment decisions. Complementary to this work, we study how information disclosure (endogenously) creates heterogeneous interests among investors facing coordination problems, and how disclosure of different kinds of bank-specific information affects investors' decisions and the stability of the banking system.

The second strand is the literature on global games. Pioneered by Carlsson and van Damme (1993), this literature is elegantly surveyed by Morris and Shin (2003). In particular, our work is more closely related to the literature on global games with heterogeneous players. For example: Corsetti et al. (2004) characterize the impact of a large trader on a population of small ones; Frankel et al. (2003) prove equilibrium uniqueness for a large class of these games; and Sákovics and Steiner (2012) provide a criterion that can be used to find the optimal targets for a variety of interventions in regime-change games with heterogeneous agents. Based on Sákovics and Steiner (2012), Drozd and Serrano-Padial (2018) discuss a credit-enforcement cycle driven by the collective default of borrowers, and Leister et al. (2017) studies strategic interaction in networks, and Serrano-Padial (2020) explore global games with heterogeneous agents based on potential maximization. Invoking Morris and Yang (2021), Dai and Yang (2020) study the role of organizations in coordinating the actions of individuals with heterogeneous interests. Related to the approach developed in Dai and Yang (2020), this paper studies an information-design problem: how the informational heterogeneity created by the regulator's disclosure affects the stability of banking systems.

Some papers also study systemic bank runs in the setup of global games. For example, Liu (2019) studies the interaction between bank runs and asset prices. Goldstein et al. (2020) use a Diamond-Dybvig style setup featuring within-bank and cross-bank strategic complementarity among depositors and find that an increase in heterogeneity among banks makes all banks more stable, given the existence of cross-bank strategic uncertainty. The heterogeneity in their model is exogenous, while in our model it stems from information disclosure and is thus the objective of information design. In addition, our work is based on different economic mechanisms. Theirs rests on the interaction between within- and cross-bank strategic uncertainty, while ours is based on the reallocation of aggregate systemic risk across banks, and on its interaction with the reallocation of expected systemic vulnerability and of expected idiosyncratic cost, respectively, all of which operate across banks. In our model, within-bank strategic uncertainty is precluded by the assumption of bank-level representative investors. In addition to illustrating the new mechanism, we further derive its implications for the regulator's design of optimal disclosure, and point out the qualitatively different effects of disclosing different kinds of bank-specific information.

2 Model Setup

2.1 Agents

We consider a three-date economy consisting of a regulator ("she"), a continuum of banks and a continuum of investors. Only the regulator and the investors are active players, all of whom are risk neutral. Both the gross discount rate and the gross risk-free rate are normalized to one. At date 0, the regulator designs rules for the disclosure of bank-specific information from all banks to investors. The total mass of banks is normalized to 1. Each bank i has a representative investor, henceforth called investor i.[5] At date 1, each investor i ("he") chooses to stay ($l^i = 1$) or to run ($l^i = 0$) based on the information available to him by then. If he runs, he secures the one unit of consumption good invested in bank i's long-term project before date 0, and bank i definitely fails at date 2. If he stays, he receives R units of consumption good from the project if bank i survives at date 2, and nothing if it fails then.

2.2 Banks' Survival Probabilities

To focus on how the regulator's information design affects investors' actions, we assume that the probability that bank i survives at date 2, P^i, follows the following reduced form, and

[5] The investor here refers to banks' wholesale investors and large depositors who are not fully insured through depositor insurance or collateral. Since we are focusing on the coordination problem at the level of the whole banking system, we intentionally assume a representative investor to mute the coordination problem within each individual bank. Adding the coordination problem at the level of individual banks does not change the qualitative results but complicates the analysis.

abstract from the details of its microfoundation:[6]

$$P^i = \frac{1}{R}\left[\theta - r^i \cdot a\left(l\right) + 1 - c^i\right] \ . \tag{1}$$

The "fundamental" θ is an aggregate state of the economy capturing all exogenous factors that simultaneously affect the survival probability of all banks, such as macroeconomic conditions. We assume that θ is distributed over $\left[\underline{\theta}, \bar{\theta}\right] \subset \mathbb{R}$.

The second term in equation (1) is our key addition to capture the strategic interaction among banks. In particular, $l \triangleq \int l^i di$ is the mass of all investors in the economy who stay ("stayers"), which can be interpreted as outside liquidity. The loss function a, which is decreasing in l, captures the systemic risk faced by all banks. Hereafter, we directly refer to $a\left(l\right)$ as systemic risk. We allow for a generic functional form of a, as long as it is positive and Lipschitz continuous almost everywhere. The coefficient r^i captures the vulnerability of bank i to systemic risk. This could be due to the heterogeneity in banks' asset liquidity or network structure in terms of asset commonality or cross holdings. We assume that $r^i = \underline{r} > 0$ with probability q^r and $r^i = \bar{r} > \underline{r}$ with probability $1 - q^r$. Let $\mathbb{E}r = q^r \underline{r} + \left(1 - q^r\right)\bar{r}$.

The idiosyncratic "cost" c^i captures all exogenous factors that only affect the survival probability of bank i. We assume that $c^i = \underline{c}$ with probability q^c and $c^i = \bar{c} > \underline{c}$ with probability $1 - q^c$. Let $\mathbb{E}c = q^c \underline{c} + \left(1 - q^c\right)\bar{c}$.

The following parametric restriction is needed to guarantee that the survival probability P^i is always in $[0, 1]$:

$$-1 \leq \underline{\theta} - \bar{r}a\left(0\right) - \bar{c} < \bar{\theta} - \underline{r}a\left(1\right) - \underline{c} \leq R - 1 \ .$$

Given the survival probability P^i, the incremental payoff for investor i from staying relative to running is

$$\pi \triangleq P^i R - 1 = \theta - r^i a\left(l\right) - c^i \ . \tag{2}$$

[6] The setup is designed such that the net return to investor i's investment follows the two-factor model in (2) analogous to those in the empirical asset-pricing literature. The loading on the exogenous factor θ is normalized to one, and that on the endogenous factor $-a(l)$ is r^i. The expected idiosyncratic cost $-\mathbb{E}c$ can be viewed as "alpha" and $\mathbb{E}c - c^i$ as the residual. Like a standard factor pricing model, equation (2) can be viewed as a decomposition of all factors affecting the survival of bank i, given its investor's action. We adopt such a factor model because it is technically convenient and easy to interpret. Our main results hold qualitatively beyond this particular functional form.

2.3 The Regulator's Information Design

The focus of this paper is on the regulator's optimal design of disclosure rules ("disclosures" hereafter) at date 0 about relevant bank-specific information (i.e., r^i or c^i) to mitigate systemic bank runs caused by strategic uncertainty. To highlight our main insights, we assume in our baseline model that investors rely completely on the regulator's disclosure to learn about r^i and c^i: without her disclosure, they know only their expected values, $\mathbb{E}r$ and $\mathbb{E}c$. We show in Section 6 that our main insights are robust to the relaxation of this assumption. To better contrast disclosures of bank-specific information in different dimensions, we assume that r^i and c^i are independently distributed, so that disclosures about r^i do not reveal information about c^i, and vice versa. We discuss the case of correlated information in Section 6.

A disclosure specifies how scores are assigned to banks based on their r^i and c^i, so that investors can only distinguish between banks with different scores, but not between those with the same score. Without loss of generality, any disclosure about r^i and c^i with n scores can be represented by the conditional means of r^i and c^i for each score and the mass of banks receiving that score; i.e., with $\{(r_k, c_k, w_k)\}_{k=1}^n$, where $r_k = \mathbb{E}[r^i|\text{score } k]$, $c_k = \mathbb{E}[c^i|\text{score } k]$, and w_k is the mass of banks receiving score k. By construction, $w_k \in [0,1]$ for all k, and $\sum_{k=1}^n w_k = 1$. As a well known result in the literature of information design, a disclosure $\{(r_k, c_k, w_k)\}_{k=1}^n$ is feasible if and only if it satisfies Bayesian plausibility; i.e., $r_k \in [\underline{r}, \bar{r}]$ and $c_k \in [\underline{c}, \bar{c}]$ for all k, $\sum_{k=1}^n w_k r_k = \mathbb{E}r$, and $\sum_{k=1}^n w_k c_k = \mathbb{E}c$. For both expositional convenience and practical consideration, we focus on finite disclosures that assign a finite number of scores to banks. In Section 4.5, we present our results concerning the limiting case where infinitely many scores are allowed.

Henceforth, superscripts denote exogenous objects, and subscripts denote conditional means given a disclosure and the resulting endogenous objects. For ease of presentation, we refer directly to r_k, c_k, or (r_k, c_k) as a "score". So a bank with a lower score is believed by investors to be stronger. We refer to such a bank as "informationally stronger," and the heterogeneity in scores as "informational heterogeneity." In contrast to "scores," we refer to $r^i \in \{\underline{r}, \bar{r}\}$, $c^i \in \{\underline{c}, \bar{c}\}$, or $(r^i, c^i) \in \{\underline{r}, \bar{r}\} \times \{\underline{c}, \bar{c}\}$ as a "type," a bank with a lower type as "physically stronger," and the heterogeneity in types as "physical heterogeneity." In addition, we use "a type-r^i investor," "a score-r_k investor," and "a score-(r_k, c_k) investor" or simply "a score-k investor" to denote the representative investor of a type-r^i bank, that of a score-r_k bank, and that of a score-(r_k, c_k) bank, respectively. Moreover, we refer to disclosures that potentially reveal something about banks' systemic vulnerabilities but nothing about idiosyncratic costs (i.e., with $c_k = \mathbb{E}c$ for all k) as "disclosures in dimension r," and "disclosures in dimension c" are defined analogously.[7]

2.4 Information about the Fundamental

At date 0, the regulator and all investors share a common prior of the fundamental θ, represented by a probability density function $h(\cdot)$. At date 1, the investor of each bank i observes a private signal about θ, $x^i = \theta + \sigma \cdot \varepsilon^i$, where ε^i is independent and identically distributed according to a probability density function $\phi(\cdot)$, and the parameter σ determines the magnitude of the signal noise, which captures the magnitude of fundamental uncertainty (about θ) faced by investors. An investor's signal can be understood as his private information or opinion about the macroeconomy. The probability density functions $h(\cdot)$ and $\phi(\cdot)$ are continuous, bounded, and fully supported over $\left[\underline{\theta}, \bar{\theta}\right]$ and $(-\infty, +\infty)$, respectively.[8] As is common in the global games literature, we assume the existence of dominance regions. That is, when θ is sufficiently low (high), it is the dominant strategy of any investor to run (to stay), irrespective of the actions of other investors; i.e.,

$$\underline{\theta} - \underline{r}a(1) - \underline{c} < 0 < \bar{\theta} - \bar{r}a(0) - \bar{c} \,. \tag{3}$$

Note that a stayer's payoff (2) is strictly increasing in the total mass of stayers. This creates motives for an investor to coordinate his decision with others in the game at date 1. However, the idiosyncracy of signal noise prevents investors from perfectly knowing others' signal realizations and thereby inferring their actions. As highlighted by the global games literature, strategic uncertainty (about others' actions) as such could persist and thus lead to miscoordination among investors, even if fundamental uncertainty vanishes (i.e., $\sigma \to 0$). To explore the impact of the regulator's information design on strategic uncertainty and to sharpen its implication on the stability of the banking system, we follow the convention of the global games literature and focus on the limit of $\sigma \to 0$.[9] This also guarantees equilibrium

[7] We do not use "disclosures about r^i (c^i)" to avoid the confusion with the treatment of a specific bank.

[8] Unbounded support of $\phi(\cdot)$ provides convenience of exposition. Our results remain valid for bounded support with minor modification.

[9] See Corsetti et al. (2004), Goldstein and Pauzner (2005) and He et al. (2019) for examples.

uniqueness. As is well known in this literature, it is without loss of generality in this limit to focus on symmetric equilibria with switching strategies. That is, an investor stays if and only if he observes x^i above a switching cutoff $\hat{x}(r_k, c_k)$, which depends on the score (r_k, c_k) assigned to his bank by the regulator. Therefore, we say a bank is *immune from runs* in a state θ if $\hat{x}(r_k, c_k) \leq \theta$ and *subject to runs* in θ if $\hat{x}(r_k, c_k) > \theta$.

2.5 The Regulator's Objectives

Finally, we introduce the regulator's objective. A disclosure $\{(r_k, c_k, w_k)\}_{k=1}^n$ results in a set of limiting switching cutoffs $\{\hat{x}_k\}_{k=1}^n$, such that all banks with the kth score are subject to runs almost surely if $\theta < \hat{x}_k$ and immune from runs almost surely if $\theta > \hat{x}_k$.[10] Suppose $\{\hat{x}_k\}_{k=1}^n$, the set of the cutoffs, has T distinct elements ranked as $\theta_1 < \theta_2 < \ldots < \theta_T$. Note that $T \leq n$ by definition. For $i \in \{1, \cdots, T\}$, let $K_i = \sum_{\{k|\hat{x}_k \leq \theta_i\}} w_k$ denote the mass of banks whose cutoffs are no greater than θ_i. For ease of notation, define $\theta_{T+1} = \bar{\theta}$. Then, the mass of banks immune from runs in state θ is essentially

$$K\left(\theta; \{K_j, \theta_j\}_{j=1}^T\right) \triangleq \sum_{i=1}^T K_i \cdot 1_{\{\theta_i \leq \theta < \theta_{i+1}\}} .$$

We refer to $K\left(\cdot; \{K_j, \theta_j\}_{j=1}^T\right)$ as a stability scheme.

Stability schemes can be partially ordered according to first-order stochastic dominance (FOSD). A stability scheme that is first-order stochastically dominated by another has a greater mass of banks immune from runs than does the latter under *any* circumstance (i.e., any value of θ). Therefore, the regulator prefers disclosure A to disclosure B if the stability scheme resulting from A is first-order stochastically dominated by that resulting from B.

In reality, a regulator may have a variety of potentially conflicting objectives. For example, in addition to maximizing the total number of banks immune from runs, she may care about the survival of particular banks. Together with different importance weights of outcomes, given different fundamental θ, combinations of those objectives result in different preferences over disclosures. To obtain results robust to the fine details of the regulator's preferences, in our baseline analysis we characterize the general properties of the optimal

[10]Here, we use the term "almost surely" because we are talking about the limiting switching cutoffs at vanishing noise.

disclosure under this minimum requirement of FOSD. As an application in stress tests, Section 5 illustrates the central role of the results of our baseline analysis in the construction of the regulator's optimal disclosure given a practical objective function.

3 An Intuitive Illustration

This section illustrates the main idea of the paper using an example of binary-score disclosures, $\{(r_k, c_k, w_k)\}_{k=1}^{2}$, with fixed masses w_1 and w_2. By construction, $w_1 + w_2 = 1$. Disclosures in dimension r in this context refer to those with $r_1 \leq \mathbb{E}r \leq r_2$ but $c_1 = \mathbb{E}c = c_2$, and disclosures in dimension c refer to those with $r_1 = \mathbb{E}r = r_2$ but $c_1 \leq \mathbb{E}c \leq c_2$. For concreteness of illustration, in this section, the regulator is assumed to maximize the probability that all banks are immune from runs.

3.1 Preview of Results

Let

$$A(l) \triangleq \int_0^l a\left(\tilde{l}\right) d\tilde{l}. \tag{4}$$

If an investor believes that the mass of stayers $\tilde{l}$ is uniformly distributed on $[l_1, l_2]$, then the systemic risk that he expects is $\frac{A(l_2)-A(l_1)}{l_2-l_1}$. By definition, $A(0) = 0$. Proposition 1 characterizes the equilibrium switching cutoffs given disclosures in dimensions r and c, respectively, as illustrated in Figure 1. Expected systemic risk under uniform beliefs plays an important role in Proposition 1, which we will explain in Section 3.3.

Proposition 1. *For disclosures in dimension r,*

- *if* $r_2/r_1 \leq \frac{A(w_1)}{w_1} \frac{1-w_1}{A(1)-A(w_1)}$, *then*

$$\hat{x}_1 = \hat{x}_2 = \mathbb{E}c + \left(\mathbb{E}\left[\frac{1}{r}\right]\right)^{-1} A(1); \tag{5}$$

- *if* $r_2/r_1 > \frac{A(w_1)}{w_1} \frac{1-w_1}{A(1)-A(w_1)}$, *then*

$$\hat{x}_1 = \mathbb{E}c + r_1 \frac{A(w_1)}{w_1} < \mathbb{E}c + r_2 \frac{A(1) - A(w_1)}{1 - w_1} = \hat{x}_2\ . \tag{6}$$

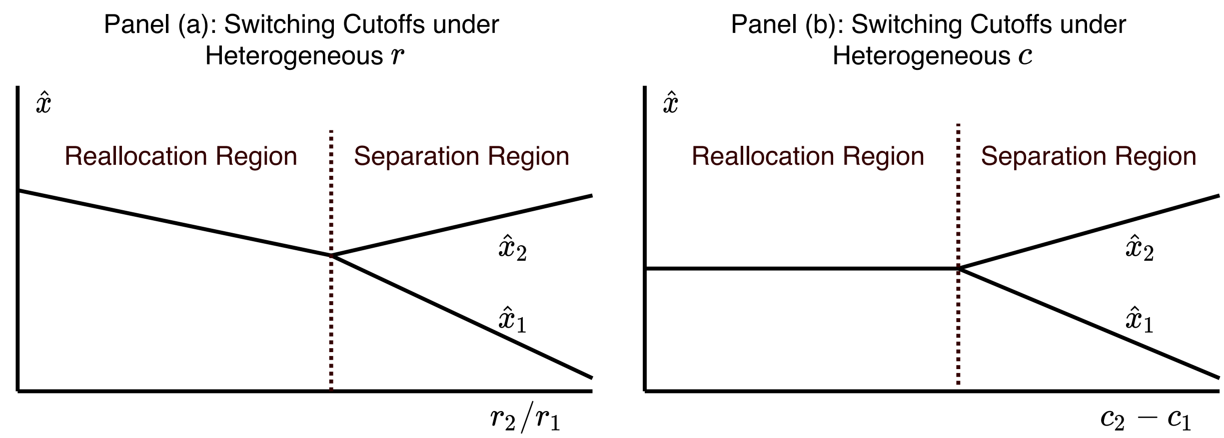

Figure 1: Switching Cutoffs Resulting from Binary-Score Disclosure Rules

For disclosures in dimension c,

- *if* $c_2 - c_1 \leq \frac{A(w_1)}{w_1} - \frac{A(1)-A(w_1)}{1-w_1}$ *, then*

$$\hat{x}_1 = \hat{x}_2 = \mathbb{E}c + \mathbb{E}r \cdot A(1); \tag{7}$$

- *if* $c_2 - c_1 > \frac{A(w_1)}{w_1} - \frac{A(1)-A(w_1)}{1-w_1}$ *, then*

$$\hat{x}_1 = \frac{A(w_1)}{w_1}\mathbb{E}r + c_1 < \frac{A(1) - A(w_1)}{1 - w_1}\mathbb{E}r + c_2 = \hat{x}_2\ . \tag{8}$$

Two patterns are evident in Figure 1. First, for disclosures in either dimension, we have $\hat{x}_1 = \hat{x}_2$ when disclosures are not that informative (i.e., when the corresponding measures of informational heterogeneity are in the reallocation regions[11]), and $\hat{x}_1 < \hat{x}_2$ when disclosures are sufficiently informative (i.e., when measures of informational heterogeneity are in the separation regions). Second, the common cutoff in the reallocation region resulting from disclosures in dimension r is strictly decreasing in informational heterogeneity (measured by r_2/r_1), while its counterpart in dimension c is constant. Recall that all banks are immune from runs if and only if $\theta \geq \max\{\hat{x}_1, \hat{x}_2\}$, and the optimal disclosures should thus minimizes $\max\{\hat{x}_1, \hat{x}_2\}$. Consequently,

Proposition 2. *The optimal binary-score disclosure in dimension r maximizes informational*

[11] We will explain why the regions are so named in Section 3.3.

heterogeneity r_2/r_1 provided that $\hat{x}_1 = \hat{x}_2$, while nondisclosure is an optimal binary-score disclosure in dimension c.

Proposition 2 articulates the qualitative difference between optimal disclosures in different dimensions, which is the core result of this paper. Informational heterogeneity generated by disclosures in dimension r (i.e., about banks' systemic vulnerabilities) could reduce both $\hat{x}_1$ and $\hat{x}_2$, and thus improve the *whole banking system*'s resilience to adverse shocks. The optimal disclosure in that dimension takes full advantage of such improvement. That is, if it is feasible to increase r_2/r_1 to the boundary between the reallocation and the separation regions (i.e., $\frac{A(w_1)}{w_1}\frac{1-w_1}{A(1)-A(w_1)}$, given by Proposition 1), then this is the optimal disclosure in dimension r. If physical heterogeneity $\bar{r}/\underline{r}$ is so low that this disclosure is not feasible, then it is optimal in dimension r to disclose as much as possible. However, such improvement is absent from disclosures in dimension c, and thus nondisclosure is optimal.

3.2 Equilibrium Switching Cutoffs

As the basis for our illustration, this subsection provides a brief derivation of equilibrium switching cutoffs (11) given a disclosure. For a given magnitude of fundamental uncertainty σ, let $\hat{x}_i^\sigma$ denote a score-(r_i, c_i) investor's switching cutoff. Then, the probability that he stays conditional on fundamental θ is

$$m_i^\sigma(\theta) \triangleq \Pr(x^i > \hat{x}_i^\sigma|\theta) = 1 - \Phi\left(\frac{\hat{x}_i^\sigma - \theta}{\sigma}\right) . \tag{9}$$

As usual, we adopt the law of large numbers convention[12] so that the total mass of stayers is

$$M^\sigma(\theta) \triangleq \sum_i w_i m_i^\sigma(\theta) . \tag{10}$$

If the investor instead chooses a different switching cutoff $\hat{x}_i^\sigma + \delta$, then his expected payoff is

$$\int_{\underline{\theta}}^{\bar{\theta}} [\theta - r_i a(M^\sigma(\theta)) - c_i]\, m_i^\sigma(\theta - \delta) h(\theta)\, d\theta .$$

[12]The law of large numbers is not well defined for a continuum of random variables (Sun (2006)). Our convention is equivalent to assuming that opponents' play is the limit of play of finite selections from the population.

It is optimal for him to choose $\delta = 0$ in equilibrium, which requires that

$$\int_{\underline{\theta}}^{\bar{\theta}} [\theta - r_i a(M^\sigma(\theta)) - c_i] (m_i^\sigma)'(\theta) h(\theta) d\theta = 0 .$$

Since $m_i^\sigma(\theta)$ converges to $1_{\{\theta \geq \hat{x}_i\}}$ as $\sigma \to 0$, we have

$$\hat{x}_i^\sigma - r_i \int_{\theta=\underline{\theta}}^{\bar{\theta}} a(M^\sigma(\theta)) dm_i^\sigma(\theta) - c_i = O(\sigma) . \tag{11}$$

The system of equations (11) for both scores (r_i, c_i) yields the limiting cutoffs $\hat{x}_i = \lim_{\sigma \to 0} \hat{x}_i^\sigma$, $i \in \{1, 2\}$ in Proposition 1. Since the prior density $h(\cdot)$ is continuous, when investors' private signals are very accurate (i.e., when σ is small), the impact of the prior is absorbed by the $O(\sigma)$ on the right-hand side of (11), and the posterior probability density of the fundamental θ, conditional on $x^i = \hat{x}_i^\sigma$, is approximately $\frac{1}{\sigma}\phi\left(\frac{\hat{x}_i^\sigma - \theta}{\sigma}\right)$. Note that $dm_i^\sigma(\theta) = \frac{1}{\sigma}\phi\left(\frac{\hat{x}_i^\sigma - \theta}{\sigma}\right) d\theta$, and $a(M^\sigma(\theta))$ is the systemic risk that a bank faces given θ. Thus,

$$\int_{\theta=\underline{\theta}}^{\bar{\theta}} a(M^\sigma(\theta)) dm_i^\sigma(\theta) \tag{12}$$

is approximately (up to $O(\sigma)$) the systemic risk expected by a score-(r_i, c_i) investor whose signal realization equals his switching cutoff ("a cutoff score-i investor" hereafter). In addition, the first term in (11), $\hat{x}_i^\sigma$, is approximately (up to $O(\sigma)$) the expectation of θ conditional on $x^i = \hat{x}_i^\sigma$. Therefore, the system of equations (11), which determine the limiting cutoffs $\hat{x}_i$ and thus the equilibrium, equalize all cutoff score-i investor's expected payoff from staying (see (2)) to zero, the payoff from running.

Equation (11) indicates that disclosures in both dimensions affect score-i investors' switching cutoff in two ways. First, disclosures differentiate banks with different scores that are otherwise believed by investors to be homogenous, subject to the Bayesian plausibility constraints

$$\sum_i w_i r_i = \mathbb{E}r , \tag{13}$$

and

$$\sum_i w_i c_i = \mathbb{E}c . \tag{14}$$

In other words, disclosures in their respective dimensions "reallocate" the constant expected systemic vulnerability $\mathbb{E}r$ or the constant expected idiosyncratic cost $\mathbb{E}c$ across banks. Second, they "allocate" different expected systemic risk to cutoff investors of banks that receive different scores. Last and most importantly, the two interact only in disclosures in dimension r. The next two subsections elaborate the last two points concerning systemic risk, respectively.

3.3 Reallocation of Constant Aggregate Systemic Risk

This subsection shows that disclosures in either dimension result in an assortative matching: both reallocate more of the constant aggregate systemic risk to informationally stronger banks (i.e., banks with better scores), and more so if the resulting informational heterogeneity is greater.

3.3.1 Constant Aggregate Systemic Risk

From (12), the aggregate systemic risk expected by all cutoff investors is given by

$$\sum_i w_i \int_{\theta=\underline{\theta}}^{\bar{\theta}} a(M^\sigma(\theta))dm_i^\sigma(\theta) = \int_{\theta=\underline{\theta}}^{\bar{\theta}} a(M^\sigma(\theta))dM^\sigma(\theta) = \int_0^1 a(M^\sigma)dM^\sigma = A\,(1)\,, \tag{15}$$

where $A(\cdot)$ is given by (4), the first equality in (15) is due to (10), and the second is due to Assumption (3). Note that it is a constant regardless of disclosures.

The second equality in (15) enables us to understand this result from the perspective of an average investor, who thinks of the whole economy as consisting of average investors like himself but who receive private signals with independent noise. As a well known equilibrium outcome in the global game literature, when his signal realization equals his switching cutoff, he holds the Laplacian belief regarding the actions of other average investors in the limit of $\sigma \to 0$. That is, he believes that the fraction of them that are staying is uniformly distributed on $[0,1]$,[13] and thus the systemic risk that he expects is $\int_0^1 a(M^\sigma)dM^\sigma$. While derived from our specific setup, the constant-aggregate-systemic-risk condition (15) echoes

[13] To see this, since the noise of private signals is i.i.d., when the noise is small, an average investor believes that the proportion of average investors whose signal realization is less than his own roughly follows the uniform distribution over $[0,1]$. When his signal is at his cutoff, this proportion equals the proportion of stayers out of all investors who share the same switching cutoff.

the belief constraint in alternative settings in Sákovics and Steiner (2012) and Serrano-Padial (2020); i.e., the weighted average strategic belief is the uniform belief on $[0,1]$.

3.3.2 Reallocation of Systemic Risk

Now we show that when σ is small, as disclosures in dimension r or c increase informational heterogeneity and thus increase the relative distance in switching cutoffs, $\Delta^{\sigma}_{1,2} \triangleq (\hat{x}^{\sigma}_2 - \hat{x}^{\sigma}_1)/\sigma$, the proportion of stayers out of all score-1 investors expected by a cutoff score-1 investor is roughly invariant, but that out of all score-2 investors expected by the cutoff score-1 investor strictly decreases. Therefore, the systemic risk expected by a cutoff score-1 investor, $\int_{\theta=\underline{\theta}}^{\bar{\theta}} a(M^{\sigma}(\theta))dm^{\sigma}_1(\theta)$, increases with $\Delta^{\sigma}_{1,2}$. A symmetric argument and an opposite conclusion hold for a cutoff score-2 investor.

Recall from (11) that equilibrium cutoffs are determined by cutoff investors' indifference. A cutoff score-i investor's belief about the fundamental θ induces his belief about the proportion m_j of stayers out of all score-j investors[14] through (9):

$$\Pr\left[m_j \leq \tilde{m}_j | x^i = \hat{x}^{\sigma}_i\right] = 1 - \Phi\left(-\Delta^{\sigma}_{i,j} + \Phi^{-1}\left(1 - \tilde{m}_j\right)\right) + O\left(\sigma\right) \ . \tag{16}$$

When σ is small, as long as $\Delta^{\sigma}_{i,j}$ is finite, the distribution (16) is non-degenerated, reflecting the *strategic uncertainty* faced by cutoff score-i investors about score-j ones. By definition, we have $\Delta^{\sigma}_{i,i} = 0$ regardless of disclosures. Thus, by (16), a cutoff score-i investor's belief about m_i is always uniform, invariant with disclosures (up to $O(\sigma)$).

Now consider $i \neq j$, and how a greater $\Delta^{\sigma}_{1,2}$ increases the systemic risk expected by a cutoff score-1 investor. Since the left-hand side of (16) is increasing in $\Delta^{\sigma}_{1,2}$, his belief about m_2 resulting from a large $\Delta^{\sigma}_{1,2}$ is first-order stochastically dominated by that resulting from a small $\Delta^{\sigma}_{1,2}$. Intuitively, a cutoff score-1 investor believes that the fundamental θ is around $\hat{x}^{\sigma}_1$ with high probability. A larger $\Delta^{\sigma}_{1,2}$ makes him believe that θ is more likely to be below $\hat{x}^{\sigma}_2$, and thus makes him more pessimistic about the chance that score-2 investors stay. Since the systemic risk is decreasing in the total mass of stayers, increasing $\Delta^{\sigma}_{1,2}$ increases the systemic risk he expects. The opposite holds for a cutoff score-2 investor. As such, more of the constant aggregate systemic risk is reallocated to score-1 investors.

Such reallocation of systemic risk happens only when there is strategic uncertainty be-

[14] By the law of large numbers, this proportion equals the probability that a score-j investor stays.

tween score-1 and score-2 investors; i.e., before $\Delta^{\sigma}_{1,2}$ reaches infinity, for which the reallocation regions in Figure 1 are named. In this case, the limiting switching cutoffs, $\hat{x}_1$ and $\hat{x}_2$, must coincide. Once $\Delta^{\sigma}_{1,2}$ approaches infinity, from (16) we have that

$$\lim_{\Delta^{\sigma}_{1,2}\to+\infty} \Pr\left[m_2 \le \tilde{m}_2 | x^1 = \hat{x}^{\sigma}_1\right] = 1$$

for any $\tilde{m}_2 \in (0,1]$; i.e., a cutoff score-1 investor no longer faces strategic uncertainty about score-2 investors: he believes that they are running almost surely, and thus that stayers come only from score-1 investors. The systemic risk that he expects reaches its maximum, $\int_{\theta=\underline{\theta}}^{\bar{\theta}} a(w_1 m^{\sigma}_1(\theta)) dm^{\sigma}_1(\theta) = \frac{A(w_1)}{w_1}$, and stays there as the limiting switching cutoffs diverge (as in the separation regions in Figure 1). Meanwhile, a symmetric argument shows that a cutoff score-2 investor believes that score-1 investors stay almost surely, and thus the systemic risk he expects stays at its minimum, $\int_{\theta=\underline{\theta}}^{\bar{\theta}} a(w_1 \cdot 1 + w_2 m^{\sigma}_2(\theta)) dm^{\sigma}_2(\theta) = \frac{A(1)-A(w_1)}{1-w_1}$. In this case, disclosures affect switching cutoffs only through the direct impact of scores, as in (6) and (8).

3.4 Assortative Matching is Beneficial Only in Dimension r

We now explain why only disclosures in dimension r can improve the stability of all banks, captured by a reduction in the average cutoff (17) derived from (11) using Bayesian plausibility constraint (14):

$$\sum_i w_i \hat{x}^{\sigma}_i = \sum_i w_i r_i \int_{\theta=\underline{\theta}}^{\bar{\theta}} a(M^{\sigma}(\theta)) dm^{\sigma}_i(\theta) + \mathbb{E}c + O(\sigma). \tag{17}$$

Note that its limit as $\sigma \to 0$ gives rise to the common switching cutoffs (5) and (7) for disclosures in the respective dimensions, as in the reallocation regions in Figure 1.

Recall from the previous subsection that disclosures in dimension r result in an assortative matching that allocates more systemic risk $\int_{\theta=\underline{\theta}}^{\bar{\theta}} a(M^{\sigma}(\theta)) dm^{\sigma}_i(\theta)$ to banks that are less vulnerable to it (i.e., with lower r_i), given their constant population aggregates (i.e., constraints (13) and (15)). Through the interaction of systemic vulnerability and expected systemic risk, represented by the first term of (17), such disclosures reduce the average cutoff of all investors, and thus improve the stability of all banks.

Yet, such interaction is absent from disclosures in dimension c, since they do not differentiate banks regarding their systemic vulnerability; i.e., $r_1 = \mathbb{E}r = r_2$, and do not reduce the aggregate systemic risk given by (15). Indeed, (17) becomes

$$\sum_i w_i \hat{x}_i^\sigma = \mathbb{E}r \cdot A(1) + \mathbb{E}c + O(\sigma),$$

which is independent of the distribution of scores in dimension c, and coincides with (7).

3.5 Additional Remarks

For expositional convenience, this section has focused on binary disclosures with fixed w_1 and w_2. In Section 4, where such restrictions are lifted, nondisclosure is still optimal in dimension c, but optimal disclosures in dimension r exploit the insight in our intuitive illustration to the extreme: the regulator assigns as many scores as possible, such that for any two scores r_i and r_j, we have $\hat{x}_i = \hat{x}_j$ while $\Delta_{i,j} \triangleq \lim_{\sigma \to 0} \Delta_{i,j}^\sigma = +\infty$, unless such practice is restricted by physical heterogeneity, where full disclosure is optimal.

4 Robust Disclosures

Based on the key messages from the binary-score illustration in Section 3, this section probes into the design of disclosures in dimensions r and c, respectively, that result in stability schemes not first-order stochastically dominating those resulting from any other disclosures. In particular, Section 4.1 solves for the equilibrium given a finite disclosure, and introduces the concepts of entanglement, separation and adjacency that characterize the strategic relationship between investors of banks receiving different scores. Section 4.2 introduces the concept of robust disclosures for bank groups, and shows that an optimal disclosure must be a collection of robust disclosures. We then characterize robust disclosures for arbitrary bank groups in each dimension. Section 4.3 shows that for any bank group, nondisclosure is always a robust disclosure in dimension c. Section 4.4 shows that for bank groups with weak physical heterogeneity, the robust disclosure in dimension r is full disclosure. For bank groups with strong physical heterogeneity, the robust disclosure assigns as many adjacent scores as possible. For such bank groups, Section 4.5 constructs a limiting robust disclosure with infinitely many scores, and shows that it outperforms all finite disclosures, and the sequence of robust disclosures converges to it as the number of scores allowed approaches infinity.

4.1 Equilibrium Given a Disclosure

Consider a disclosure $\{(r_i, c_i; w_i)\}_{i=1}^n$ with n different scores and associated mass $w_1, w_2, ..., w_n$, respectively. Again, it is without loss of generality to focus on symmetric equilibria in which all investors are playing switching strategies. Let $\hat{x}_i^\sigma$ be the switching cutoff of score-(r_i, c_i) investors for $\sigma > 0$, and $\Delta_{i,j}^\sigma = \left(\hat{x}_j^\sigma - \hat{x}_i^\sigma\right)/\sigma$. Then, the probability that a score-(r_i, c_i) investor chooses to stay if the fundamental is θ, $m_i^\sigma(\theta)$, is still given by (9); the total mass of stayers, $M^\sigma(\theta)$, is still given by (10); and $\hat{x}_i^\sigma$ still satisfies (11).

Proposition 3 characterizes the equilibrium in the limit $\sigma \to 0$, which generalizes the equilibrium characterization in Section 3 to any finite disclosure.

Proposition 3. *As $\sigma \to 0$, $\forall i, j \in \{1, 2, \dots, n\}$, $\hat{x}_i^\sigma \to \hat{x}_i$ and $\Delta_{i,j}^\sigma \to \Delta_{i,j}$, where $\{\hat{x}_i, \Delta_{i,j}\}_{i,j=1}^n$ satisfy the system of equations*

$$\hat{x}_i = c_i + r_i \int_0^1 a \left(\sum_{j=1}^n w_j \left[1 - \Phi\left(\Phi^{-1}(1 - m_i) + \Delta_{i,j}\right)\right] \right) dm_i \ , \tag{18}$$

with

$$\Delta_{i,j} \begin{cases} = +\infty, & \text{if } \hat{x}_j > \hat{x}_i \\ = -\infty, & \text{if } \hat{x}_j < \hat{x}_i \\ \in [-\infty, +\infty], & \text{if } \hat{x}_j = \hat{x}_i \end{cases} , \tag{19}$$

and

$$-\Delta_{i,j} = \Delta_{j,i} = \sum_{k=j+1}^{i} \Delta_{k-1,k} \ . \tag{20}$$

Conversely, if $\{\hat{x}_i, \Delta_{i,j}\}_{i,j=1}^n$ satisfies this system of equations, then investors' switching cut-offs converge to $\{\hat{x}_i\}_{i=1}^n$ under the disclosure $\{(r_i, c_i; w_i)\}_{i=1}^n$ as $\sigma \to 0$.

As a generalization of (11), equation (18) characterizes investors' limiting cutoffs. Equations (19) and (20) follow the definition of $\Delta_{i,j}^\sigma$. Notably, these conditions are not only necessary but also sufficient for $\{\hat{x}_i\}_{i=1}^n$ to be the limiting cutoffs. Sufficiency is especially important for an information design problem, because it guarantees that the disclosure derived from these conditions can surely induce the desired equilibrium.

Why does the equilibrium operate in this manner? We exemplify the rationale with two-score disclosures in dimension c (i.e., $r_1 = \mathbb{E}r = r_2$ but $c_1 < \mathbb{E}c < c_2$) for expositional convenience. A similar argument applies to disclosures in dimension r. For any $\sigma > 0$, the equilibrium cutoffs $(\hat{x}_1^\sigma, \hat{x}_2^\sigma)$ should equalize all cutoff investors' expected payoffs from staying to zero, the payoff from running. Then by (11), we obtain

$$\hat{x}_1^\sigma - \mathbb{E}r \cdot \int_{m_1^\sigma=0}^{1} a\left(w_1 m_1^\sigma + w_2\left[1 - \Phi\left(\Phi^{-1}(1-m_1^\sigma) + \Delta_{1,2}^\sigma\right)\right]\right) dm_1^\sigma - c_1 = O\left(\sigma\right) \qquad (21)$$

and

$$\hat{x}_2^\sigma - \mathbb{E}r \cdot \int_{m_2^\sigma=0}^{1} a\left(w_1\left[1 - \Phi\left(\Phi^{-1}(1-m_2^\sigma) - \Delta_{1,2}^\sigma\right)\right] + w_2 m_2^\sigma\right) dm_2^\sigma - c_2 = O\left(\sigma\right) \ , \qquad (22)$$

where the left-hand sides of the equations are the expected payoffs of cutoff score-1 and cutoff score-2 investors, respectively.

The first term of the expected payoff, $\hat{x}_i^\sigma$, corresponds to a score-c_i investor's expectation of the fundamental θ conditional on receiving his cutoff signal $\hat{x}_i^\sigma$. The cutoff investors of a score that induces a lower cutoff are more pessimistic about the fundamental than those of a score inducing a higher cutoff. This corresponds to the *fundamental channel*, and its magnitude largely depends on the *absolute distance* between the cutoffs, $\hat{x}_2^\sigma - \hat{x}_1^\sigma$. As explained in Section 3.3, the second term of the expected payoff is the systemic risk expected by a cutoff score-c_i investor. The cutoff investors of a score inducing a lower cutoff expect more systemic risk than those of a score inducing a higher cutoff. This corresponds to a *strategic channel,* whose magnitude largely depends on the *relative distance* between the cutoffs, $\Delta_{1,2}^\sigma \triangleq (\hat{x}_2^\sigma - \hat{x}_1^\sigma)/\sigma$.

First, according to the directions of the two channels, we can readily see that $\hat{x}_1^\sigma < \hat{x}_2^\sigma$. Otherwise, compared with a cutoff score-c_2 investor, a cutoff score-c_1 investor is more optimistic about the fundamental and expects less systemic risk, but his bank is also informationally stronger (i.e., $c_1 < c_2$), rendering his expected payoff strictly higher than a cutoff score-c_2 investor's. Then, it is impossible to make cutoff investors of both scores indifferent, contradicting the equilibrium condition.

Next, let's think about what happens when σ is small. As $\hat{x}_2^\sigma - \hat{x}_1^\sigma$ increases from 0, $\Delta_{1,2}^\sigma$ increases quickly and results in substantial difference in the systemic risk expected by the cutoff investors of the two scores. It is possible that even when $\hat{x}_2^\sigma - \hat{x}_1^\sigma$ is of the order $O(\sigma)$, $\Delta_{1,2}^\sigma$ is large so that the cutoff investors of the two scores have very different expectations of systemic risk. Recall from Section 3.3 that the difference reaches its maximum when $\Delta_{1,2}^\sigma$ approaches infinity, which corresponds to the case that a cutoff score-1 (score-2) investor becomes almost certain that score-2 (score-1) investors are running (staying).

If $c_2 - c_1$ is not too large, as in the reallocation regions in Figure 1, the maximum difference in expected systemic risk, $\int_{m_1^\sigma=0}^1 a(w_1 m_1^\sigma)\, dm_1^\sigma - \int_{m_2^\sigma=0}^1 a(w_1 + w_2 m_2^\sigma)\, dm_2^\sigma = \frac{A(w_1)}{w_1} - \frac{A(1)-A(w_1)}{1-w_1}$, multiplied by the common systemic vulnerability $\mathbb{E}r$ is greater than $c_2 - c_1$. Then a small $\hat{x}_2^\sigma - \hat{x}_1^\sigma$ (of the order $O(\sigma)$) suffices to make up the difference $c_2 - c_1$ in the expected payoffs of the investors through the strategic channel. Therefore, we end up with an equilibrium in which investors have almost the same cutoff but the systemic risk expected by cutoff investors is quite different: $\hat{x}_2^\sigma - \hat{x}_1^\sigma = O(\sigma)$ but $\Delta_{1,2}^\sigma > 0$. In this case, only the strategic channel comes into effect, and the fundamental channel is negligible.

If $c_2 - c_1$ is large, as in the separation regions in Figure 1, the strategic channel alone is not enough to make up the difference $c_2 - c_1$ in the expected payoffs. Then the fundamental channel is required to kick in, so that the indifference conditions (21) and (22) can be maintained. As a result, we end up with an equilibrium in which investors have substantially different cutoffs: $\hat{x}_2^\sigma - \hat{x}_1^\sigma > 0$ and $\Delta_{1,2}^\sigma \to \infty$ at vanishing σ. In this case, both channels have substantial impacts, and the strategic channel is exhausted, which is reflected by $\Delta_{1,2} = +\infty$ as in (19).

Without loss of generality, henceforth we reorder the scores such that $\Delta_{i-1,i} \geq 0$ for all i. Based on Proposition 3, we define the concepts of entanglement, separation and adjacency. Figure 2 illustrates the concepts with a disclosure only in dimension r; i.e., with $c_i = \mathbb{E}c$ for all i.

Definition 1. For a pair of scores (r_i, c_i) and (r_j, c_j) with $i < j$,

- if $\Delta_{i,j} < +\infty$, then we must have $\hat{x}_i = \hat{x}_j$, and we say scores (r_i, c_i) and (r_j, c_j) are *entangled*;
- if $\hat{x}_j > \hat{x}_i$, then we say scores (r_i, c_i) and (r_j, c_j) are *separate*;
- if $\Delta_{i,j} = +\infty$ and $\hat{x}_i = \hat{x}_j$, then we say scores (r_i, c_i) and (r_j, c_j) are *adjacent*.

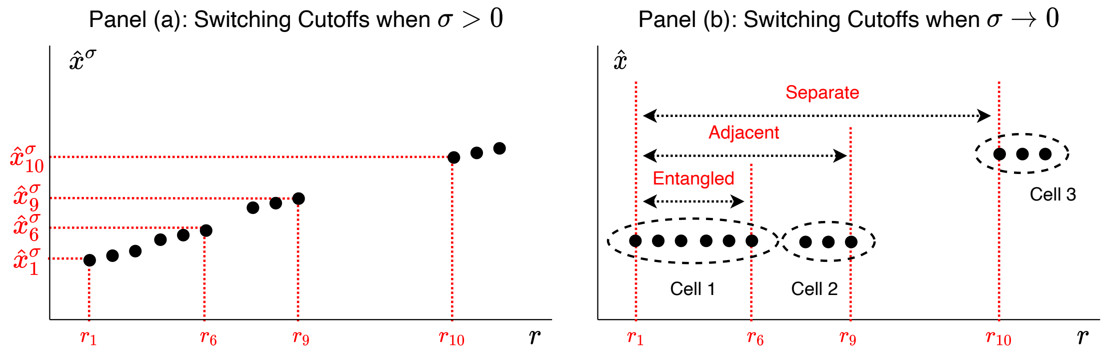

Figure 2: Switching Cutoffs Given a Finite Disclosure Rule

Recall from (16) and the discussion above that the strategic uncertainty between investors of banks receiving scores (r_i, c_i) and (r_j, c_j) is determined by $(\hat{x}^\sigma_j - \hat{x}^\sigma_i)/\sigma$, which converges to $\Delta_{i,j}$ as $\sigma \to 0$. Entanglement refers to the situation in which the investors of banks receiving two scores have substantial strategic uncertainty, as for r_1 and r_6 in Figure 2(b). Separation refers to the situation in which the investors of banks with the two scores have distinct cutoffs. Necessarily, there is no strategic uncertainty between them, as for r_1 and r_{10} in Figure 2(b). Adjacency refers to the knife-edge situation in which the investors of banks with the two scores have the same limiting cutoff but no strategic uncertainty, as for r_1 and r_9 in Figure 2(b).

Moreover, entanglement defines an equivalence relation on scores, and divides all investors into several partition cells: there is strategic uncertainty between the investors of banks with scores in the same cell, but not between those of banks with scores in different cells. As illustrated in Figure 2(b), investors in different partition cells share the same switching cutoff if they are adjacent (as Cell 1 and Cell 2), and have different switching cutoffs if they are separate (as Cell 1 and Cell 3). Proposition 4 shows that given the partition defined by entanglement, we can obtain explicit expressions for the switching cutoffs in the limiting case.

Proposition 4. *Given a finite disclosure, the limiting Bayes Nash equilibrium is characterized by a consecutive* $Z-$*partition of* $\{1, \dots, n\}$, $\{\{i : p_z \leq i < p_{z+1}\} \,|z = 1, 2, \dots, Z\}$, *with* $1 = p_1 < \cdots < p_z < \cdots < p_{Z+1} = n + 1$, *such that*

- *Scores in the same partition cell are all entangled with each other;*
- *Scores in different partition cells are adjacent or separate; and*

- *If $i \in \{p_z, p_z+1, \cdots, p_{z+1}-1\}$, then*

$$\hat{x}_i = \left(\sum_{j=p_z}^{p_{z+1}-1} \frac{w_j}{r_j}\right)^{-1} \left[\sum_{j=p_z}^{p_{z+1}-1} \frac{c_j}{r_j} w_j + A\left(\sum_{j=1}^{p_{z+1}-1} w_j\right) - A\left(\sum_{j=1}^{p_z-1} w_j\right)\right], \tag{23}$$

where $A(\cdot)$ is given by (4). Moreover, $\{\hat{x}_i\}_{i=1}^n$ is a weakly increasing sequence.

4.2 Robust Disclosures

Recall from Section 2.5 that optimal disclosures yield stability schemes not first-order stochastically dominating those resulting from any other feasible disclosure. In this subsection, we formally introduce the concept of robust disclosures and show that an optimal disclosure must be a combination of robust disclosures.

As the first step, we restrict our attention to disclosures with no more than $n \geq 2$ scores.[15] Since the set of n-score disclosures is bounded and closed, optimal disclosures exist. Now suppose an optimal disclosure induces $T \leq n$ distinct limiting switching cutoffs in equilibrium, ranked as $\theta_1 < \theta_2 < \ldots < \theta_T$. Then we can regard a disclosure as a collection of sub-disclosures, each of which is imposed on a group of banks whose investors share the same limiting switching cutoff. Intuitively, given other investors' equilibrium behavior, any sub-disclosure of an optimal disclosure should minimize the corresponding investors' cutoffs by and large. Based on this idea, we define (κ, t)-robust disclosures for a bank group and show that any sub-disclosure of an optimal disclosure must be a certain (κ, t)-robust disclosure for the bank group.

Definition 2. A *bank group* (W, Q^r, Q^c) refers to a mass W of banks, mass Q^r of which are $\underline{r}$-type banks and mass Q^c of which are $\underline{c}$-type banks.

Definition 3. A (κ, t)*-robust disclosure* in dimension r or c for a bank group (W, Q^r, Q^c) is a disclosure with no more than t scores in that dimension for this group, that minimizes the maximum switching cutoffs of its investors, given that among all banks (of mass $1 - W$) outside this group, mass κ are immune from runs and the rest are subject to runs almost surely.

[15] Note that $n = 1$ means no disclosure.

A robust disclosure is so named for two reasons. First, the bank group as a whole is immune from runs only if its weakest constituent is immune; i.e., the bank whose investor's switching cutoff is the maximum among the whole group. By minimizing this maximum cutoff, a (κ, t)-robust disclosure maximizes the stability of the weakest constituent, and thus of the whole group, to adverse fundamental shocks. Second, this approach is in the spirit of maxmin expected utility theory of Gilboa and Schmeidler (1989), and Hansen and Sargent (2001) show its connection to the robust-control theory.

Proposition 5. *Suppose $K\left(\cdot;\{K_i,\theta_i\}_{i=1}^T\right)$ is a stability scheme resulting from an n-score optimal disclosure. Let t_i denote the number of scores whose corresponding investors share the limiting switching cutoff θ_i. Then for any i, the sub-disclosure for the bank group consisting of all the banks whose investors share the switching cutoff θ_i must be the (K_{i-1}, t_i)-robust disclosure for the group.*

Proposition 5 confirms that the regulator's optimal disclosure must be a combination of (κ, t)-robust disclosures. To see the intuition, consider an optimal disclosure and the resulting stability scheme $K\left(\cdot;\{K_i,\theta_i\}_{i=1}^T\right)$. Consider the bank group consisting of all the banks whose investors share the switching cutoff θ_i, whose mass is $K_i - K_{i-1}$. As long as their signal realizations are in $(\theta_{i-1},\theta_{i+1})$, which includes θ_i, these investors think that among investors outside this bank group, whose mass is $1-(K_i-K_{i-1})$, those with cutoffs no greater than θ_{i-1}, whose mass is K_{i-1}, stay almost surely, and the rest run almost surely. This is precisely the condition on the outsiders of this bank group for its (K_{i-1}, t_i)-robust disclosure. Denote the maximum switching cutoffs under the (K_{i-1}, t_i)-robust disclosure by θ_i'. By definition, we have $\theta_i' \leq \theta_i$. But the optimality of the original disclosure implies that $\theta_i' = \theta_i$. Otherwise, we can replace the original sub-disclosure for this bank group with its (K_{i-1}, t_i)-robust disclosure, without changing the original sub-disclosures for its outsiders. It can be shown that under this alternative disclosure, investors of this bank group would have switching cutoffs no more than θ_i', while other investors' switching cutoffs do not increase relative to their original levels. This results in a stability scheme that is first-order stochastically dominated by $K\left(\cdot;\{K_j,\theta_j\}_{j=1}^T\right)$, violating the optimality of the original disclosure.

In the rest of the section, we derive (κ, t)-robust disclosures in dimensions c and r respectively for $t \geq 2$.[16] Since the mass of stayers outside the bank group is fixed at κ, we can equivalently consider the investors of the given bank group (W, Q^r, Q^c) as playing a coordination game only among themselves, where the systemic risk they face when the mass of stayers among them is l is

$$a_\kappa(l) \triangleq a(l + \kappa)$$

instead of $a(l)$. We then define

$$A_\kappa(l) \triangleq \int_0^l a_\kappa(w)dw = A(l + \kappa) - A(\kappa) \ .$$

By definition, $A_\kappa(0) = 0$. Thus, the equilibrium of the game is still characterized by Propositions 3 and 4, with $a(l)$ and $A(l)$ replaced by $a_\kappa(l)$ and $A_\kappa(l)$, respectively.

4.3 Robust Disclosures in Dimension c

First, we consider disclosures in dimension c, where $r_i = \mathbb{E}r$ for all i. As discussed in Section 3 and implied by (23), the average switching cutoff of the whole bank group is not affected by disclosures in dimension c, so the maximum of the switching cutoffs is minimized as long as all investors in the bank group have the same switching cutoff. This is because holding constant the expected idiosyncratic cost of the group, $\frac{[Q^c \cdot \underline{c} + (W - Q^c) \cdot \bar{c}]}{W}$, the resulting assortative matching, which reallocates more of the constant aggregate systemic risk, $A_\kappa(W)$, to banks believed to have lower idiosyncratic costs, is not conducive to mitigating systemic runs.

Proposition 6. *Nondisclosure is a (κ, t)-robust disclosure in dimension c for any bank group (W, Q^r, Q^c), in which all investors share the switching cutoff*

$$\hat{x}_c(W, Q^c, \kappa) = \frac{[Q^c \cdot \underline{c} + (W - Q^c) \cdot \bar{c}]}{W} + \mathbb{E}r \cdot \frac{A_\kappa(W)}{W} \ . \qquad (24)$$

4.4 Robust Disclosures in Dimension r

Now we consider disclosures in dimension r, where $c_i = \mathbb{E}c$ for all i. In the two-score setup in Section 3, the maximum of the two cutoffs reaches its minimum when informational heterogeneity in dimension r is maximized, provided that the two cutoffs coincide. This is because

[16] Note that $t = 1$ means no disclosure.

of beneficial assortative matching: holding constant the expected systemic vulnerability of the bank group, more of the constant aggregate systemic risk is reallocated to investors of banks that are believed to be less vulnerable to systemic risk. This improves the stability of all banks. We show in this subsection that this conclusion also holds for the design of a (κ, t)-robust disclosure. For notational convenience, we suppress Q^c in (W, Q^r, Q^c) in the rest of this section.

As discussed in Section 3, if the heterogeneity between two bank scores is sufficiently large, their investors will have different switching cutoffs. Then, reducing the heterogeneity can always lower the maximum of the switching cutoffs until they become equal. Proposition 7 confirms that this finding holds generally.

Proposition 7. *In a (κ, t)-robust disclosure in dimension r, all scores must be either entangled or adjacent to each other.*

According to Proposition 7, we focus on disclosures that result in the same switching cutoffs for all investors, and robust disclosures are those that minimize the common switching cutoff. By (23), the common switching cutoff is

$$\hat{x}_i = \mathbb{E}c + \left(\sum_j \frac{w_j}{r_j}\right)^{-1} \cdot A_\kappa \left(\sum_j w_j\right) ,$$

which generalizes (5). As an integral over $a_\kappa(\cdot) > 0$, $A_\kappa\left(\sum_j w_j\right) > 0$ and thus $\hat{x}_i$ is strictly increasing in $\left(\sum_j \frac{w_j}{r_j}\right)^{-1}$. Lemma 1 establishes that when two scores are entangled, marginally increasing informational heterogeneity in dimension r with a mean-preserving spread of scores reduces $\left(\sum_j \frac{w_j}{r_j}\right)^{-1}$ and thus $\hat{x}_i$. This captures the essence of the beneficial assortative matching discussed in Section 3.3.

Lemma 1. *Suppose $r_i' \leq r_i \leq r_j \leq r_j'$, $w_i r_i + w_j r_j = w_i' r_i' + w_j' r_j'$, and $w_i + w_j = w_i' + w_j'$. Then we have*

$$\frac{w_i}{r_i} + \frac{w_j}{r_j} \leq \frac{w_i'}{r_i'} + \frac{w_j'}{r_j'},$$

and the equality holds if and only if $r_i' = r_i$ and $r_j' = r_j$.

Proof. The proof is straightforward from the convexity of $f(r) = 1/r$ in $(0, +\infty)$. □

Hence, a (κ, t)-robust disclosure should maximize the heterogeneity in dimension r, provided that all investors have the same switching cutoff. Yet, the extent of such maximization is restricted by the original physical heterogeneity among banks, as captured by $\bar{r}/\underline{r}$. To capture this innate constraint, we define strong/weak κ-heterogeneity in dimension r.

Definition 4. A bank group (W, Q^r) has weak κ-heterogeneity if

$$\bar{r}/\underline{r} \leq \frac{A_\kappa(Q^r)}{Q^r} \frac{W - Q^r}{A_\kappa(W) - A_\kappa(Q^r)} .$$

Otherwise, it has strong κ-heterogeneity.

If a bank group (W, Q^r) has weak κ-heterogeneity, then with mass κ of banks outside the group immune from runs almost surely, all investors in the group share the same switching cutoff, even if the informational heterogeneity is maximized with full disclosure. Thus,

Proposition 8. *If a bank group (W, Q^r) has weak κ-heterogeneity, then its (κ, t)-robust disclosure in dimension r is full disclosure.*

The rest of this section focuses on (κ, t)-robust disclosures for bank groups with strong κ-heterogeneity. Proposition 9 characterizes their key properties.

Proposition 9. *For any bank group with strong κ-heterogeneity, all scores of a (κ, t)-robust disclosure in dimension r must be adjacent to each other. Moreover, the resulting common switching cutoff is strictly decreasing in t.*

The first property reflects two considerations. By Lemma 1, we want to maximize informational heterogeneity in dimension r. However, due to the strong κ-heterogeneity, too much informational heterogeneity will make scores separate, which is not desirable according to Proposition 7. Therefore, we stop at adjacency, the knife-edge case that investors' cutoffs are about to diverge.

The second property implies that allowing for more scores always reduces the common switching cutoff induced by a robust disclosure. To see this, note that for any (κ, t)-robust disclosure, we can regard one of its scores as two scores with the same mass but forced to coincide by the constraint on total number of scores. If one more score is allowed, the regulator can at least make these two scores also adjacent, which further reduces the common switching cutoff.

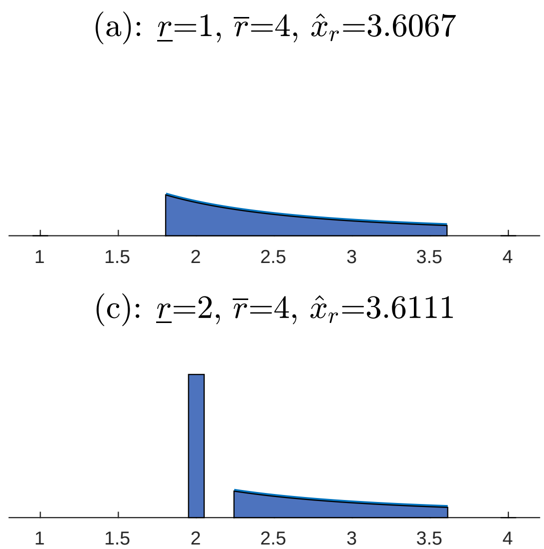

Figure 3: Limiting robust disclosures

4.5 The Limiting Robust Disclosure for Bank Groups with Strong κ-heterogeneity

By Proposition 9, for a bank group with strong κ-heterogeneity, the regulator would prefer to implement a sub-disclosure with as many scores as possible to lower the common cutoff. What if she is allowed to assign as many scores as she likes? In this subsection, we construct the limiting robust disclosure and study its properties.

For a bank group (W, Q^r), denote the candidate for the limit of its (κ, t)-robust disclosure as $t \to +\infty$ by a function

$$\Omega(\cdot; W, Q^r, \kappa) : [\underline{r}, \overline{r}] \to [0, W] \ ,$$

where $\Omega(r; W, Q^r, \kappa)$ represents the mass of banks whose scores are less than or equal to r. Figure 3 illustrates its structure with $\mathbb{E}r = 2.5$, $\mathbb{E}c = 0$, and $a(l) = 2 - l$.

The essence of adjancency is the maximization of informational heterogeneity under the constraint of a common switching cutoff. Consider first the situation where $\underline{r}$ is so low and $\bar{r}$ is so high that they do not restrain the regulator through Bayesian plausibility from pushing such maximization to the extreme, as illustrated in Figure 3(a). First, such maximization reduces the mass of banks sharing the same score to zero, so that $\Omega(\cdot; W, Q^r, \kappa)$ is continuous. Otherwise, further reduction of the common switching cutoff is feasible by replacing a score of positive mass with its mean-preserving spread. Second, such maximization also eliminates the strategic uncertainty faced by all investors: when a score-r investor is indifferent, he believes that among investors in (W, Q^r), those with scores $r' < r$, whose mass is $\Omega(r; W, Q^r, \kappa)$, stays almost surely, and the others run almost surely, so that his switching cutoff, which he shares with all investors due to adjacency of scores, is

$$\hat{x}_r(W, Q^r, \kappa) \equiv \mathbb{E}c + r \cdot a_\kappa(\Omega(r; W, Q^r, \kappa)). \tag{25}$$

In this situation, the right-hand side of (25) must be constant over all r in the support of Ω,[17] and we refer to (25) as the common-switching-cutoff constraint. Lastly, the total mass of banks and the mass of type-$\underline{r}$ banks given by Ω must be consistent with (W, Q^r); i.e., $\Omega(\bar{r}; W, Q^r, \kappa) = W$ and $\int_{r=\underline{r}}^{\bar{r}} r \cdot d\Omega(r; W, Q^r, \kappa) = \underline{r} \cdot Q^r + \bar{r} \cdot [W - Q^r]$. There is a unique Ω satisfying all these properties. To avoid distracting readers with technicalities, we relegate its detailed construction to Section A of the Appendix.

Now consider the situation where $\bar{r}$ is low so that it restrains the regulator from further spreading scores beyond it, as illustrated in Figure 3(b). In this situation, a positive mass has to be "piled" at $\bar{r}$. Note that there is a gap between $\bar{r}$ and the supremum of the continuous component of the distribution, r^+. Strategic complementarity between all investors and the strategic uncertainty between investors of these score-$\bar{r}$ banks leads to the isolation of these banks from the rest. To see this, while both a cutoff score-r^+ investor and a cutoff score-$\bar{r}$ investor believe that non score-$\bar{r}$ investors are staying almost surely, the cutoff score-r^+ investor believes that all score-$\bar{r}$ investors are running almost surely, while the cutoff score-$\bar{r}$ investor believes that the proportion of score-$\bar{r}$ investors who stay is uniformly distributed in $[0, 1]$. Thus, as long as the mass piled at $\bar{r}$ is positive, there is a non-infinitesimal difference in the systemic risk that they expect, and the common-switching-cutoff constraint (25) requires a gap between r^+ and $\bar{r}$. Similar phenomena occur when only $\underline{r}$ is restrictive, as in Figure 3(c), and when both $\underline{r}$ and $\bar{r}$ are restrictive, as in Figure 3(d). When $\underline{r}$ and $\bar{r}$ become so restrictive that (W, Q^r) has weak κ-heterogeneity, the continuous component of the distribution vanishes, consistent with Proposition 8 that full disclosure is the (κ, t)-robust disclosure for (W, Q^r) in dimension r for all t.

Proposition 10. *Consider disclosures in dimension r for any bank group (W, Q^r) with strong*

[17] In general, (25) holds for all scores of mass zero in limiting robust disclosures.

κ-heterogeneity.

- *$\hat{x}_r(W, Q^r, \kappa)$ is the infimum of the switching cutoffs of (κ, t)-robust disclosures in dimension r for all $t \geq 1$.*
- *(κ, t)-robust disclosures in dimension r converge to $\Omega(\cdot; W, Q^r, \kappa)$ as $t \to \infty$, in the sense that the distance between their quantile functions converges to 0 in the L^1-norm.*

Proposition 10 provides two pieces of important information. First, as the number of scores t goes to infinity, the common cutoff of (κ, t)-robust disclosures converges downward to $\hat{x}_r(W, Q^r, \kappa)$. Hence, $\hat{x}_r(W, Q^r, \kappa)$ can be considered as the (asymptotically) lowest cutoff the regulator can achieve with sufficient scores. Second, $\Omega(\cdot; W, Q^r, \kappa)$ is indeed the limit of (κ, t)-robust disclosures as the number of scores t goes to infinity.

It is worth noting that although $\Omega(\cdot; W, Q^r, \kappa)$ involves infinitely many scores and thus is not feasible in a practical design problem, it still provides a meaningful benchmark. First, for sufficiently large t, (κ, t)-robust disclosures are arbitrarily close to $\Omega(\cdot; W, Q^r, \kappa)$. Second, $\Omega(\cdot; W, Q^r, \kappa)$ can be explicitly characterized, so that it serves as a tractable tool that enables us to study the nature of optimal disclosures with many scores. Therefore, we refer to $\Omega(\cdot; W, Q^r, \kappa)$ as the κ-robust disclosure for the bank group (W, Q^r). In addition, the notion of a κ-robust disclosure can also be extended to disclosures in dimension c (which is nondisclosure, Proposition 6), and to those in dimension r for bank groups with weak κ-heterogeneity (which is full disclosure, Proposition 8). In the application in Section 5, we characterize optimal disclosures based on κ-robust disclosures in all these cases.

5 An Application: Public Disclosure of Stress-Test Results

Since the 2007-08 financial crisis, a vigorous debate has arisen concerning whether and how the regulator should disclose the results of the stress tests of individual financial institutions. The essence of this debate is the design of optimal public disclosure of bank-specific information that mitigates systemic bank runs. This section complements existing discussions with two novel implications due to the presence of systemic risk. Theoretically, this section also demonstrates the critical role of robust disclosures developed in Section 4 in the construction of optimal disclosures, given a complete preference of the regulator that respects the partial order in Section 2.5.

5.1 Optimal Disclosures

In practice, the regulator is concerned about whether banks are able to withstand negative economic shocks. As manifested by the design of stress tests, the regulator often focuses on hypothetical adverse scenarios and makes policies accordingly to improve financial stability in these scenarios. Motivated by this observation, we assume that the regulator's objective is to maximize the mass of banks immune from runs in a hypothetical adverse scenario, where the fundamental θ equals an exogenous $\hat{\theta}$.

Proposition 11. *Consider optimal disclosures in dimension r. Let $\hat{x}_r$ be given by (25). Suppose $\overline{r}/\underline{r} > \frac{A(q^r)}{q^r}\frac{1-q^r}{A(1)-A(q^r)}$, where*

- *If $\hat{\theta} \geq \hat{x}_r(q^r, q^r, 0)$, a mass $1-W\left(\hat{\theta}, q^r\right)$ of type-$\overline{r}$ banks are fully revealed and subject to runs at $\hat{\theta}$ while the remaining banks are revealed as specified by their 0-robust disclosure and are immune from runs at $\hat{\theta}$. Here, $W\left(\hat{\theta}, q^r\right)$ is the maximum W in $[q^r, 1]$ such that $\hat{x}_r(W, q^r, 0) \leq \hat{\theta}$.*
- *If $\hat{\theta} < \hat{x}_r(q^r, q^r, 0)$, no bank is immune from runs at $\hat{\theta}$ regardless of disclosures.*

Suppose $\overline{r}/\underline{r} \leq \frac{A(q^r)}{q^r}\frac{1-q^r}{A(1)-A(q^r)}$. All banks are revealed as specified by their 0-robust disclosure. If $\hat{\theta} \geq \hat{x}_r(1, q^r, 0)$, they are all immune from runs at $\hat{\theta}$; otherwise, they are all subject to runs at $\hat{\theta}$.

Again, we adopt the law of large numbers convention[18] so that q^r is also the mass of type-$\underline{r}$ banks in the system. When physical heterogeneity in systemic vulnerabilities is strong, investors of different banks may have different switching cutoffs. This makes room for the regulator to reduce the cutoffs of some banks and prevent runs on them at the cost of increasing the cutoffs of the others through disclosures. Since the regulator cares only about the mass of banks immune from runs at $\hat{\theta}$, all sacrificed banks must be physically weak and fully revealed, and the corresponding 0-robust disclosure is made to "preserved" banks to maximize their joint resilience. When physical heterogeneity in systemic vulnerabilites is weak, all investors always share the same cutoff in equilibrium, so in any state, banks are either all immune from or all subject to runs. As implied by Proposition 5, the 0-robust

[18] See Footnote 12.

disclosure for all banks maximizes their resilience. Proposition 12 follows an analogous logic. Again, by the law of large numbers convention, q^c is also the mass of type-$\underline{c}$ banks in the system.

Proposition 12. *Consider optimal disclosures in dimension c. Let $\hat{x}_c$ be given by (24). Suppose $\overline{c} - \underline{c} > \left[\frac{A(q^c)}{q^c} - \frac{A(1)-A(q^c)}{1-q^c}\right]\mathbb{E}r$.*

- *If $\hat{\theta} \geq \hat{x}_c(q^c, q^c, 0)$, a mass $1 - W\left(\hat{\theta}, q^c\right)$ of type-$\overline{c}$ banks are fully revealed and subject to runs at $\hat{\theta}$ while the remaining banks are revealed as specified by their 0-robust disclosure and immune from runs at $\hat{\theta}$. Here, $W\left(\hat{\theta}, q^c\right)$ is the maximum W in $[q^c, 1]$ such that $\hat{x}_c(W, q^c, 0) \leq \hat{\theta}$.*
- *If $\hat{\theta} < \hat{x}_c(q^c, q^c, 0)$, no bank is immune from runs at $\hat{\theta}$ regardless of disclosures.*

Suppose $\overline{c} - \underline{c} \leq \left[\frac{A(q^c)}{q^c} - \frac{A(1)-A(q^c)}{1-q^c}\right]\mathbb{E}r$. All banks are revealed as specified by their 0-robust disclosure. If $\hat{\theta} \geq \hat{x}_c(1, q^c, 0)$, they are all immune from runs at $\hat{\theta}$; otherwise, they are all subject to runs at $\hat{\theta}$.

Figures 4[19] and 5[20] illustrate how the respective optimal disclosures in dimensions r and c evolve with the deterioration of the average quality of the banking system, represented by a reduction in q^r or q^c. Banks immune from runs are indicated in blue, whose mass is represented by w, and those subject to runs are denoted in red.

5.2 Implications of Systemic Risk

A key addition of our model to the standard Bayesian persuasion model is systemic risk and the consequent strategic complementarity between the investors of different banks. This subsection stipulates two novel implications of this addition.

First, the optimal disclosure of systemic vulnerability differs qualitatively from that of idiosyncratic factors (i.e., "cost") for banks immune from runs. Recall from Section 3 that the former reallocates more of the constant aggregate systemic risk to the banks known to be less vulnerable to it, while such beneficial assortative matching is absent in the latter. As a result, the former maximally differentiates banks immune from runs provided that they are

[19]For all panels, $(\underline{r}, \overline{r}) = (2, 3)$, $\mathbb{E}c = 0$, $a(l) = 2 - l$, and $\overline{r}/\underline{r} > \frac{A(q^r)}{q^r}\frac{1-q^r}{A(1)-A(q^r)}$ is satisfied.

[20]For all panels, $(\underline{c}, \overline{c}) = (0, 2)$, $\mathbb{E}r = 2.5$, $a(l) = 2 - l$, $\hat{\theta} = 4.75$and $\overline{c} - \underline{c} > \left[\frac{A(q^c)}{q^c} - \frac{A(1)-A(q^c)}{1-q^c}\right]\mathbb{E}r$ is satisfied.

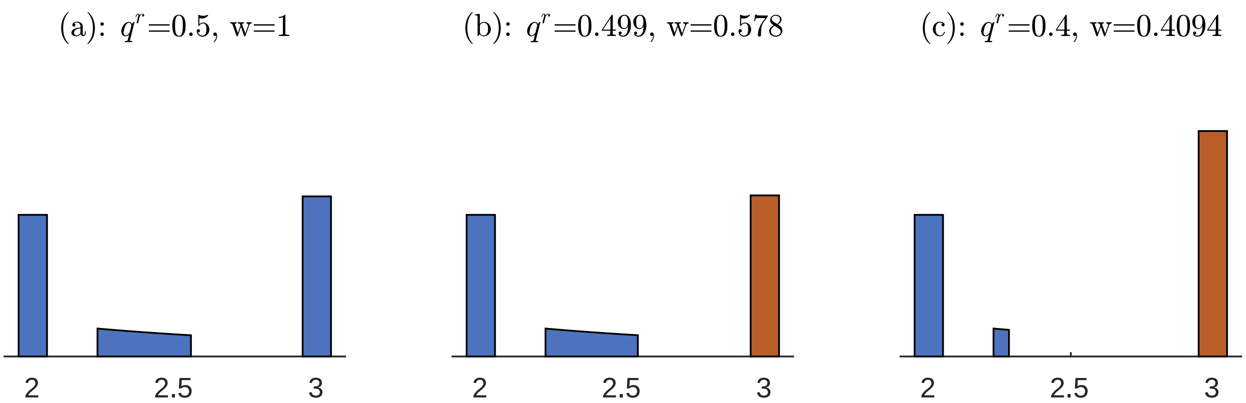

Figure 4: Optimal disclosures in dimension r

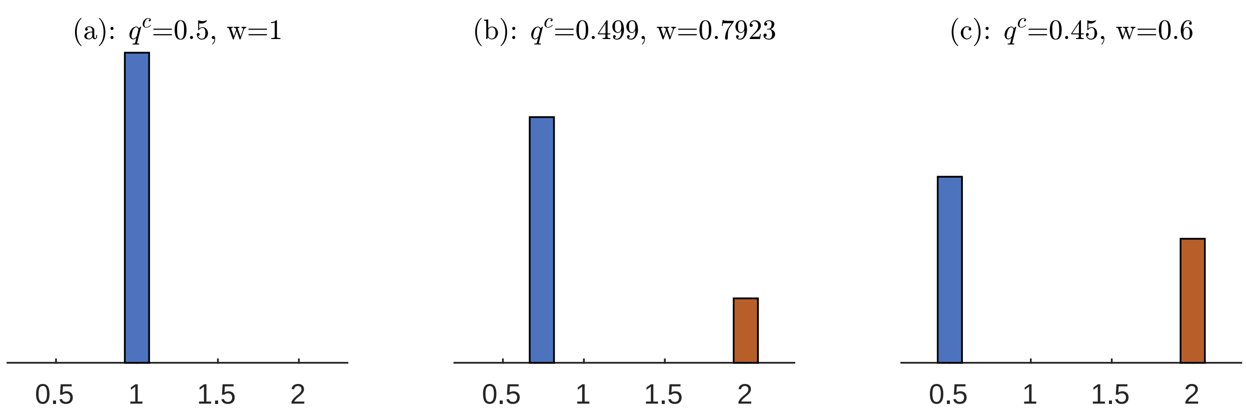

Figure 5: Optimal disclosures in dimension c

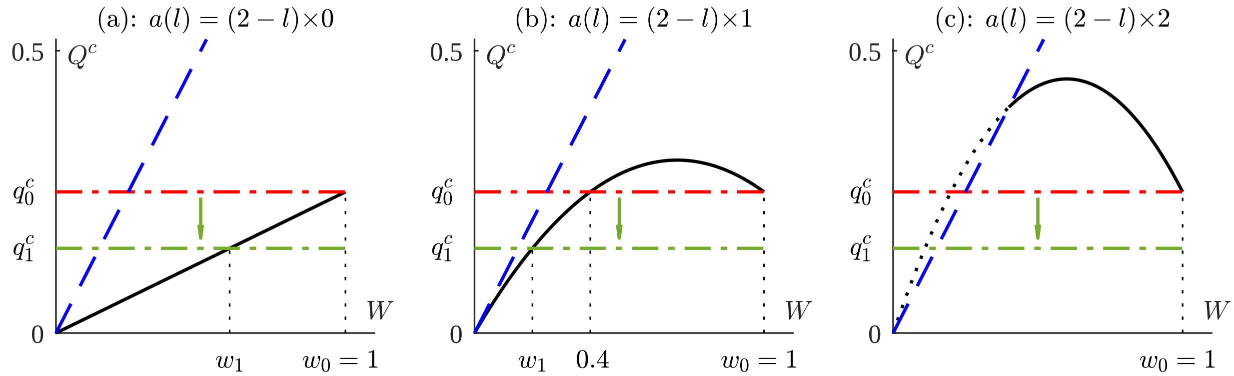

Figure 6: The minimum mass of type-$\underline{c}$ banks

equally resilient (see Propositions 7-9), while the latter does not entail such differentiation (see Proposition 6). This distinction is driven by the systemic risk in the banking system: if systemic risk is absent, so is the assortative matching, and differentiation of preserved banks is no longer beneficial, as in a standard Bayesian persuasion model.

Second, when the quality of the banking system deteriorates, the regulator may have to face a sudden run on a huge mass of banks rather than gradually abandoning weak banks. That is, the mass w of banks immune from runs under the optimal disclosure may experience a negative discontinuous jump from 1, as the average quality of the whole banking system, measured by q^r or q^c, falls below a critical level that could protect all banks from runs. (The corresponding thresholds in Figures 4 and 5 are $q^r = 0.5$ and $q^c = 0.5$, respectively.)

Figure 6 illustrates the role of systemic risk in shaping this implication with disclosures in dimension c. To figure out the maximum mass w of banks that can be immune from runs at $\hat{\theta}$ given q^c, and subsequently how a fall in q^c affects w, we first consider the following question: as a candidate for the desired group of immune banks, suppose banks outside a group of mass W are run almost surely at $\hat{\theta}$ (just like banks outside the desired group), what is the minimum mass Q^c of physically strong (i.e., type-$\underline{c}$) banks required in the group for it to be immune from runs, given $\hat{\theta}$?[21] The answer is given by (24), with κ and $\hat{x}_c(W, Q^c, \kappa)$ fixed at 0 and $\hat{\theta}$, respectively. The black solid curves plot Q^c against W accordingly, given the same $\hat{\theta} = 4.5$ but different systemic risk functions $a(\cdot)$, respectively.[22]

[21] That is, to guarantee that the common switching cutoff $\hat{x}_c$ resulting from its 0-robust disclosure is no greater than $\hat{\theta}$.

[22] For the three panels, we scale up $a(\cdot)$ while keeping the expected systemic vulnerability equal to $\mathbb{E}r = 2.5$.

Figure 6(a) depicts the benchmark without systemic risk by fixing $a(\cdot) \equiv 0$. There, a constant average quality of the bank group, and thus a constant Q^c/W, are required for the group to be preserved given the same $\hat{\theta}$. Thus, the black solid curve is a straight line passing through the origin. To keep the whole system from runs, Q^c must reach the critical level q_0^c indicated by the red dot-dashed line. The total mass q^c of physically strong banks in the system determines the maximum mass w of immune banks through the constraint $Q^c \leq q^c$. If $q^c > q_0^c$, the constraint slacks and $w = 1$. Now consider a fall in q^c from q_0^c to q_1^c, as shown by the downward shift of the horizontal dot-dashed line indicated by the green arrow. The constraint $Q^c \leq q^c$ binds in the process, and the mass w of immune banks adjust *linearly* from $w_0 = 1$ down to w_1, the level corresponding to $Q^c = q_1^c$ on the black solid line. Thus, without systemic risk, there is no discontinuous jump in w.

Systemic risk may break the monotonic relation between Q^c and W, as illustrated in Figures 6(b) and (c): since banks outside the group are run almost surely at $\hat{\theta}$, a reduction in W (e.g., from 1 to 0.9 in both figures) raises the systemic risk expected by all investors in the group at $\hat{\theta}$, and requires a higher Q^c to compensate. This results in a discontinuous jump in w following a fall in q^c from the critical level q_0^c analogous to that in Figure 6(a). In Figure 6(b) in this process, even if the magnitude of the fall in q^c is infinitesimal, the resulting reduction in the mass w of immune banks, $w_0 - w_1$, is greater than $1 - 0.4 = 0.6$! When systemic risk is sufficiently high, another constraint by construction, $Q^c \leq W$, further worsens the situation. In all panels, this constraint means that the black solid curve cannot go beyond the blue dashed 45-degree line: $Q^c = W$. In Figure 6(c), this is violated for low values of W corresponding to the black dotted curve segment. For these values, even if $Q^c = W$, the runs on banks (of mass $1 - W$) outside this group raises the systemic risk expected by investors in this group by so much, that their switching cutoff cannot be reduced to $\hat{\theta}$. This implies a downward jump in w of magnitude 1: once q^c falls below the critical level q_0^c, no bank is immune from runs regardless of disclosures.

To ensure that the same $\hat{\theta} = 4.5$ is feasible for disclosures under a uniformly higher systemic risk $a(\cdot)$, we lower $(\underline{c}, \overline{c})$ accordingly. Specifically, we pick $\overline{c} = 5 - 1.5 \times \mathbb{E}r \times \frac{a(l)}{2-l}$ and $\underline{c} = \overline{c} - 2$.

6 Discussion

This section discusses the robustness of our results to the relaxation of two assumptions. Mathematical details are relegated to the Internet Appendix due to length limitations.

In our baseline model, banks' systemic vulnerabities and idiosyncratic costs are assumed to be independent, allowing the regulator to disclose information in only one dimension. This facilitates our analysis of the different impact of disclosures in different dimensions. In reality, different bank-specific information could be correlated, so that a disclosure in one dimension automatically reveals information in the other. In our first robustness check, we allow for correlation between bank types in different dimensions. We obtain that if types in different dimensions are not too negatively correlated, or if allocation of systemic risk is sufficiently important,[23] disclosures in dimension r can still be beneficial due to the same mechanism of assortative matching as in the baseline model, and disclosures in dimension c affect investors' strategies only through the information they reveal in dimension r.

In our baseline model, investors' priors are assumed to be uninformative about bank-specific information in both dimensions c and r. In reality, investors may have information sources about their banks other than the regulator's disclosures. In our second robustness check, we consider the possibility of informative common priors in dimensions r and c, and show that disclosures in dimension c can only hurt the stability of a bank group (i.e., increase the maximum of its investors' switching cutoffs), while there is always a disclosure in dimension r that improves its stability. In this sense, our main results are robust to this possibility.

7 Conclusion

This paper studies how the disclosure of bank-specific information can mitigate systemic bank runs through a novel channel: the reallocation of systemic risk across banks. We find that regardless of disclosure, the aggregate systemic risk expected by all banks is constant, and that the disclosure of bank-specific information differentiates banks by their resilience to adverse shocks, and results in an assortative matching: it reallocates systemic risk from weak banks to strong ones. However, the disclosure of different kinds of bank-specific infor-

[23] A precise if-and-only-if condition is given in the Internet Appendix.

mation have qualitatively different impacts. The disclosure of information concerning banks' vulnerability to systemic risk could improve the stability of all banks, because it reallocates more of the constant aggregate systemic risk to banks that are believed to be less vulnerable to such risk. However, the resulting assortative matching from the disclosure of banks' idiosyncratic factors is not conducive to mitigating systemic bank runs.

Throughout the paper, we have focused on disclosures in either dimension, but not both. This enables us to highlight the dependence of optimal disclosure on the nature of the information, which is ignored in the literature. Once the joint design of disclosures in both dimensions with more than two scores is allowed, in addition to the values of scores in each dimension, the regulator can flexibly design the correlation structure between them. While interesting, this would significantly complicate the analysis, as exemplified by the binary-score setup in Section C.1.2 of the Internet Appendix. We leave this promising but technically challenging work for future research.

References

Bouvard, M., P. Chaigneau, and A. D. Motta (2015). Transparency in the financial system: Rollover risk and crises. *The Journal of Finance 70*(4), 1805–1837.

Carlsson, H. and E. van Damme (1993). Global games and equilibrium selection. *Econometrica 61*(5), 989–1018.

Corsetti, G., A. Dasgupta, S. Morris, and H. S. Shin (2004). Does one soros make a difference? a theory of currency crises with large and small traders. *The Review of Economic Studies 71*(1), 87–113.

Dai, L. and M. Yang (2020). Organizations and coordination in a diverse population.

Drozd, L. A. and R. Serrano-Padial (2018). Financial contracting with enforcement externalities. *Journal of Economic Theory 178*, 153 – 189.

Frankel, D. M., S. Morris, and A. Pauzner (2003). Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory 108*(1), 1 – 44.

Gilboa, I. and D. Schmeidler (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics 18*(2), 141–153.

Goldstein, I. and C. Huang (2016). Bayesian persuasion in coordination games. *American Economic Review 106*(5), 592–96.

Goldstein, I., A. Kopytov, L. Shen, and H. Xiang (2020). Bank heterogeneity and financial stability. *NBER Working Paper* (w27376).

Goldstein, I. and Y. Leitner (2018). Stress tests and information disclosure. *Journal of Economic Theory 177*, 34 – 69.

Goldstein, I. and A. Pauzner (2005). Demandšcdeposit contracts and the probability of bank runs. *The Journal of Finance 60*(3), 1293–1327.

Goldstein, I. and H. Sapra (2014). Should banks' stress test results be disclosed? an analysis of the costs and benefits. *Foundations and Trends in Finance 8*(1), 1–54.

Hansen, L. P. and T. J. Sargent (2001). Robust control and model uncertainty. *The American Economic Review 91*(2), 60–66.

He, Z., A. Krishnamurthy, and K. Milbradt (2019). A model of safe asset determination. *American Economic Review 109*(4), 1230–62.

Inostroza, N. and A. Pavan (2020). Persuasion in global games with application to stress testing.

Leister, M., Y. Zenou, and J. Zhou (2017). Coordination on networks. CEPR Discussion Papers 12392, C.E.P.R. Discussion Papers.

Liu, X. (2019). A dynamic model of systemic bank runs.

Morris, S. and H. Shin (2003). Global games: Theory and applications. In M. Dewatripont, L. Hansen, and S. Turnovsky (Eds.), *Advances in Economics and Econometrics: Proceedins of the Eight World Congress of the Econometric Society*, pp. 56–114. Cambridge: Cambridge University Press.

Morris, S. and M. Yang (2021). Coordination and continuous stochastic choice. *Review of Economic Studies, forthcoming*.

Orlov, D., P. Zryumov, and A. Skrzypacz (2018). Design of macro-prudential stress tests.

Serrano-Padial, R. (2020). Large games with heterogeneous players. *Available at SSRN 3541957*.

Sákovics, J. and J. Steiner (2012). Who matters in coordination problems? *American Economic Review 102*(7), 3439–61.

Sun, Y. (2006). The exact law of large numbers via fubini extension and characterization of insurable risks. *Journal of Economic Theory 126*(1), 31–69.

Williams, B. (2017). Stress tests and bank portfolio choice.

Appendix

A The Construction of Limiting κ-Robust Disclosures

We take two steps to construct the limiting robust disclosure $\Omega(r; W, Q^r, \kappa)$. In Step 1, for any bank group with a total mass W of banks, we construct an auxiliary disclosure $\tilde{\Omega}(r; W, \hat{X}, \kappa)$ that takes the following form: there is a continuous component with support $[r_-, r_+] \subset [\underline{r}, \bar{r}]$, a mass $\underline{m} \geq 0$ piled at $\underline{r}$, and a mass $\bar{m} \geq 0$ at $\bar{r}$, such that all scores are adjacent to each other and their common cutoff is $\hat{X}$. In Step 2, we show that there exists a unique value of $\hat{X}$, denoted by $\hat{x}_r(W, Q^r, \kappa)$, such that the total mass $\tilde{Q}$ of type-$\underline{r}$ banks implied by this auxiliary disclosure is exactly Q^r. We then define $\Omega(r; W, Q^r, \kappa)$ as $\tilde{\Omega}(r; W, \hat{x}_r(W, Q^r, \kappa), \kappa)$.

A.1 Constructing auxiliary disclosures

Given the total mass W of banks, if $\hat{X} \in \left[\underline{r}\frac{A_\kappa(W)}{W} + \mathbb{E}c, \overline{r}\frac{A_\kappa(W)}{W} + \mathbb{E}c\right]$, we say $\hat{X}$ is a feasible switching cutoff.[24] Define a distribution of scores $r \in [\underline{r}, \overline{r}]$, whose cumulative distribution function is $\tilde{\Omega}(\cdot; W, \hat{X}, \kappa)$: $[\underline{r}, \overline{r}] \to [0, W]$ such that

$$\tilde{\Omega}(r; W, \hat{X}, \kappa) = \begin{cases} \underline{m}, & \text{if } r \in (\underline{r}, r_-) \\ \underline{m} + \int_{r_-}^{r} w(\tau)d\tau, & \text{if } r \in (r_-, r_+] \\ \underline{m} + \int_{r_-}^{r_+} w(\tau)d\tau, & \text{if } r \in (r_+, \overline{r}) \\ \underline{m} + \int_{r_-}^{r_+} w(\tau)d\tau + \overline{m}, & \text{if } r = \overline{r} \end{cases}, \tag{26}$$

where

$$\underline{m} \begin{cases} = 0, & \text{if } \hat{X} \geq \mathbb{E}c + \underline{r}a_\kappa(0) \\ \text{satisfies } \hat{X} = \mathbb{E}c + \underline{r}\frac{A_\kappa(\underline{m})}{\underline{m}} & \text{if } \hat{X} < \mathbb{E}c + \underline{r}a_\kappa(0) \end{cases},$$

$$\overline{m} \begin{cases} = 0, & \text{if } \hat{X} \leq \mathbb{E}c + \overline{r}a_\kappa(W) \\ \text{satisfies } \hat{X} = \mathbb{E}c + \overline{r}\frac{A_\kappa(W) - A_\kappa(W - \overline{m})}{\overline{m}} & \text{if } \hat{X} > \mathbb{E}c + \overline{r}a_\kappa(W) \end{cases},$$

and $w(\cdot)$ is such that for any $r \in [r^-, r^+]$,

$$\hat{X} = \mathbb{E}c + r \cdot a_\kappa\left(\underline{m} + \int_{r_-}^{r} w(\tau)d\tau\right), \tag{27}$$

and that the total mass

$$\tilde{\Omega}(\overline{r}; W, \hat{X}, \kappa) = \underline{m} + \int_{r_-}^{r_+} w(\tau)d\tau + \overline{m} = W.$$

The construction of $\tilde{\Omega}(\cdot; W, \hat{X}, \kappa)$ ensures that all different scores are adjacent. To see this, first observe that by construction, all investors share the same switching cutoff $\hat{X}$. In addition, equation (27) indicates that any investor whose score is in the "continuous component" $[r^-, r^+]$ of the distribution $\tilde{\Omega}(\cdot; W, \hat{X}, \kappa)$ faces no strategic uncertainty. That is, when he is indifferent between running and staying, he believes that stayers exactly consist of investors with scores lower than his own, whose mass is $\underline{m} + \int_{r_-}^{r} w(\tau)d\tau$.

[24] Only switching cutoffs in this range are feasible. Given the total mass W of the bank group, if all banks are of type-$\underline{r}$, then the common switching cutoff of their investors is $\underline{r}\frac{A_\kappa(W)}{W} + \mathbb{E}c$, which is the lowest feasible target switching cutoff. Similarly, $\overline{r}\frac{A_\kappa(W)}{W} + \mathbb{E}c$ is the highest feasible target switching cutoff.

Moreover, physical heterogeneity may restrict informational heterogeneity; i.e., scores cannot go beyond $[\underline{r}, \bar{r}]$. If $\underline{r}$ is so high that $\hat{X} < \mathbb{E}c + \underline{r}a_\kappa(0)$, some mass out of W has to be piled at $r = \underline{r}$. These are type-$\underline{r}$ banks whose type is fully revealed and whose mass $\underline{m}$ is such that their investors face strategic uncertainty only among themselves. The high value of $\underline{r}$ impedes the annihilation of such uncertainty.

Similarly, consider the situation where $\bar{r}$ is too low for a given W (recall that $a_\kappa(\cdot)$ is decreasing), such that $\hat{X} > \mathbb{E}c + \bar{r}a_\kappa(W)$. While assigning higher scores to banks hurts their investors' confidence, this adverse effect is dominated by the beneficial assortative matching, given the large mass of banks to be dealt with. So the regulator would "spread" the score beyond $\bar{r}$ if feasible, which is impeded by the low value of $\bar{r}$. Again, the mass $\bar{m}$ piled at $\bar{r}$ (consisting of type-$\bar{r}$ banks whose type is fully revealed) is such that their investors face strategic uncertainty only among themselves.

A.2 Determining the Common Cutoff

For any $W \in [0,1]$ and any feasible switching cutoff $\hat{X}$, the distribution $\tilde{\Omega}(r; W, \hat{X}, \kappa)$ implies a mass of type-$\underline{r}$ banks $\tilde{Q}\left(\hat{X}; W, \kappa\right) \triangleq \frac{\bar{r}W - \int_{r=\underline{r}}^{\bar{r}} r \cdot d\tilde{\Omega}(r; W, \hat{X}, \kappa)}{\bar{r} - \underline{r}}$. The auxiliary disclosure constructed in Step 1 is continuous in nature. To enhance the stability of the bank group with given total mass W; i.e., to induce a lower $\hat{X}$, the auxiliary disclosure has to assign more scores with low r, which requires a larger mass of type-$\underline{r}$ banks out of the fixed total mass W. This monotonicity further implies that there is a unique value of $\hat{X}$, denoted by $\hat{x}_r(W, Q^r, \kappa)$, that is consistent with the mass Q^r of type-$\underline{r}$ banks in the bank group (W, Q^r).

Lemma 2. *$\tilde{Q}\left(\hat{X}; W, \kappa\right)$ is continuous and strictly decreasing in $\hat{X}$. Thus, there exists a unique $\hat{x}_r(W, Q^r, \kappa)$ such that $\tilde{Q}(\hat{x}_r(W, Q^r, \kappa); W, \kappa) = Q^r$. In addition, $\hat{x}_r(W, Q^r, \kappa)$ is continuous and decreasing in Q^r.*

B Proofs

Proposition 1 is a direct corollary of Proposition 4 proved later, and **Proposition 2** is straightforward from Proposition 1. Proofs of all the lemmas introduced in the Appendix are relegated to the Internet Appendix.

Proof of Proposition 3

We take three steps to prove the proposition. The first two steps are summarized by the following two lemmas.

Lemma 3. *For any infinite sequence* $\{\sigma_m\}_{m=1}^{+\infty}$ *of* σ *that converges to 0, there exists an infinite subsequence* $\{\sigma_m^4\}_{m=1}^{+\infty}$ *such that all* $\hat{x}_i^{\sigma_m^4}$ *and* $\Delta_{i-1,i}^{\sigma_m^4}$ *either converge to finite numbers or go to infinity. Moreover, their limits* $\{(\hat{x}_i^0, \Delta_{j,i}^0)\}_{j,i\in\{1,2,\ldots,n\}}$ *satisfy the equation system consisting of* (18), (19), *and* (20).

Lemma 4. *The equation system consisting of* (18), (19), *and* (20) *has a unique solution.*

Based on the two lemmas, we prove that the equation system is the necessary and sufficient condition for $\{\hat{x}_i\}_{i=1}^n$ to be the limits of the cutoffs as $\sigma \to 0$.

Suppose as $\sigma \to 0$, $\{\hat{x}_i^\sigma\}_{i=1}^n$ do not converge to $\{\hat{x}_i^0\}_{i=1}^n$. That means, there exists ϵ and an infinite sequence $\{\sigma_m\}_{m=1}^{+\infty}$ such that $\max_i |\hat{x}_i^{\sigma_m} - \hat{x}_i^0| > \epsilon$. However, according to Part I and Part II, there exists an infinite subsequence $\{\sigma_m^4\}_{m=1}^{+\infty}$ of $\{\sigma_m\}_{m=1}^{+\infty}$ such that $\{\hat{x}_i^{\sigma_m^4}\}_{i=1}^n$ converges to $\{\hat{x}_i^0\}_{i=1}^n$. Contradiction! Therefore, as $\sigma \to 0$, $\{\hat{x}_i^\sigma\}_{i=1}^n$ converge to $\{\hat{x}_i^0\}_{i=1}^n$.

On the other hand, if $\{\hat{x}_i\}_{i=1}^n$ satisfies the equation system, when the disclosure is implemented, as $\sigma \to 0$, $\{\hat{x}_i^\sigma\}_{i=1}^n$ must converge to the solution of the equation system, which is uniquely $\{\hat{x}_i\}_{i=1}^n$.

Proof of Proposition 4

For any z, the scores in the partition cell $[p_z, p_{z+1})$ all have the same cutoff in the limiting case. Denote it by $\hat{x}(z)$. Then for any $i \in [p_z, p_{z+1})$,

$$\begin{aligned}\hat{x}(z) &= \mathbb{E}c + r_i \int_0^1 a\left(\sum_{\{j:\Delta_{j,i}=+\infty\}} w_j + \sum_{\{j:|\Delta_{i,j}|<\infty\}} w_j\left[1-\Phi\left(\Phi^{-1}(1-m_i)-\Delta_{j,i}\right)\right]\right) dm_i \\ &= \mathbb{E}c + r_i \int_0^1 a\left(\sum_{j=1}^{p_z-1} w_j + \sum_{j=p_z}^{p_{z+1}-1} w_j\left[1-\Phi\left(\Phi^{-1}(1-m_i)-\Delta_{j,i}\right)\right]\right) dm_i,\end{aligned}$$

so

$$\frac{\hat{x}(z)-\mathbb{E}c}{r_i} = \int_0^1 a\left(\sum_{j=1}^{p_z-1} w_j + \sum_{j=p_z}^{p_{z+1}-1} w_j\left[1-\Phi\left(\Phi^{-1}(1-m_i)-\Delta_{j,i}\right)\right]\right) dm_i.$$

For any real number μ_i, we can replace m_i with $1 - \Phi(\mu_i - y)$ and write the right-hand side as an integral with respect to y over $(-\infty, +\infty)$, i.e.,

$$\begin{aligned}
\frac{\hat{x}(z) - \mathbb{E}c}{r_i} &= \int_{y=-\infty}^{+\infty} a\left(\sum_{j=1}^{p_z-1} w_j + \sum_{j=p_z}^{p_{z+1}-1} w_j\left[1 - \Phi\left(\Phi^{-1}\left(\Phi\left(\mu_i - y\right)\right) - \Delta_{j,i}\right)\right]\right) d\left[1 - \Phi\left(\mu_i - y\right)\right] \\
&= \int_{y=-\infty}^{+\infty} a\left(\sum_{j=1}^{p_z-1} w_j + \sum_{j=p_z}^{p_{z+1}-1} w_j\left[1 - \Phi\left(\mu_i - y - \Delta_{j,i}\right)\right]\right) d\left[1 - \Phi\left(\mu_i - y\right)\right] \qquad (28)
\end{aligned}$$

Specifically, due to Equation (20), we can pick $\{\mu_i\}_{i=1}^n$ such that $\mu_j - \mu_i = \Delta_{i,j}$.

Multiplying Equation (28) by w_j and sum over $[p_z, p_{z+1})$, we obtain

$$\begin{aligned}
\sum_{i=p_z}^{p_{z+1}-1} \frac{\hat{x}(z) - \mathbb{E}c}{r_i} w_i &= \sum_{i=p_z}^{p_{z+1}-1} \int_{y=-\infty}^{+\infty} a\left(\sum_{j=1}^{p_z-1} w_j + \sum_{j=p_z}^{p_{z+1}-1} w_j\left[1 - \Phi\left(\mu_i - y - \Delta_{j,i}\right)\right]\right) d\left[w_i - w_i\Phi\left(\mu_i - y\right)\right] \\
&= \int_{y=-\infty}^{+\infty} a\left(\sum_{j=1}^{p_z-1} w_j + \sum_{j=p_z}^{p_{z+1}-1} w_j\left[1 - \Phi\left(\mu_j - y\right)\right]\right) d\left\{\sum_{i=p_z}^{p_{z+1}-1}\left[w_i - w_i\Phi\left(\mu_i - y\right)\right]\right\} \\
&= \int_{\omega=0}^{\sum_{i=p_z}^{p_{z+1}-1} w_i} a\left(\sum_{j=1}^{p_z-1} w_j + \omega\right) d\omega = A\left(\sum_{j=1}^{p_{z+1}-1} w_j\right) - A\left(\sum_{j=1}^{p_z-1} w_j\right).
\end{aligned}$$

so

$$\hat{x}(z) = \mathbb{E}c + \left(\sum_{j=p_z}^{p_{z+1}-1} \frac{w_j}{r_j}\right)^{-1}\left[A\left(\sum_{j=1}^{p_{z+1}-1} w_j\right) - A\left(\sum_{j=1}^{p_z-1} w_j\right)\right].$$

Proof of Proposition 5

Suppose the proposition does not hold. Specifically, the sub-disclosure for the bank group whose investors have the switching cutoff θ_k is not the (K_{k-1}, t_k)-robust disclosure for the group. Notice that the maximum of the switching cutoffs is also θ_k. Denote the maximum of the switching cutoffs under the (K_{k-1}, t_k)-robust disclosure by $\hat{x}_{(K_{k-1}, t_k)}$. Then by the definition, $\hat{x}_{(K_{k-1}, t_k)} < \theta_k$. We show that if sub-disclosure is replaced with the (K_{k-1}, t_k)-robust disclosure for the group, all investors have weakly lower cutoffs, and a positive mass of them have strictly lower cutoffs.

Part I: investors with cutoffs smaller than θ_k under the original disclosure.

Suppose these investors correspond to the first m scores. The investors' cutoffs are

$\{\hat{x}_i, \Delta_{i,j}\}_{j,i=1}^m$ under the original disclosure and $\{\hat{x}'_i, \Delta'_{i,j}\}_{j,i=1}^m$ under the new disclosure respectively. Since $\Delta_{i,j} = +\infty$ for $i \leq m$ and $j > m$, $\{\hat{x}_i\}_{i=1}^m$ satisfies

$$\hat{x}_i = c_i + r_i \int_0^1 a \left(\sum_{j=1}^m w_j \left[1 - \Phi\left(\Phi^{-1}(1-m_i) - \Delta_{j,i}\right)\right] \right) dm_i.$$

Suppose there exists $i \leq m$ such that $\hat{x}'_i > \hat{x}_i$ and such i constitutes the set $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_L\}$ where $\tau_1 < \tau_2 \ldots < \tau_L$. Consider $i \in \mathcal{T}$. There must exist $j \leq m$ such that $\Delta'_{j,i} < \Delta_{j,i}$; otherwise, $\hat{x}'_i \leq \hat{x}_i$. Let $\xi(i)$ be the smallest j such that $\Delta'_{j,i} < \Delta_{j,i}$.

Note that for $j \notin \mathcal{T}$, since $\hat{x}'_j \leq \hat{x}_j$ and $\hat{x}'_i > \hat{x}_i$, $\Delta'_{j,i} \geq \Delta_{j,i}$. Hence, $\xi(i) \in \mathcal{T}$. Since $\xi(\tau_1) \in \mathcal{T}$, $\xi(\tau_1) > \tau_1$. Consider $\xi^{(2)}(\tau_1) = \xi(\xi(\tau_1))$. It must be in $\mathcal{T}$. By the definition of $\xi(\tau_1)$, for any $j \in \mathcal{T}$ and $j < \xi(\tau_1)$, $\Delta'_{j,\tau_1} \geq \Delta_{j,\tau_1}$, and $\Delta'_{\xi(\tau_1),\tau_1} < \Delta_{\xi(\tau_1),\tau_1}$. So, for these j,

$$\Delta'_{j,\xi(\tau_1)} = \Delta'_{j,\tau_1} - \Delta'_{\xi(\tau_1),\tau_1} > \Delta_{j,\tau_1} - \Delta_{\xi(\tau_1),\tau_1} = \Delta_{j,\xi(\tau_1)},$$

which implies $\xi(\xi(\tau_1)) > \xi(\tau_1)$. Iterating the procedure, we end up with an infinite sequence $\{\xi^{(j)}(\tau_1)\}_{j=1}^{+\infty}$ in $\mathcal{T}$. This is impossible because $\mathcal{T}$ is a finite set. Therefore, for $i \leq m$, $\hat{x}'_i \leq \hat{x}_i$.

Part II: investors with cutoffs equal to θ_k under the original disclosure.

Denote the (K_{k-1}, t_k)-robust disclosure for the group by $\{(r'_i, c'_i, w'_i)\}_{i=m+1}^{m+t_k}$. By the definition of the (K_{k-1}, t_k)-robust disclosure and $K_{k-1} = \sum_{j=1}^m w_j$, there exists $\{\hat{x}''_i, \Delta''_{i,j}\}_{i,j=m+1}^{m+t_k}$ such that $\hat{x}''_i \leq \hat{x}_{(K_{k-1},t_k)}$, where

$$\hat{x}''_i = c'_i + r'_i \int_0^1 a \left(\sum_{j=1}^m w_j + \sum_{j=m+1}^{m+t_k} w'_j \left[1 - \Phi\left(\Phi^{-1}(1-m_i) - \Delta''_{j,i}\right)\right] \right) dm_i,$$

$$\Delta''_{i-1,i} \begin{cases} = +\infty, & \text{if } \hat{x}''_i > \hat{x}''_{i-1} \\ = -\infty, & \text{if } \hat{x}''_i < \hat{x}''_{i-1} \\ \in [-\infty, +\infty], & \text{if } \hat{x}''_i = \hat{x}_{i-1}" \end{cases},$$

and $-\Delta''_{i,j} = \Delta''_{j,i} = \sum_{u=j+1}^i \Delta''_{u-1,u}$. Denote by $\{\hat{x}'_i, \Delta'_{i,j}\}_{j,i=m+1}^{m+t_k}$ the cutoffs of the t_k scores specified by the (K_{k-1}, t_k)-robust disclosure under the new disclosure. Suppose there exists $m+1 \leq i \leq m+t_k$ such that $\hat{x}'_i > \max\{\hat{x}_{(K_{k-1},t_k)}, \theta_{k-1}\}$ and such i's constitute a set $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_L\}$ where $\tau_1 < \tau_2 \ldots < \tau_L$. Consider $i \in \mathcal{T}$. There must exist $m+1 \leq j \leq m+t_k$

such that $\Delta'_{j,i} < \Delta''_{j,i}$; otherwise,

$$\hat{x}'_i \leq c'_i + r'_i \int_0^1 a \left(\sum_{j=1}^{m} w_j + \sum_{j=m+1}^{m+t_k} w'_j \left[1 - \Phi\left(\Phi^{-1}(1-m_i) - \Delta''_{j,i} \right) \right] \right) dm_i = \hat{x}''_i \leq \hat{x}_{(K_{k-1}, t_k)}.$$

Let $\xi(i)$ be the smallest j such that $\Delta'_{j,i} < \Delta''_{j,i}$. Similar to Part I, we will encounter contradiction. Hence, for $m+1 \leq i \leq m+t_k$, $\hat{x}'_i \leq \max\{\hat{x}_{(K_{k-1},t_k)}, \theta_{k-1}\} < \theta_k$.

Part III: investors with cutoffs greater than θ_k under the original disclosure.

Note that all other investors have cutoffs smaller than θ_k under the new disclosure. Since the equation system in Proposition 3 has a unique solution, the cutoffs of these investors must be the same under the new disclosure as they are under the original disclosure.

Proof of Proposition 6

According to (23) and $r_i = \mathbb{E}r$ for all i, if $i \in \{p_k, p_k+1, ..., p_{k+1}-1\}$,

$$\sum_{j=p_k}^{p_{k+1}-1} w_j \hat{x}_j = \sum_{j=p_k}^{p_{k+1}-1} w_j \hat{x}_i = \sum_{j=p_k}^{p_{k+1}-1} w_j c_j + \mathbb{E}r \left[A_\kappa \left(\sum_{j=1}^{p_{k+1}-1} w_j \right) - A_\kappa \left(\sum_{j=1}^{p_k - 1} w_j \right) \right].$$

Then the average cutoff is

$$\frac{\sum_{j=1}^{t} w_j \hat{x}_j}{W} = \frac{\sum_{j=1}^{t} w_j c_j}{W} + \mathbb{E}r \frac{A_\kappa(W)}{W} = \frac{[Q^c \cdot \underline{c} + (W - Q^c) \cdot \bar{c}]}{W} + \mathbb{E}r \cdot \frac{A_\kappa(W)}{W},$$

which is also a lower bound of the maximum cutoff. Under nondisclosure, all investors have the same cutoff, so the maximum cutoff achieves this lower bound.

Proof of Proposition 7

For any k, the scores in the partition cell $[p_k, p_{k+1})$ all have the same cutoff in the limiting case. Denote it by $\hat{x}(k)$. If there are two scores that have different cutoffs, then there must exist k such that $\hat{x}(k) < \hat{x}(k+1)$. Let k_{max} be the maximum among them. Then $\hat{x}(k_{max}+1) = \hat{x}(k_{max}+2) = \ldots = \hat{x}(K)$.

Let $\tilde{x}$ be the root of

$$\sum_{i=p_{k_{max}+1}}^{p_{K+1}-1} \frac{(\tilde{x}-\mathbb{E}c)w_i r_i}{\hat{x}(k_{max}+1)-\mathbb{E}c} + \sum_{i=p_{k_{max}}}^{p_{k_{max}+1}-1} \frac{(\tilde{x}-\mathbb{E}c)w_i r_i}{\hat{x}(k_{max})-\mathbb{E}c} = \sum_{i=p_{k_{max}}}^{p_{K+1}-1} w_i r_i.$$

Then $\hat{x}(k_{max}) < \tilde{x} < \hat{x}(k_{max}+1)$.

Consider an alternative disclosure with

$$r_i' = \begin{cases} r_i, & \forall i < p_{k_{max}} \\ \frac{(\tilde{x}-\mathbb{E}c)r_i}{\hat{x}(k_{max})-\mathbb{E}c} & \forall p_{k_{max}} \le i < p_{k_{max}+1} \\ \frac{(\tilde{x}-\mathbb{E}c)r_i}{\hat{x}(k_{max}+1)-\mathbb{E}c} & \forall i \ge p_{k_{max}+1} \end{cases}$$

and the same mass w_i for each score as the original disclosure. We guess the equilibrium is $(\hat{x}_1', \ldots, \hat{x}_n')$, where $\hat{x}_i' = \hat{x}_i$ if $i < p_{k_{max}}$, $\hat{x}_i' = \tilde{x}$ if $\forall i \ge p_{k_{max}}$, and its $\{\Delta_{i,j}'\}_{j,i=1}^t$ is the same as the original one $\{\Delta_{i,j}\}_{j,i=1}^t$. To verify that this is indeed the equilibrium, we need to show that $\forall i, j$,

$$\hat{x}_i' = \mathbb{E}c + r_i' \int_0^1 a_\kappa \left(\sum_{j=1}^n w_j \left[1 - \Phi\left(\Phi^{-1}(1-m_i) - \Delta_{j,i}' \right) \right] \right) dm_i,$$

and $\hat{x}_i' = \hat{x}_j'$ if Δ_{ij}' is finite. It is easy to see

$$\begin{aligned} \frac{\hat{x}_i' - \mathbb{E}c}{r_i'} = \frac{\hat{x}_i - \mathbb{E}c}{r_i} &= \int_0^1 a_\kappa \left(\sum_{j=1}^n w_j \left[1 - \Phi\left(\Phi^{-1}(1-m_i) - \Delta_{j,i} \right) \right] \right) dm_i \\ &= \int_0^1 a_\kappa \left(\sum_{j=1}^n w_j \left[1 - \Phi\left(\Phi^{-1}(1-m_i) - \Delta_{j,i}' \right) \right] \right) dm_i. \end{aligned}$$

In addition, for $i, j < p_{k_{max}}$, if Δ_{ij}' is finite, Δ_{ij} is finite, then $\hat{x}_i' = \hat{x}_i = \hat{x}_j = \hat{x}_j'$; for $i < p_{k_{max}}$ and $j \ge p_{k_{max}}$, Δ_{ij}' is infinite; for $i, j \ge p_{k_{max}}$, $\hat{x}_i' = \hat{x}_j'$. So, this is indeed the equilibrium, and it has the same partition structure as the original one. This alternative disclosure has a strictly lower maximum cutoff; i.e., $\max_i \{\hat{x}_i'\} = \tilde{x} < \hat{x}(k_{max}+1) = \max_i \{\hat{x}_i\}$. To prevent such decrease in the maximum cutoff, the (κ, t)-robust disclosure must result in all scores being either entangled or adjacent to each other.

Proof of Proposition 8

According to Proposition 4, the switching cutoff of investors in the k-th partition cell is

$$\hat{x}(k) = \mathbb{E}c + \left(\sum_{j=p_k}^{p_{k+1}-1} \frac{w_j}{r_j}\right)^{-1} \left[A_\kappa\left(\sum_{j=1}^{p_{k+1}-1} w_j\right) - A_\kappa\left(\sum_{j=1}^{p_k-1} w_j\right)\right].$$

So, the maximum cutoff

$$\max_i \{\hat{x}_i\} \geq \mathbb{E}c + \left(\sum_{j=p_k}^{p_{k+1}-1} \frac{w_j}{r_j}\right)^{-1} \left[A_\kappa\left(\sum_{j=1}^{p_{k+1}-1} w_j\right) - A_\kappa\left(\sum_{j=1}^{p_k-1} w_j\right)\right]$$
$$\Leftrightarrow \left(\sum_{j=p_k}^{p_{k+1}-1} \frac{w_j}{r_j}\right)\left[\max_i \{\hat{x}_i\} - \mathbb{E}c\right] \geq A_\kappa\left(\sum_{j=1}^{p_{k+1}-1} w_j\right) - A_\kappa\left(\sum_{j=1}^{p_k-1} w_j\right).$$

Summing over k, we obtain

$$\left(\sum_{i=1}^{t} \frac{w_i}{r_i}\right)\left[\max_i \{\hat{x}_i\} - \mathbb{E}c\right] \geq A_\kappa(W) \Rightarrow \max_i \{\hat{x}_i\} \geq \mathbb{E}c + \frac{A_\kappa(W)}{\sum_{i=1}^{t} \frac{w_i}{r_i}}.$$

By Lemma 1, we obtain

$$\frac{w_i}{r_i} \leq \frac{w_i\overline{r} - w_i r_i}{\overline{r} - \underline{r}}\frac{1}{\underline{r}} + \frac{w_i r_i - w_i\underline{r}}{\overline{r} - \underline{r}}\frac{1}{\overline{r}} \Rightarrow \sum_{i=1}^{t} \frac{w_i}{r_i} \leq \frac{Q^r}{\underline{r}} + \frac{W - Q^r}{\overline{r}}.$$

The last inequality holds if any r_i is not equal to $\underline{r}$ or $\overline{r}$. So,

$$\max_i \{\hat{x}_i\} \geq \mathbb{E}c + \frac{A_\kappa(W)}{\frac{Q}{\underline{r}} + \frac{W-Q}{\overline{r}}}.$$

Consider the disclosure $r_1 = \underline{r}$, $r_2 = \overline{r}$, $w_1 = Q^r$ and $w_2 = W - Q^r$. By $\underline{r}\frac{A_\kappa(Q^r)}{Q^r} \geq \overline{r}\frac{A_\kappa(W)-A_\kappa(Q^r)}{W-Q^r}$, they must have the same cutoff, $\mathbb{E}c + \frac{A_\kappa(W)}{\frac{Q}{\underline{r}}+\frac{W-Q}{\overline{r}}}$. Therefore, the robust disclosure is a uniquely full revelation of information.

Proof of Proposition 9

Suppose k_{max} is the greatest score that is entangled with another score. We prove the statement for any $k_{max} \geq 2$ by mathematical induction.

First, consider $k_{max} = 2$ and $t = 2$. The two scores are entangled and have a common cutoff $\mathbb{E}c + \frac{A_\kappa(W)}{\frac{w_1}{r_1}+\frac{w_2}{r_2}}$. Consider an alternative disclosure r'_1 and r'_2 that satisfy $w_1 r_1 + w_2 r_2 = w'_1 r'_1 + w'_2 r'_2$, $w_1 + w_2 = w'_1 + w'_2$, and $r'_1 = r_1 - \delta(r_1 - \underline{r})$, $r'_2 = r_2 + \delta(\overline{r} - r_2)$. Since r'_1 and r'_2 are entangled when $\delta = 0$ and are separate when $\delta = 1$, there must exist a $\delta' > 0$ such that they are adjacent; i.e., $r'_1 \frac{A_\kappa(w'_1)}{w'_1} = r'_2 \frac{A_\kappa(w_1+w_2)-A_\kappa(w'_1)}{w_1+w_2-w'_1}$. Note for any $\delta > 0$, $\frac{w_1}{r_1} + \frac{w_2}{r_2} < \frac{w'_1}{r'_1} + \frac{w'_2}{r'_2}$. So when r'_1 and r'_2 are adjacent, their maximum cutoff is strictly lower than the original one.

Second, consider $k_{max} = 2$ and $t \geq 3$. Only r_1 and r_2 are entangled. The scores $2 \sim t$ are adjacent. All scores have the common cutoff $\hat{x}_3 = \mathbb{E}c + r_3 \frac{A_\kappa(w_1+w_2+w_3)-A_\kappa(w_1+w_2)}{w_3}$. Consider the mean-preserving spread of $\{(r_1, w_1), (r_2, w_2)\}$, $\{(r'_1, w'_1), (r'_2, w'_2)\}$ where $r'_1 = r_1$, $r'_2 = r_2 + \delta$, $w'_1 r'_1 + w'_2 r'_2 = w_1 r_1 + w_2 r_2$, and $w'_1 + w'_2 = w_1 + w_2$. We show that there exists δ' such that $r'_1 \frac{A_\kappa(w'_1)}{w'_1} = r'_2 \frac{A_\kappa(w_1+w_2)-A_\kappa(w'_1)}{w_1+w_2-w'_1}$. Note that $\delta' \in [0, +\infty)$. To see this, on one hand, since only r_1 and r_2 are entangled under the original disclosure, when $\delta = 0$, $r'_1 \frac{A_\kappa(w'_1)}{w'_1} > r'_2 \frac{A_\kappa(w_1+w_2)-A_\kappa(w'_1)}{w_1+w_2-w'_1}$. On the other hand, $\frac{A_\kappa(w_1+w_2)-A_\kappa(w'_1)}{w_1+w_2-w'_1} \geq a_\kappa(w_1 + w_2)$ and $r'_1 \frac{A_\kappa(w'_1)}{w'_1} < r_1 a_\kappa(0)$, when δ is sufficiently large, $r'_1 \frac{A_\kappa(w'_1)}{w'_1} < r'_2 \frac{A_\kappa(w_1+w_2)-A_\kappa(w'_1)}{w_1+w_2-w'_1}$. Therefore, there exists $\delta' > 0$ such that the equation holds. Consider an alternative disclosure $(r'_1, r'_2, r_3, \ldots r_t)$ that replaces $\{(r_1, w_1), (r_2, w_2)\}$ with the $\{(r'_1, w'_1), (r'_2, w'_2)\}$ associated with δ' while keeping $\{(r_i, w_i)\}_{i=3}^n$ unchanged. Since

$$
\begin{aligned}
r'_1 \frac{A_\kappa(w'_1)}{w'_1} &= r'_2 \frac{A_\kappa(w_1+w_2) - A_\kappa(w'_1)}{w_1 + w_2 - w'_1} = \left(\frac{w'_1}{r'_1} + \frac{w_1 + w_2 - w'_1}{r'_2}\right)^{-1} \cdot A_\kappa(w_1 + w_2) \\
&< \left(\frac{w_1}{r_1} + \frac{w_2}{r_2}\right)^{-1} \cdot A_\kappa(w_1 + w_2) = r_3 \frac{A_\kappa(w_1 + w_2 + w_3) - A_\kappa(w_1 + w_2)}{w_3},
\end{aligned}
$$

it is straightforward to see that under the alternative disclosure, the scores $3 \sim t$ are still adjacent to each other and their common switching cutoff is still $\hat{x}_3$; the scores 1 and 2 are adjacent and have a common cutoff strictly lower than $\hat{x}_3$. That means, each partition cell has only one score. Following the proof of Lemma 7, we can find a disclosure under which all scores are adjacent in this case and the maximum cutoff is strictly lower.

Next, suppose the statement holds for $k_{max} < k (\geq 3)$ and any t. Consider $k_{max} = k$

and any t. Consider an alternative disclosure that replaces (r_{k-1}, r_k) with one of its mean-preserving spread (r'_{k-1}, r'_k) while keeping other scores unchanged. Specifically, $r'_{k-1} = r_{k-1} - \delta$ and $r'_k = r_k$. Consider the process that δ increases until $r'_{k-1} = r_{k-2}$ or r'_{k-1} and r'_k are adjacent, whichever is first. We conjecture that the cutoff of r'_k is always decreasing and $r_{k+1}, \ldots r_t$ have the same cutoff as under the original disclosure. Suppose the partition cell that contains r'_{k-1} and r'_k consists of the scores $\{n(\delta), \ldots, k-1, k\}$. Denote it as $P(\delta)$. In this process, $P(\delta)$ may experience three kinds of changes.

1. $P(\delta)$ does not change. Then the cutoff of r'_k

$$\hat{x}'_k = \mathbb{E}c + \left(\sum_{j=n(\delta)}^{k-2} \frac{w_j}{r_j} + \frac{w_{k-1}}{r'_{k-1}} + \frac{w_k}{r'_k} \right)^{-1} \left[A_\kappa \left(\sum_{j=1}^{k} w_j \right) - A_\kappa \left(\sum_{j=1}^{n(\Delta)-1} w_j \right) \right]$$

   is strictly decreasing in δ.

2. $P(\delta)$ absorbs some scores below $n(\delta)$. Denote the $P(\delta)$ before and after the change by $P(\delta_-)$ and $P(\delta_+)$. In the instant when the change happens, the score $n(\delta_-) - 1$ must be adjacent to $P(\delta_-)$ and the scores $\{n(\delta_+), \ldots, n(\delta_-) - 1\}$ are either entangled or adjacent. So, in this instant,

$$\hat{x}'_k = \mathbb{E}c + \left(\sum_{j=n(\delta_-)}^{k-2} \frac{w_j}{r_j} + \frac{w_{k-1}}{r'_{k-1}} + \frac{w_k}{r'_k} \right)^{-1} \left[A_\kappa \left(\sum_{j=1}^{k} w_j \right) - A_\kappa \left(\sum_{j=1}^{n(\Delta_-)-1} w_j \right) \right],$$

   and also

$$\hat{x}'_k = \mathbb{E}c + \left(\sum_{j=n(\delta_+)}^{k-2} \frac{w_j}{r_j} + \frac{w_{k-1}}{r'_{k-1}} + \frac{w_k}{r'_k} \right)^{-1} \left[A_\kappa \left(\sum_{j=1}^{k} w_j \right) - A_\kappa \left(\sum_{j=1}^{n(\Delta_+)-1} w_j \right) \right].$$

   This implies that $\hat{x}'_k$ has no jump when the change happens.

3. $P(\delta)$ drop some scores in $\{n(\delta), \ldots k-2\}$. Following the same analysis of the second case, $\hat{x}'_k$ has no jump when the change happens.

We have verified the conjecture. Then, no matter for what reason the process stops, we end up with a new disclosure with $k_{max} \leq k-1$. If $k = t$, its maximum cutoff is strictly lower than the original one. If $k < t$, when the process stops, $\hat{x}'_k < \hat{x}_k \le \hat{x}_{k+1}$. Following the proof of Lemma 7, we can find another disclosure whose maximum cutoff is strictly lower than $\hat{x}_{k+1}$ and $k_{max} \le k - 1$.

We can iterate this procedure finite times and end up with a disclosure with all scores adjacent and its maximum cutoff strictly lower than that of the original one.

Proof of Proposition 10

By Proposition 9, we focus on disclosures with all scores adjacent. For any disclosure like this, suppose their common cutoff is $\hat{x}$. Then $\forall k$, $\hat{x} = \mathbb{E}c + r_k \frac{A_\kappa(\sum_{i=1}^{k} w_i) - A_\kappa(\sum_{i=1}^{k-1} w_i)}{w_k}$, where $\sum_{i=1}^{t} w_i = W$ and $\sum_{i=1}^{t} w_i r_i = (W - Q^r)\underline{r} + Q^r \overline{r}$.

Part I: $\hat{x}_r(W, Q^r, \kappa)$ is smaller than the maximum cutoff under any finite disclosure.

Consider $\tilde{\Omega}(r; W, \hat{x}, \kappa)$ that is defined in Appendix A.1. Since $r_1 \ge \underline{r}$ and $r_1 \frac{A_\kappa(w_1)}{w_1} = \hat{x} - \mathbb{E}c = \underline{r}\frac{A_\kappa(\underline{m})}{\underline{m}}$, it is easy to see that $\underline{m} \le w_1$. Similarly, $\overline{m} \le w_t$. Then for any k, there exists $\tilde{r}_k$ such that $\tilde{\Omega}(\tilde{r}_k; W, \hat{x}, \kappa) = \sum_{i=1}^{k} w_i$. For the part of $\tilde{\Omega}(r; W, \hat{x}, \kappa)$ over $[\tilde{r}_{k-1}, \tilde{r}_k]$, we have

$$\begin{aligned}(\hat{x} - \mathbb{E}c) \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} \frac{1}{r} d\tilde{\Omega}(r; W, \hat{x}, \kappa) &= \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} a_\kappa\left(\tilde{\Omega}(r; W, \hat{x}, \kappa)\right) d\tilde{\Omega}(r; W, \hat{x}, \kappa) \\ &= A_\kappa\left(\sum_{i=1}^{k} w_i\right) - A_\kappa\left(\sum_{i=1}^{k-1} w_i\right) = \frac{\hat{x} - \mathbb{E}c}{r_k} w_k. \end{aligned} \tag{29}$$

Since by Cauchy-Schwarz inequality,

$$\int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} \frac{1}{r} d\tilde{\Omega}(r; W, \hat{x}, \kappa) \cdot \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} r d\tilde{\Omega}(r; W, \hat{x}, \kappa) \ge w_k^2 = \frac{w_k}{r_k} w_k r_k,$$

we have $\int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} r d\tilde{\Omega}(r; W, \hat{x}, \kappa) \ge w_k r_k$. Summing over k, we have

$$\underline{r} \cdot \tilde{Q}\left(\hat{X}; W, \kappa\right) + \bar{r} \cdot \left[W - \tilde{Q}\left(\hat{X}; W, \kappa\right)\right] = \int_{r=\underline{r}}^{\overline{r}} r d\tilde{\Omega}(r; W, \hat{x}, \kappa) \ge \sum_{i=1}^{t} w_i r_i = \underline{r} \cdot Q^r + \bar{r} \cdot (W - Q^r),$$

so $\tilde{Q}(\hat{x};W,\kappa) \leq Q^r$. Since $\tilde{Q}\left(\hat{X};W,\kappa\right)$ is decreasing in $\hat{X}$, $\hat{x} \geq \hat{x}_r(W,Q^r,\kappa)$. Therefore, $\hat{x}_r(W,Q^r,\kappa)$ is the lower bound of the maximum cutoff.

Part II: there exists a sequence of t-score disclosures $\left\{\Omega_{(t)}\right\}$ such that their maximum cutoff converges to $\hat{x}_r(W,Q^r,\kappa)$ as $t \to +\infty$.

For any feasible $\hat{X}$, consider a t-score disclosure $(r_1,\dots r_t)$ as follow: $(r_1,w_1) = (\underline{r},\underline{m})$, $(r_t,w_t) = (\overline{r},\overline{m})$; for $2 \leq k \leq t-1$, $w_k = \delta = \frac{W-w_1-w_t}{t-2}$, and r_k satisfies $\hat{X} = \mathbb{E}c + r_k\frac{A_\kappa(w_1+(k-1)\delta)-A_\kappa(w_1+(k-2)\delta)}{\delta}$.

Let $S(\hat{X};t) \triangleq w_1\underline{r} + \sum_{i=2}^{t-1}\delta r_i + w_t\overline{r}$ be the sum of r. $S(\hat{X};t)$ is continuous in $\hat{X}$. There exists an $\hat{x}$ such that $S(\hat{x};t) = \underline{r}\cdot Q^r + \overline{r}\cdot(W-Q^r)$. In this case, the disclosure is feasible and all scores are adjacent and have the common cutoff $\hat{x}$. Denote this disclosure by $\Omega_{(t)}$. Let $S(\hat{x}) \triangleq \int_{r=\underline{r}}^{\overline{r}} r d\tilde{\Omega}(r;W,\hat{x},\kappa)$.

Next, we compare $S(\hat{x};t)$ with $S(\hat{x})$. Similar to the above, suppose $\tilde{r}_k$ satisfies $\tilde{\Omega}(\tilde{r}_k;W,\hat{x},\kappa) = \sum_{i=1}^{k} w_i = w_1 + (k-1)\delta$. Then We have

$$(\hat{x}-\mathbb{E}c)\int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k}\frac{1}{r}d\tilde{\Omega}(\tilde{r}_k;W,\hat{x},\kappa) = A_\kappa(w_1+(k-1)\delta) - A_\kappa(w_1+(k-2)\delta) = \frac{\hat{x}-\mathbb{E}c}{r_k}\delta$$

$$S(\hat{x}) - S(\hat{x};t) = \int_{r=r^-}^{r^+} r d\tilde{\Omega}(r;W,\hat{x},\kappa) - \sum_{k=2}^{t-1}\delta r_k = \sum_{k=2}^{t-1}\left[\int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} r d\tilde{\Omega}(r;W,\hat{x},\kappa) - \delta r_k\right].$$

Note that

$$\begin{aligned}
\left[\int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} r d\tilde{\Omega}(r;W,\hat{x},\kappa) - \delta r_k\right]\frac{\delta}{r_k} &= \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} r d\tilde{\Omega}(r;W,\hat{x},\kappa)\cdot\int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k}\frac{1}{r}d\tilde{\Omega}(r;W,\hat{x},\kappa) - \delta^2 \\
&= \int_{z=\tilde{r}_{k-1}}^{\tilde{r}_k}\int_{y=\tilde{r}_{k-1}}^{\tilde{r}_k} y\cdot\frac{1}{z}d\tilde{\Omega}(y;W,\hat{x},\kappa)d\tilde{\Omega}(z;W,\hat{x},\kappa) - \delta^2 \\
&\leq \int_{z=\tilde{r}_{k-1}}^{\tilde{r}_k}\int_{y=\tilde{r}_{k-1}}^{\tilde{r}_k}\frac{\tilde{r}_k}{\tilde{r}_{k-1}}d\tilde{\Omega}(y;W,\hat{x},\kappa)d\tilde{\Omega}(z;W,\hat{x},\kappa) - \delta^2 \\
&= \frac{\tilde{r}_k - \tilde{r}_{k-1}}{\tilde{r}_{k-1}}\delta^2
\end{aligned}$$

Since

$$w(r) = \frac{da_\kappa^{-1}\left(\frac{\hat{x}-\mathbb{E}c}{r}\right)}{dr} = \frac{1}{-a'_\kappa\left[a_\kappa^{-1}\left(\frac{\hat{x}-\mathbb{E}c}{r}\right)\right]}\frac{\hat{x}-\mathbb{E}c}{r^2} \geq \frac{1}{\sup\{-a'_\kappa\}}\underline{r}\frac{A_\kappa(W)}{W}\frac{1}{\overline{r}^2} > 0$$

is bounded from below by a positive number, $\inf\{w(r)\}$ exists and is positive. Then

$$\delta = \int_{\tilde{r}_{k-1}}^{\tilde{r}_k} w(r)dr \geq (\tilde{r}_k - \tilde{r}_{k-1})\inf\{w(r)\} \Rightarrow \frac{\tilde{r}_k - \tilde{r}_{k-1}}{\tilde{r}_{k-1}} \leq \frac{\delta}{\tilde{r}_{k-1}\inf\{w(r)\}} \leq \frac{\delta}{\underline{r}\inf\{w(r)\}},$$

so

$$\int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} rd\tilde{\Omega}(r;W,\hat{x},\kappa) - \delta r_k \leq \frac{r_k}{\underline{r}\inf\{w(r)\}}\delta^2 \leq \frac{\overline{r}}{\underline{r}\inf\{w(r)\}}\delta^2.$$

Summing over k, we obtain

$$S(\hat{x}) - S(\hat{x};t) \leq \frac{(t-2)\overline{r}}{\underline{r}\inf\{w(r)\}}\delta^2 \leq \frac{W\overline{r}}{\underline{r}\inf\{w(r)\}}\delta.$$

As $t \to +\infty$, we have $\delta \to 0$, so

$$S(\hat{x}) \to \underline{r}\cdot Q^r + \bar{r}\cdot(W - Q^r) \Rightarrow \tilde{Q}\left(\hat{x};W,\kappa\right) \to Q^r.$$

Since $\hat{x}_r\left(W,Q^r,\kappa\right)$ is continuous in Q^r, as $t \to +\infty$, $\hat{x}_r\left(W,\tilde{Q}\left(\hat{x};W,\kappa\right),\kappa\right) \to \hat{x}_r\left(W,Q^r,\kappa\right)$, i.e., $\hat{x} \to \hat{x}_r\left(W,Q^r,\kappa\right)$. So, by increasing the number of scores, we can make the common cutoff of $\Omega_{(t)}$, which is also its maximum cutoff, arbitrarily close to $\hat{x}_r\left(W,Q^r,\kappa\right)$.

Part III: The quantile functions of (κ,t)-robust disclosures converge to that of $\Omega(\cdot;W,Q^r,\kappa)$ in L^1-norm as $t\to\infty$.

Denote the (κ,t)-robust disclosure by $\{(r_i;w_i)\}_{i=1}^t$ and its corresponding common cutoff as $\hat{x}$. Suppose $\tilde{r}_k$ satisfies $\tilde{\Omega}(\tilde{r}_k;W,\hat{x},\kappa) = \sum_{i=1}^k w_i$. As $t \to +\infty$, $\hat{x} \to \hat{x}_r\left(W,Q^r,\kappa\right)$. By the continuity of $\tilde{\Omega}(\cdot;W,\hat{x},\kappa)$ in $\hat{x}$, it is easy to see that the quantile functions of $\tilde{\Omega}(\cdot;W,\hat{x},\kappa)$ converge to that of $\Omega(\cdot;W,Q^r,\kappa)$ in L^1-norm as $t\to\infty$. Then our goal is to prove that as $t \to +\infty$, $\sum_{k=1}^t \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} |r-r_k|\, d\tilde{\Omega}(r;W,\hat{x},\kappa) \to 0$. Since

$$\sum_{k=1}^t \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} \left|\frac{1}{r} - \frac{1}{r_k}\right| d\tilde{\Omega}(r;W,\hat{x},\kappa) \geq \frac{1}{\overline{r}^2}\sum_{k=1}^t \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} |r-r_k|\, d\tilde{\Omega}(r;W,\hat{x},\kappa),$$

it suffices to prove that as $t \to +\infty$, $\sum_{k=1}^{t} \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} \left| \frac{1}{r} - \frac{1}{r_k} \right| d\tilde{\Omega}(r; W, \hat{x}, \kappa) \to 0$.

Consider any $1 \leq k \leq t$. Let $\Lambda_k \equiv \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} \left| \frac{1}{r} - \frac{1}{r_k} \right| d\tilde{\Omega}(r; W, \hat{x}, \kappa)$. Following (29), $\int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} \frac{1}{r} d\tilde{\Omega}(r; W, \hat{x}, \kappa) = \frac{w_k}{r_k}$. Then it is easy to see $\tilde{r}_{k-1} \leq r_k \leq \tilde{r}_k$. Suppose y and z satisfy $\int_{r=\tilde{r}_{k-1}}^{r_k} \frac{1}{r} d\tilde{\Omega}(r; W, \hat{x}, \kappa) = \frac{1}{y} \int_{r=\tilde{r}_{k-1}}^{r_k} d\tilde{\Omega}(r; W, \hat{x}, \kappa)$ and $\int_{r=r_k}^{\tilde{r}_k} \frac{1}{r} d\tilde{\Omega}(r; W, \hat{x}, \kappa) = \frac{1}{z} \int_{r=r_k}^{\tilde{r}_k} d\tilde{\Omega}(r; W, \hat{x}, \kappa)$. Then $y \leq r_k \leq z$. Denoting $\int_{r=\tilde{r}_{k-1}}^{r_k} d\tilde{\Omega}(r; W, \hat{x}, \kappa)$ by u, we have

$$\frac{1}{y} u + \frac{1}{z}(w_k - u) = \frac{w_k}{r_k}, \tag{30}$$

$$\left(\frac{1}{y} - \frac{1}{r_k}\right) u + \left(\frac{1}{r_k} - \frac{1}{z}\right)(w_k - u) = \Lambda_k. \tag{31}$$

Moreover, by Cauchy-Schwarz inequality

$$\begin{aligned} \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} r d\tilde{\Omega}(r; W, \hat{x}, \kappa) &= \int_{r=\tilde{r}_{k-1}}^{r_k} r d\tilde{\Omega}(r; W, \hat{x}, \kappa) + \int_{r=r_k}^{\tilde{r}_k} r d\tilde{\Omega}(r; W, \hat{x}, \kappa) \\ &\geq \frac{u^2}{\int_{r=\tilde{r}_{k-1}}^{r_k} \frac{1}{r} d\tilde{\Omega}(r; W, \hat{x}, \kappa)} + \frac{(w_k - u)^2}{\int_{r=r_k}^{\tilde{r}_k} \frac{1}{r} d\tilde{\Omega}(r; W, \hat{x}, \kappa)} = yu + z(w_k - u). \end{aligned}$$

Next, we derive a lower bound of $yu + z(w_k - u) - w_k r_k$. From (30) and (31), we can obtain $\frac{1}{y} = \frac{1}{r_k} + \frac{\Lambda_k}{2u}$ and $\frac{1}{z} = \frac{1}{r_k} - \frac{\Lambda_k}{2(w_k - u)}$. Then

$$yu + z(w_k - u) - w_k r_k = \frac{u}{\frac{1}{r_k} + \frac{\Lambda_k}{2u}} + \frac{w_k - u}{\frac{1}{r_k} - \frac{\Lambda_k}{2(w_k - u)}} - w_k r_k = r_k^2 \Lambda_k \left[-\frac{u}{2u + r_k \Lambda_k} + \frac{(w_k - u)}{2(w_k - u) - r_k \Lambda_k} \right].$$

Its derivative with respect to u is $r_k^2 \Lambda_k \left[-\frac{r_k \Lambda_k}{(2u + r_k \Lambda_k)^2} + \frac{r_k \Lambda_k}{(2w_k - 2u - r_k \Lambda_k)^2} \right]$, which is increasing in u. The minimum is attained at $\frac{r_k \Lambda_k}{(2u + r_k \Lambda_k)^2} = \frac{r_k \Lambda_k}{(2w_k - 2u - r_k \Lambda_k)^2} \Leftrightarrow u = \frac{w_k - r_k \Lambda_k}{2}$. Hence, $yu + z(w_k - u) - w_k r_k \geq \frac{r_k^3 \Lambda_k^2}{w_k}$.

Further, $\sum_{k=1}^{t} \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} r d\tilde{\Omega}(r; W, \hat{x}, \kappa) \geq \sum_{k=1}^{t} \left(w_k r_k + \frac{r_k^3 \Lambda_k^2}{w_k} \right) = \sum_{k=1}^{t} w_k r_k + \sum_{k=1}^{t} \frac{r_k^3 \Lambda_k^2}{w_k}$. By Cauchy-Schwarz inequality, $\sum_{k=1}^{t} \frac{r_k^3 \Lambda_k^2}{w_k} \cdot \sum_{k=1}^{t} \frac{w_k}{r_k^3} \geq \left(\sum_{k=1}^{t} \Lambda_k \right)^2$, so $\sum_{k=1}^{t} \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} r d\tilde{\Omega}(r; W, \hat{x}, \kappa) - \sum_{k=1}^{t} w_k r_k \geq \left(\sum_{k=1}^{t} \frac{w_k}{r_k^3} \right)^{-1} \left(\sum_{k=1}^{t} \Lambda_k \right)^2 \geq \frac{\underline{r}^3}{W} \left(\sum_{k=1}^{t} \Lambda_k \right)^2$.

As $t \to +\infty$, $\hat{x} \to \hat{x}_r(W, Q^r, \kappa)$, so $\sum_{k=1}^{t} \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} r d\tilde{\Omega}(r; W, \hat{x}, \kappa) \to \underline{r} \cdot Q^r + \bar{r} \cdot (W - Q^r) = \sum_{k=1}^{t} w_k r_k$. This implies $\sum_{k=1}^{t} \int_{r=\tilde{r}_{k-1}}^{\tilde{r}_k} \left| \frac{1}{r} - \frac{1}{r_k} \right| d\tilde{\Omega}(r; W, \hat{x}, \kappa) = \sum_{k=1}^{t} \Lambda_k \to 0$.

Proof of Proposition 11

Since $\kappa \equiv 0$ in this proof, we suppress the last argument of $\hat{x}_r$ for notational convenience.

Part I: $\overline{r}/\underline{r} \leq \frac{A(q^r)}{q^r}\frac{1-q^r}{A(1)-A(q^r)}$.

If $\hat{\theta} \geq \hat{x}_r(1,q^r) = \mathbb{E}c + \left(\frac{q^r}{\underline{r}} + \frac{1-q^r}{\overline{r}}\right)^{-1} \cdot A(1)$, by full revelation, all investors have a common switching cutoff $\hat{x}_r(1,q^r)$, so all banks are immune from runs. Moreover, full revelation minimizes the common switching cutoff of all banks.

Next, consider the case $\hat{\theta} < \hat{x}_r(1,q^r)$. Suppose the proposition does not hold. That means, under a disclosure, a group of banks (W,Q) can be immune from runs. It is easy to see that $\underline{r}\frac{A(q^r)}{q^r} \geq \frac{A(1)}{\frac{q^r}{\underline{r}}+\frac{1-q^r}{\overline{r}}} \geq \overline{r}\frac{A(1)-A(q^r)}{1-q^r}$.

If $\overline{r}/\underline{r} > \frac{A(Q)}{Q}\frac{W-Q}{A(W)-A(Q)}$, then (W,Q) has strong 0-heterogeneity. According to the construction of its 0-robust disclosure, $\hat{x}_r(W,Q) \geq \mathbb{E}c + \underline{r}\lim_{x\to\underline{m}}\frac{A(x)}{x} \geq \mathbb{E}c + \underline{r}\frac{A(q^r)}{q^r} > \hat{\theta}$. Contradiction!

If $\overline{r}/\underline{r} \leq \frac{A(Q)}{Q}\frac{W-Q}{A(W)-A(Q)}$, (W,Q) has weak 0-heterogeneity, and its maximum cutoff cannot be smaller than $Ec + \frac{A(W)}{\frac{Q}{\underline{r}}+\frac{W-Q}{\overline{r}}}$. Since

$$\frac{A(1)-A(W)}{\frac{1-W}{\overline{r}}} < \frac{A(W)-A(Q)}{\frac{W-Q}{\overline{r}}} \leq \frac{A(W)}{\frac{Q}{\underline{r}}+\frac{W-Q}{\overline{r}}} \leq \frac{A(Q)}{\frac{Q}{\underline{r}}},$$

$\frac{A(1)}{\frac{q^r}{\underline{r}}+\frac{1-q^r}{\overline{r}}} \leq \frac{A(1)}{\frac{Q}{\underline{r}}+\frac{1-Q}{\overline{r}}} = \frac{1}{\frac{Q}{\underline{r}}+\frac{W-Q}{\overline{r}}+\frac{1-W}{\overline{r}}}\left[A(W) + A(1) - A(W)\right] < \frac{A(W)}{\frac{Q}{\underline{r}}+\frac{W-Q}{\overline{r}}}$. So $\mathbb{E}c + \frac{A(W)}{\frac{Q}{\underline{r}}+\frac{W-Q}{\overline{r}}} > \hat{\theta}$. Contradiction!

Part II: $\overline{r}/\underline{r} > \frac{A(q^r)}{q^r}\frac{1-q^r}{A(1)-A(q^r)}$.

We prove Proposition 11 for the strong 0-heterogeneity case based on Lemma 5.

Lemma 5. *Suppose* $\overline{r}/\underline{r} > \frac{A(q^r)}{q^r}\frac{1-q^r}{A(1)-A(q^r)}$.

- *For any* $Q \in [0,q^r]$ *and* $\hat{\theta} \in [\hat{x}_r(Q,Q), \hat{x}_r(1,q^r))$, $\hat{x}_r(W,Q) = \hat{\theta}$ *has a unique solution in* $[Q,1)$, $W(\hat{\theta},Q)$.
- *Moreover,* $\lim_{r\to\overline{r}-}\Omega\left(r; W(\hat{\theta},Q), Q, 0\right) = W(\hat{\theta},Q)$, *and* $W(\hat{\theta},Q)$ *is continuous and strictly increasing in* Q *and* $\hat{\theta}$.

If $\hat{\theta} \geq \hat{x}_r(1, q^r)$, the 0-robust disclosure for the whole banking system, $\Omega(\cdot; 1, q^r, 0)$, can ensure that all banks survive.

If $\hat{\theta} < \hat{x}_r(q^r, q^r)$, then $\hat{\theta} < \mathbb{E}c + \underline{r}\frac{A(q^r)}{q^r}$. This implies that for any bank group with no more than a mass q^r of $\underline{r}$-type banks, there does not exist a 0-robust disclosure such that the common cutoff is not higher than $\hat{\theta}$. So, no bank can be immune from runs.

Next, we consider $\hat{\theta} \in [\hat{x}_r(q^r, q^r), \hat{x}_r(1, q^r))$. First, any group of banks (W, Q) that has a cutoff smaller than $\hat{x}_r(1, q^r)$ must have strong 0-heterogeneity. Suppose not, i.e., $\overline{r}/\underline{r} \leq \frac{A(Q)}{Q}\frac{W-Q}{A(W)-A(Q)}$. Then its lowest cutoff is

$$\mathbb{E}c + \frac{A(W)}{\frac{Q}{\underline{r}} + \frac{W-Q}{\overline{r}}} \geq \mathbb{E}c + \overline{r}\frac{A(W) - A(Q)}{W - Q} \geq \mathbb{E}c + \overline{r}\frac{A(1) - A(q^r)}{1 - q^r} > \hat{x}_r(1, q^r).$$

Second, consider any bank group (W, Q) that can be immune from runs. Suppose its maximum cutoff under its 0-robust disclosure is $\theta' \leq \hat{\theta}$. Then W solves $\hat{x}_r(W, Q) = \theta'$ and $W \leq 1$. By the proof of Lemma 5, we know that it must be $W = W(\theta', Q)$. Since $W(\theta, Q)$ is strictly increasing in θ and Q, $W \leq W(\hat{\theta}, q^r)$. And $W(\hat{\theta}, q^r)$ can be attained uniquely by the 0-robust disclosure for bank group $(W(\hat{\theta}, q^r), q^r)$, which consists of measure q^r of $\underline{r}$-type banks and measure $W(\hat{\theta}, q^r) - q^r$ of $\overline{r}$-type banks.

Proof of Proposition 12

Since $\kappa \equiv 0$ in this proof, we suppress the last argument of $\hat{x}_c$ for notational convenience.

Part I: $\overline{c} - \underline{c} \leq \left[\frac{A(q^c)}{q^c} - \frac{A(1)-A(q^c)}{1-q^c}\right]\mathbb{E}r$

Suppose there exists a disclosure such that banks have different switching cutoffs. Because the average switching cutoff of all investors is always $\hat{x}_c(1, q^c)$, there exists a group of banks (W, Q) such that $(W, Q) \leq (1, q^c)$ and $\hat{x}_c(W, Q) < \hat{x}_c(1, q^c)$, i.e.,

$$\begin{aligned} &\frac{Q}{W}\left[\underline{c} + \mathbb{E}r\frac{A(Q)}{Q}\right] + \frac{W-Q}{W}\left[\overline{c} + \mathbb{E}r\frac{A(W) - A(Q)}{W - Q}\right] \\ &< q^c\left[\underline{c} + \mathbb{E}r\frac{A(q^c)}{q^c}\right] + (1 - q^c)\left[\overline{c} + \mathbb{E}r\frac{A(1) - A(q^c)}{1 - q^c}\right]. \end{aligned}$$

If $\underline{c} + \mathbb{E}r\frac{A(Q)}{Q} < \overline{c} + \mathbb{E}r\frac{A(W)-A(Q)}{W-Q}$, $\hat{x}_c(W, Q) > \underline{c} + \mathbb{E}r\frac{A(Q)}{Q} \geq \underline{c} + \mathbb{E}r\frac{A(q^c)}{q^c} \geq \hat{x}_c(1, q^c)$.

Contradiction!

If $\underline{c} + \mathbb{E}r\frac{A(Q)}{Q} \geq \overline{c} + \mathbb{E}r\frac{A(W)-A(Q)}{W-Q}$, then $\overline{c} + \mathbb{E}r\frac{A(W)-A(Q)}{W-Q} \leq \hat{x}_c\left(W, Q\right)$. Since

$$\begin{aligned}\hat{x}_c\left(1, Q\right) =& Q\left[\underline{c} + \mathbb{E}r\frac{A(Q)}{Q}\right] + (W - Q)\left[\overline{c} + \mathbb{E}r\frac{A(W) - A(Q)}{W - Q}\right] + (1 - W)\left[\overline{c} + \mathbb{E}r\frac{A(1) - A(W)}{1 - W}\right] \\ =& W\hat{x}_c\left(W, Q\right) + (1 - W)\left[\overline{c} + \mathbb{E}r\frac{A(1) - A(W)}{1 - W}\right] \leq \hat{x}_c\left(W, Q\right),\end{aligned}$$

$\hat{x}_c\left(1, q^c\right) \leq \hat{x}_c\left(1, Q\right) \leq \hat{x}_c\left(W, Q\right)$. Contradiction!

So, no matter what the disclosure is, all banks have the same cutoff, $\hat{x}_c\left(1, q^c\right)$.

Part II: $\overline{c} - \underline{c} > \left[\frac{A(q^c)}{q^c} - \frac{A(1)-A(q^c)}{1-q^c}\right]\mathbb{E}r$

If $\hat{\theta} \geq \hat{x}_c\left(1, q^c\right)$, nondisclosure can ensure all banks immune from runs.

If $\hat{\theta} < \hat{x}_c\left(1, q^c\right)$, only part of the banks can be immune from runs. Suppose a group of banks (W, Q) are. Then their average cutoff must be weakly smaller than $\hat{\theta}$, i.e.,

$$\hat{x}_c\left(W, Q\right) = \frac{Q \cdot \underline{c} + (W - Q) \cdot \overline{c}}{W} + \mathbb{E}r \cdot \frac{A\left(W\right)}{W} \leq \hat{\theta}.$$

We want to find the maximum W subject to this constraint. Notice that $\hat{x}_c\left(W, Q\right)$ is decreasing in Q. It is easy to see that the maximum W must be a solution to $\hat{x}_c\left(W, q^c\right) = \hat{\theta}$. Last, we prove the following lemma in the Internet Appendix.

Lemma 6. *For $\hat{\theta} < \hat{x}_c\left(1, q^c\right)$, $\hat{x}_c\left(W, q^c\right) = \hat{\theta}$ has a unique solution $W(\hat{\theta}, q^c)$.*

Also, notice that $\hat{x}_c\left(W, Q\right) \geq \hat{x}_c\left(W, q^c\right) \geq \hat{x}_c\left(q^c, q^c\right)$. So, if $\hat{\theta} < \hat{x}_c\left(q^c, q^c\right)$, no bank can be immune from runs under any disclosure.

C Internet Appendix

C.1 Details of Robustness Checks in Section 6

C.1.1 Informative Prior of Bank-Specific Information

In our baseline model, we assume that investors' priors are uninformative about bank-specific information in both dimensions c and r. In reality, investors may have information sources about their banks other than the regulator's disclosures. Thus, in this subsection, we consider the possibility of informative common priors in dimensions r and c, and show that disclosures in dimension c can only hurt the stability of a bank group (i.e., increase the maximum of its investors' switching cutoffs), while in dimension r there is always a disclosure that improves the group's stability. In this sense, our main results are robust to this possibility.

Specifically, consider a bank group with mass W for whom we are designing κ-robust disclosures. Suppose public information beyond the regulator's control differentiates it into n categories, $\{(r_i, c_i, w_i)\}_{i=1}^{n}$, where r_i, c_i and w_i are the expected types and mass of category i, respectively. Let $\hat{x}_i$ be the switching cutoff of investors of category i banks without disclosures. Without loss of generality, assume that the corresponding $\Delta_{i-1,i}$ given by (19) are all nonnegative, so that $\hat{x}_n \geq \hat{x}_i$ for all i. Let $P_{max} = \{i : \hat{x}_i = \hat{x}_n\}$. Then, by Proposition 4,

$$\hat{x}_n = \frac{1}{\sum_{z\in P_{max}} \frac{w_z}{r_z}} \left[\sum_{z\in P_{max}} \frac{c_z}{r_z} w_z + A_\kappa (W) - A_\kappa \left(\sum_{z\notin P_{max}} w_z \right) \right]. \tag{32}$$

In this context, a disclosure in dimension c assigns $t_i \geq 1$ scores, $(c_{i,1}, c_{i,2}, \dots c_{i,t_i})$, to category i banks. Let $w_{i,j}$ be the mass of banks receiving score $c_{i,j}$. Then by construction, $w_i = \sum_{j=1}^{t_i} w_{i,j}$, and Bayesian plausibility requires that $c_i = \frac{\sum_{j=1}^{t_i} w_{i,j} c_{i,j}}{w_i}$. Let $\hat{x}_{i,j}$ be the switching cutoff of score-$c_{i,j}$ investors.

Proposition 13. *For any disclosure in dimension c, we have* $\max_{i\in P_{max}} \{\hat{x}_{i,j}\} \geq \hat{x}_n$.

Since $\max \{\hat{x}_{i,j}\} \geq \max_{i\in P_{max}} \{\hat{x}_{i,j}\}$, Proposition 13 establishes that robust disclosures never reveal information in dimension c even if investors' common prior is informative. To understand Proposition 13, consider first the case where $\max_{i\notin P_{max}} \{\hat{x}_{i,j}\} < \min_{i\in P_{max}} \{\hat{x}_{i,j}\}$, so that the disclosure does not change the belief of cutoff investors of banks in P_{max} that all investors outside P_{max} are staying. In this case, since the disclosure changes neither the expected systemic vulnerability r_i of any bank in P_{max}, nor the aggregate systemic risk expected by all cutoff investors in P_{max}, it can be shown that the average switching cutoff of investors of banks in P_{max} is still given by the right-hand side of (32). As a result, we have $\max_{i\in P_{max}} \{\hat{x}_{i,j}\} = \hat{x}_n$ if $\hat{x}_{i,j}$ is still constant in P_{max}, and $\max_{i\in P_{max}} \{\hat{x}_{i,j}\} > \hat{x}_n$ otherwise.

If in addition, the disclosure makes $\max_{i\notin P_{max}} \{\hat{x}_{i,j}\} \geq \min_{i\in P_{max}} \{\hat{x}_{i,j}\}$, then cutoff investors of banks in P_{max} no longer believe that those not in P_{max} stay for sure. This increases the aggregate systemic risk expected by them and their average cutoff. In this case, we must have $\max_{i\in P_{max}} \{\hat{x}_{i,j}\} > \hat{x}_n$ as well.

Now consider a disclosure in dimension r defined analogously to that in dimension c. Proposition 14 instead establishes that κ-robust disclosures should still disclose information in dimension r as in the baseline model.

Proposition 14. *There exists a disclosure in dimension r such that* $\max\{\hat{x}_{i,j}\} < \hat{x}_n$.

Let $\tilde{i}$ be the category such that $c_{\tilde{i}} = \min\{c_i : \Delta_{i,n} < +\infty\}$. Consider marginally increasing informational heterogeneity of banks in category $(r_{\tilde{i}}, c_{\tilde{i}}, w_{\tilde{i}})$ in dimension r. Since score $\tilde{i}$ is entangled with score n, all scores assigned to banks in this category are still entangled with score n, so that investors share the highest cutoff under the new disclosure if and only if their banks belong to categories in P_{max}. Then, Lemma 1 implies that this new disclosure reduces the common cutoff shared by these investors as desired.

C.1.2 Correlation Between Dimensions r and c

In the baseline model, banks' systemic vulnerabities and idiosyncratic costs are assumed to be independent, allowing the regulator to disclose information in only one dimension. This facilitates our analysis of the different impact of disclosures in different dimensions. In reality, different bank-specific information could be correlated, so that a disclosure in one dimension automatically reveals information in the other. This subsection discusses this possibility by allowing for correlation between types in different dimensions in the binary-score setup in Section 3. Specifically, we consider disclosures in one dimension while allowing for arbitrary $\beta = Cov(c^i, r^i)/Var(r^i)$. In this context, score-r_i implies the corresponding $c_i = \beta \cdot (r_i - \mathbb{E}r) + \mathbb{E}c$. We show that as long as $\beta + A(1) > 0$, i.e., if types in different dimensions are not too negatively correlated, or if allocation of systemic risk is sufficiently important, disclosures in dimension r can still be beneficial due to the same mechanism of assortative matching as in the baseline model, and disclosures in dimension c affect investors' strategies only through the information they reveal in dimension r.

Consider the binary scores, $\{(r_i, c_i, w_i)\}_{i=1,2}$, with $r_1 \leq r_2$. By Proposition 4, the average switching cutoff of all investors is

$$w_1\hat{x}_1 + w_2\hat{x}_2 = \mathbb{E}c + A(1)\cdot\mathbb{E}r + [\beta + A(1)]\left[\left(\mathbb{E}\left[\frac{1}{r}\right]\right)^{-1} - \mathbb{E}r\right], \tag{33}$$

where $\mathbb{E}\left[\frac{1}{r}\right] = \sum_j \frac{w_j}{r_j}$. The sum of the first two terms of the right-hand side of (33) is the common cutoff with nondisclosure. Again, by Jensen's inequality, $\left(\mathbb{E}\left[\frac{1}{r}\right]\right)^{-1} - \mathbb{E}r < 0$ whenever $r_1 < r_2$.

Equation (33) confirms that even if bank-specific information is correlated, disclosures affect the average switching cutoff only through their impact in dimension r. Disclosures in dimension c affect the average switching cutoff only indirectly through the information in dimension r revealed due to their correlation.

If $\beta + A(1) = 0$, then the average cutoff in (33) is constantly $\mathbb{E}c + A(1) \cdot \mathbb{E}r$. Indeed, we show that in this knife-edge case, regardless of disclosures, we always have $\hat{x}_1 = \hat{x}_2$, and a cutoff score-1 investor always expects the same systemic risk, $A(1)$, as a cutoff-score-2 investor does.

If $\beta + A(1) > 0$, then the average cutoff in (33) decreases with informational heterogeneity in dimension r. Indeed, as in the baseline model, disclosures in dimension r still reallocate more of the constant aggregate systemic risk to cutoff score-1 investors, whose expected systemic vulnerability is also lower. As long as r_2/r_1 is not too large, we still have $\hat{x}_1 = \hat{x}_2$, and the common switching cutoff decreases with r_2/r_1.

If $\beta + A(1) < 0$, then a cutoff score-1 investor's expected idiosyncratic cost c_1 is so high that he expects less systemic risk than a cutoff score-2 investor, whose expected systemic vulnerability is higher. In this case, disclosures are detrimental to the banking system, and thus nondisclosure is optimal.

C.2 Proofs of Lemmas in the Appendix

Proof of Lemma 2

We show that for any r, $\tilde{\Omega}(r;W,\hat{X},\kappa)$ is strictly decreasing and continuous in $\hat{X}$. Continuity is obvious.

Consider $\hat{X} < \hat{X}'$. It is easy to see $\underline{m} \geq \underline{m}'$ and $\overline{m} \leq \overline{m}'$. For $r \in [\max\{r_-, r'_-\}, \min\{r_+, r'_+\}]$,

$$ra_\kappa\left(\tilde{\Omega}(r;W,\hat{X},\kappa)\right) = \hat{X} - \mathbb{E}c < \hat{X}' - \mathbb{E}c = ra_\kappa\left(\tilde{\Omega}(r;W,\hat{X}',\kappa)\right)$$

so $\tilde{\Omega}(r;W,\hat{X},\kappa) > \tilde{\Omega}(r;W,\hat{X}',\kappa)$.

If $r_- \leq r'_-$, for $r \in [\underline{r}, r'_-)$, $\tilde{\Omega}(r;W,\hat{X}',\kappa) = \underline{m}' \leq \underline{m} \leq \tilde{\Omega}(r;W,\hat{X},\kappa)$.

If $r_- > r'_-$, for $r \in [\underline{r}', r_-)$, $\tilde{\Omega}(r;W,\hat{X},\kappa) = \underline{m} = \tilde{\Omega}(r_-;W,\hat{X},\kappa) > \tilde{\Omega}(r_-;W,\hat{X}',\kappa) \geq \tilde{\Omega}(r;W,\hat{X}',\kappa)$. So, for $r \in [\underline{r}, \max\{r_-, r'_-\}]$, $\tilde{\Omega}(r;W,\hat{X},\kappa) > \tilde{\Omega}(r;W,\hat{X}',\kappa)$.

If $r_+ \leq r'_+$, for $r \in (r_+, \overline{r})$, $\tilde{\Omega}(r;W,\hat{X},\kappa) = W - \overline{m} \geq W - \overline{m}' \geq \tilde{\Omega}(r;W,\hat{X}',\kappa)$.

If $r_+ > r'_+$, for $r \in (r'_+, \overline{r})$, $\tilde{\Omega}(r;W,\hat{X},\kappa) \geq \tilde{\Omega}(r'_+;W,\hat{X},\kappa) > \tilde{\Omega}(r'_+;W,\hat{X}',\kappa) = \tilde{\Omega}(r;W,\hat{X}',\kappa)$. So, for $r \in (\min\{r_+, r'_+\}, \overline{r})$, $\tilde{\Omega}(r;W,\hat{X},\kappa) > \tilde{\Omega}(r;W,\hat{X}',\kappa)$.

To sum up, we obtain that $\tilde{\Omega}(r;W,\hat{X},\kappa)$ is strictly decreasing in $\hat{X}$. Since

$$\tilde{Q}\left(\hat{X};W,\kappa\right) = \frac{\overline{r}W - \int_{r=\underline{r}}^{\overline{r}} r \cdot d\tilde{\Omega}(r;W,\hat{X},\kappa)}{\overline{r} - \underline{r}} = \frac{\int_{r=\underline{r}}^{\overline{r}} \tilde{\Omega}(r;W,\hat{X},\kappa) \cdot dr}{\overline{r} - \underline{r}},$$

$\tilde{Q}\left(\hat{X};W,\kappa\right)$ is strictly decreasing and continuous in $\hat{X}$.

Note that if $\hat{X} = \mathbb{E}c + \underline{r}\frac{A_\kappa(W)}{W}$, $\tilde{\Omega}(r;W,\hat{X},\kappa) = W$ for $r \in [\underline{r}, \overline{r}]$, so $\tilde{Q}\left(\hat{X};W,\kappa\right) = W$; if $\hat{X} = \mathbb{E}c + \overline{r}\frac{A_\kappa(W)}{W}$, $\tilde{\Omega}(r;W,\hat{X},\kappa) = 0$ for $r \in [\underline{r}, \overline{r})$, so $\tilde{Q}\left(\hat{X};W,\kappa\right) = 0$. Therefore, for any $Q^r \in [0, W]$, there exists a unique $\hat{x}_r(W, Q^r, \kappa)$ such that $\tilde{Q}(\hat{x}_r(W, Q^r, \kappa); W, \kappa) = Q^r$ and $\hat{x}_r(W, Q^r, \kappa)$ is continuous and strictly decreasing in Q^r.

Proof of Lemma 3

A cutoff investor must be indifferent between staying and running in equilibrium; i.e.,

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - r_i a\left(\sum_{j=1}^{n} w_j\left[1 - \Phi\left(\frac{\hat{x}_j^\sigma - \theta}{\sigma}\right)\right]\right) - c_i\right] \phi\left(\frac{\hat{x}_i^\sigma - \theta}{\sigma}\right) h(\theta) d\theta = 0.$$

Let $y = 1 - \Phi\left(\frac{\hat{x}_i^\sigma - \theta}{\sigma}\right)$ and $\Delta_{j,i}^\sigma \triangleq \frac{\hat{x}_i^\sigma - \hat{x}_j^\sigma}{\sigma}$. Then we obtain $\int_0^1 g(\theta, \{\Delta_{j,i}^\sigma\}_{j=1}^n, y)dy = 0$, where $\theta = \hat{x}_i^\sigma - \sigma\Phi^{-1}(1-y)$ and

$$g(\theta, \{\Delta_{i,j}^\sigma\}_{j=1}^n, y) \triangleq \left[\theta - r_i a\left(\sum_{j=1}^n w_j\left[1 - \Phi\left(\Phi^{-1}(1-y) + \Delta_{i,j}^\sigma\right)\right]\right) - c_i\right] h(\theta).$$

Due to the assumption that there exist dominance regions, $\hat{x}_i^{\sigma_m} \in (\underline{\theta}, \overline{\theta})$. By Bolzano–Weierstrass theorem, there exists an infinite subsequence of $\{\sigma_m\}_{m=1}^{+\infty}$, say $\{\sigma_m^1\}_{m=1}^{+\infty}$, such that $\forall i \in \{1, 2, \ldots, n\}$, $\hat{x}_i^{\sigma_m}$ converges to a finite number $\hat{x}_i^0$. As for any $\{\Delta_{i-1,i}^{\sigma_m^1}\}_{m=1}^{+\infty}$, if it is bounded, there exists an infinite subsequence of $\{\sigma_m^1\}_{m=1}^{+\infty}$, say $\{\sigma_m^2\}_{m=1}^{+\infty}$, such that $\Delta_{i-1,i}^{\sigma_m^2}$ converges to a finite number; if it is not bounded, there exists an infinite subsequence $\{\sigma_m^3\}_{m=1}^{+\infty}$ such that it goes to $+\infty$ or $-\infty$. We can iterate this construction for all $\{\Delta_{i-1,i}^{\sigma_m^1}\}_{m=1}^{+\infty}$. Finally, we end up with an infinite subsequence $\{\sigma_m^4\}_{m=1}^{+\infty}$ such that $\hat{x}_k^{\sigma_m^4} \to \hat{x}_k^0$ and $\Delta_{j,k}^{\sigma_m^4} \to \Delta_{j,k}^0$ for all j, k. In addition, if $\hat{x}_i^0 > (<)\hat{x}_{i-1}^0$,

$$\Delta_{i-1,i}^0 = \lim_{m\to\infty} \Delta_{i-1,i}^{\sigma_m^4} = \lim_{m\to\infty} \frac{\hat{x}_i^{\sigma_m^4} - \hat{x}_{i-1}^{\sigma_m^4}}{\sigma_m^4} = \lim_{m\to\infty} \frac{\hat{x}_i^0 - \hat{x}_{i-1}^0}{\sigma_m^4} = +\infty(-\infty),$$

and

$$\Delta_{j,i}^0 = \lim_{m\to\infty} \Delta_{j,i}^{\sigma_m^4} = \lim_{m\to\infty} \sum_{k=j+1}^{i} \Delta_{k-1,k}^{\sigma_m^4} = \sum_{k=j+1}^{i} \Delta_{k-1,k}^0.$$

Next, we show $\int_{-\infty}^{\infty} g(\hat{x}_i^0, \{\Delta_{j,i}^0\}_{j=1}^n, y)dy = 0$. Consider any σ and $\epsilon > 0$. There exists $t_1 > 0$ such that $1 - \Phi(t_1) + \Phi(-t_1) < \frac{\epsilon}{R\sup_\theta\{h(\theta)\}}$. Then for any n real numbers $\{\Delta_{j,i}^\sigma\}_{j=1}^n$, finite or infinite,

$$\begin{aligned}
&\left|\int_{|\Phi^{-1}(1-y)|\geq t_1} g(\theta, \{\Delta_{j,i}^\sigma\}_{j=1}^n, y)dy\right| \\
=&\left|\int_{\left|\frac{\hat{x}_i^\sigma-\theta}{\sigma}\right|\geq t_1} \left[\theta - r_i a\left(\sum_{j=1}^n w_j\left[1 - \Phi\left(\frac{\hat{x}_i^\sigma - \theta}{\sigma} + \Delta_{i,j}^\sigma\right)\right]\right) - c_i\right] h(\theta)\frac{1}{\sigma}\phi\left(\frac{\hat{x}_i^\sigma - \theta}{\sigma}\right) d\theta\right| \\
\leq&\int_{\left|\frac{\hat{x}_i^\sigma-\theta}{\sigma}\right|\geq t_1} Rh(\theta)\frac{1}{\sigma}\phi\left(\frac{\hat{x}_i^\sigma - \theta}{\sigma}\right) d\theta \leq R\sup_\theta\{h(\theta)\}\int_{\left|\frac{\hat{x}_i^\sigma-\theta}{\sigma}\right|\geq t_1} \frac{1}{\sigma}\phi\left(\frac{\hat{x}_i^\sigma - \theta}{\sigma}\right) d\theta \\
<&R\sup_\theta\{h(\theta)\} \cdot \frac{\epsilon}{R\sup_\theta\{h(\theta)\}} = \epsilon
\end{aligned}$$

and

$$\left|\int_{|\Phi^{-1}(1-y)|\geq t_1} g(\hat{x}_i^0, \{\Delta_{j,i}^{\sigma}\}_{j=1}^n, y)dy\right| < \epsilon.$$

Since $h(\theta)$ is continuous in θ, for $\hat{x}_i^0$, there exists t_2 such that for any $\theta \in (\hat{x}_i^0 - t_2, \hat{x}_i^0 + t_2)$,

$$\left|g(\hat{x}_i^0, \{\Delta_{j,i}^{\sigma}\}_{j=1}^n, y) - g(\theta, \{\Delta_{j,i}^{\sigma}\}_{j=1}^n, y)\right| < \epsilon.$$

Since $\sigma_m^4 \to 0$ and $\hat{x}_i^{\sigma_m^4} \to \hat{x}_i^0$, when m is sufficiently large,

$$(\hat{x}_i^{\sigma_m^4} - t_1\sigma_m^4, \hat{x}_i^{\sigma_m^4} + t_1\sigma_m^4) \in (\hat{x}_i^0 - t_2, \hat{x}_i^0 + t_2).$$

There,

$$\begin{aligned}
&\left|\int_0^1 g(\theta, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y)dy - \int_0^1 g(\hat{x}_i^0, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y)dy\right| \\
<&\left|\int_{|\Phi^{-1}(1-y)|<t_1} g(\theta, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y)dy - \int_{|\Phi^{-1}(1-y)|<t_1} g(\hat{x}_i^0, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y)dy\right| + 2\epsilon \\
\leq&\int_{|\Phi^{-1}(1-y)|<t_1} \left|g(\theta, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y) - g(\hat{x}_i^0, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y)\right| dy + 2\epsilon < 3\epsilon.
\end{aligned}$$

Next, we show that when m is sufficiently large,

$$\left|\Phi\left(\Phi^{-1}(1-y) + \Delta_{i,j}^{\sigma_m^4}\right) - \Phi\left(\Phi^{-1}(1-y) + \Delta_{i,j}^0\right)\right|$$

can arbitrarily small for any y satisfying $|\Phi^{-1}(1-y)| < t_1$. If $\Delta_{j,k}^0$ is finite, because $\Delta_{j,k}^{\sigma_m^4} \to \Delta_{j,k}^0$ and $\phi(\cdot)$ is bounded, it is obvious. If $\Delta_{j,k}^0 = +\infty$, then $\Phi\left(\Phi^{-1}(1-y) - \Delta_{j,k}^0\right) = 0$. Since $\Phi\left(\Phi^{-1}(1-y) - \Delta_{j,k}^{\sigma_m^4}\right)$ decreases to 0 as $\Delta_{j,k}^{\sigma_m^4} \to +\infty$, for any $\delta > 0$, when m is sufficiently large, $\Delta_{j,k}^{\sigma_m^4}$ can be large enough such that for any y satisfying $|\Phi^{-1}(1-y)| < t_1$,

$$\Phi\left(\Phi^{-1}(1-y) - \Delta_{j,k}^{\sigma_m^4}\right) \leq \Phi(t_1 - \Delta_{j,k}^{\sigma_m^4}) < \delta.$$

The case of $\Delta_{j,k}^0 = -\infty$ follows a similar idea. Since $a(\cdot)$ is Lipschitz continuous, we obtain that when m is sufficiently large,

$$\left|g(\hat{x}_i^0, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y) - g(\hat{x}_i^0, \{\Delta_{j,i}^0\}_{j=1}^n, y)\right| < \epsilon$$

for any y satisfying $|\Phi^{-1}(1-y)| < t_1$,

$$\begin{aligned}
&\left|\int_0^1 g(\hat{x}_i^0, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y)dy - \int_0^1 g(\hat{x}_i^0, \{\Delta_{j,i}^0\}_{j=1}^n, y)dy\right| \\
\leq &\left|\int_{|\Phi^{-1}(1-y)|<t_1} g(\hat{x}_i^0, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y)dy - \int_{|\Phi^{-1}(1-y)|<t_1} g(\hat{x}_i^0, \{\Delta_{j,i}^0\}_{j=1}^n, y)dy\right| + 2\epsilon < 3\epsilon.
\end{aligned}$$

Thus, when m is sufficiently large,

$$\left|\int_0^1 g(\hat{x}_i^0, \{\Delta_{j,i}^0\}_{j=1}^n, y)dy\right| = \left|\int_0^1 g(\hat{x}_i^0, \{\Delta_{j,i}^0\}_{j=1}^n, y)dy - \int_0^1 g(\theta, \{\Delta_{j,i}^{\sigma_m^4}\}_{j=1}^n, y)dy\right| < 6\epsilon.$$

Since ϵ can be arbitrarily small, $\int_{-\infty}^{\infty} g(\hat{x}_i^0, \{\Delta_{j,i}^0\}_{j=1}^n, y)dy = 0$, i.e.,

$$\begin{aligned}
&\int_0^1 \left[\hat{x}_i^0 - r_i a\left(\sum_{j=1}^n w_j\left[1 - \Phi\left(\Phi^{-1}(1-y) - \Delta_{j,i}^0\right)\right]\right) - c_i\right] h(\hat{x}_i^0)dy = 0 \\
\Leftrightarrow &\hat{x}_i^0 = c_i + r_i \int_0^1 a\left(\sum_{j=1}^n w_j\left[1 - \Phi\left(\Phi^{-1}(1-y) - \Delta_{j,i}^0\right)\right]\right) dy
\end{aligned}$$

This proof also confirms the existence of the solution to the equation system.

Proof of Lemma 4

Suppose $\{\hat{x}_i\}_{i=1}^n$ and $\{\hat{x}'_i\}_{i=1}^n$ both satisfy the equation system and they are different in at least one element. They have $\{\Delta_{j,i}\}_{j,i\in\{1,\ldots,n\}}$ and $\{\Delta'_{j,i}\}_{j,i\in\{1,\ldots,n\}}$ respectively.

Suppose there are types with $\hat{x}'_i > \hat{x}_i$ and they constitute the set $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_L\}$ where $\tau_1 < \tau_2 \ldots < \tau_L$. Consider $i \in \mathcal{T}$. Since $g(\hat{x}'_i, \{\Delta'_{j,i}\}_{j=1}^n, y)$ is increasing in $\Delta'_{j,i}$, if $\Delta'_{j,i} \geq \Delta_{j,i}$ for any j, then

$$\int_{-\infty}^{\infty} g(\hat{x}'_i, \{\Delta'_{j,i}\}_{j=1}^n, y)dy > \int_{-\infty}^{\infty} g(\hat{x}_i, \{\Delta_{j,i}\}_{j=1}^n, y)dy \cdot \frac{h(\hat{x}'_i)}{h(\hat{x}_i)} = 0.$$

Therefore, $\Delta'_{j,i} < \Delta_{j,i}$ for some j. Let $\xi(i)$ be the smallest j such that $\Delta'_{j,i} < \Delta_{j,i}$.

Note that for $j \notin \mathcal{T}$, since $\hat{x}'_j \leq \hat{x}_j$ and $\hat{x}'_i > \hat{x}_i$, $\Delta'_{j,i} \geq \Delta_{j,i}$. So, $\xi(\tau_1) \in \mathcal{T}$, and $\xi(\tau_1) > \tau_1$. Likewise, $\xi^{(2)}(\tau_1) = \xi(\xi(\tau_1))$ must be in $\mathcal{T}$. By the definition of $\xi(\tau_1)$, for any $j \in \mathcal{T}$ and $j < \xi(\tau_1)$, $\Delta'_{j,\tau_1} \geq \Delta_{j,\tau_1}$, and $\Delta'_{\xi(\tau_1),\tau_1} < \Delta_{\xi(\tau_1),\tau_1}$. So, for these j,

$$\Delta'_{j,\xi(\tau_1)} = \Delta'_{j,\tau_1} - \Delta'_{\xi(\tau_1),\tau_1} > \Delta_{j,\tau_1} - \Delta_{\xi(\tau_1),\tau_1} = \Delta_{j,\xi(\tau_1)},$$

which implies $\xi(\xi(\tau_1)) > \xi(\tau_1)$. Iterating the procedure, we end up with an infinite sequence $\{\xi^{(m)}(\tau_1)\}_{m=1}^{+\infty}$ in $\mathcal{T}$. This is impossible because $\mathcal{T}$ is a finite set.

Therefore, the types with $\hat{x}'_i > \hat{x}_i$ do not exist; nor do the types with $\hat{x}'_i < \hat{x}_i$. The solution is unique. Note that the solution is the limits $\{\hat{x}_i^0\}_{i=1}^n$ in Lemma 3.

Proof of Lemma 5

First, we show that for any $Q \in [0, q^r]$ and $\hat{\theta} \in [\hat{x}_r(Q,Q), \hat{x}_r(1,q^r))$, there exists a $\tilde{W} \in [Q,1)$ such that $\tilde{\Omega}(\cdot;\tilde{W},\hat{\theta},0)$ contains mass Q of $\underline{r}$-type banks. Define $\check{Q}(\hat{\theta};\check{r})$ as

$$\check{Q}(\hat{\theta};\check{r}) \triangleq \underline{m}(\hat{\theta}) + \int_{r_-(\hat{\theta})}^{\check{r}} \frac{\overline{r}-r}{\overline{r}-\underline{r}} w(r;\hat{\theta})dr,$$

where

$$\underline{m}(\hat{\theta}) \begin{cases} = 0, & \text{if } \hat{\theta} \geq \mathbb{E}c + \underline{r}a(0) \\ \text{satisfies } \hat{\theta} = \mathbb{E}c + \underline{r}\frac{A(\underline{m}(\hat{\theta}))}{\underline{m}(\hat{\theta})} & \text{if } \hat{\theta} < \mathbb{E}c + \underline{r}a(0) \end{cases}$$

and $w(\cdot;\hat{\theta})$ satisfies that for any $r \in \left[r^-(\hat{\theta}), \check{r}\right]$, $\hat{\theta} = \mathbb{E}c + ra\left(\underline{m}(\hat{\theta}) + \int_{r_-(\hat{\theta})}^{r} w(\tau;\hat{\theta})d\tau\right)$. $\check{Q}(\hat{\theta};\check{r})$ represents the mass of $\underline{r}$-type banks in the robust disclosure under which the cutoff is $\hat{\theta}$ and the maximum score below $\overline{r}$ is $\check{r}$. Let $\check{\Omega}(r;\hat{\theta}) \triangleq \underline{m}(\hat{\theta}) + \max\left\{\int_{r_-(\hat{\theta})}^{r} w(\tau;\hat{\theta})d\tau, 0\right\}$. We now show that $Q \leq \check{Q}(\hat{\theta};\overline{r})$. This would imply that there exists $r_+(\hat{\theta},Q) \leq \overline{r}$ such that $Q = \check{Q}(\hat{\theta}; r_+(\hat{\theta},Q))$, and a desired $\tilde{W}$ is then given by $\tilde{W} = \check{\Omega}(r_+(\hat{\theta},Q);\hat{\theta})$.

A proof similar to that of Lemma 2 shows that $\check{\Omega}(r;\hat{\theta})$ is decreasing in $\hat{\theta}$ for $r \in [\underline{r},\overline{r}]$. Consider $\hat{\theta} < \hat{\theta}'$. It is easy to see that $\underline{m}(\hat{\theta}) \geq \underline{m}(\hat{\theta}')$. For $r \in (\max\{r_-(\hat{\theta}), r_-(\hat{\theta}')\}, \overline{r}]$, $ra\left(\check{\Omega}(r;\hat{\theta})\right) = \hat{\theta} - \mathbb{E}c < \hat{\theta}' - \mathbb{E}c = ra\left(\check{\Omega}(r;\hat{\theta}')\right)$. So $\check{\Omega}(r;\hat{\theta}) > \check{\Omega}(r;\hat{\theta}')$. If $r_-(\hat{\theta}) \leq r_-(\hat{\theta}')$, for $r \in [r_-(\hat{\theta}), r_-(\hat{\theta}'))$, $\check{\Omega}(r;\hat{\theta}') = \underline{m}(\hat{\theta}') \leq \underline{m}(\hat{\theta}) \leq \check{\Omega}(r;\hat{\theta})$. If $r_-(\hat{\theta}) > r_-(\hat{\theta}')$, for $r \in [r_-(\hat{\theta}'), r_-(\hat{\theta}))$, $\check{\Omega}(r;\hat{\theta}) = \underline{m}(\hat{\theta}) = \check{\Omega}(r_-(\hat{\theta});\hat{\theta}) > \check{\Omega}(r_-(\hat{\theta});\hat{\theta}') \geq \check{\Omega}(r;\hat{\theta}')$. So, for

$r \in [\underline{r}, \max\{r_-(\hat{\theta}), r_-(\hat{\theta}')\}]$, $\check{\Omega}(r;\hat{\theta}) > \check{\Omega}(r;\hat{\theta}')$.

Note that

$$\check{Q}(\hat{\theta};\check{r}) = \underline{m}(\hat{\theta}) + \int_{r_-}^{\check{r}} \frac{\overline{r}-r}{\overline{r}-\underline{r}} d\check{\Omega}(r;\hat{\theta}) = \underline{m}(\hat{\theta}) + \frac{\overline{r}-\check{r}}{\overline{r}-\underline{r}}\check{\Omega}(\check{r};\hat{\theta}) - \frac{\overline{r}-r_-}{\overline{r}-\underline{r}}\underline{m}(\hat{\theta}) + \int_{r_-}^{\check{r}} \frac{1}{\overline{r}-\underline{r}}\check{\Omega}(r;\hat{\theta})dr$$
$$= \frac{r_- - \underline{r}}{\overline{r}-\underline{r}}\underline{m}(\hat{\theta}) + \frac{\overline{r}-\check{r}}{\overline{r}-\underline{r}}\check{\Omega}(\check{r};\hat{\theta}) + \int_{r_-}^{\check{r}} \frac{1}{\overline{r}-\underline{r}}\check{\Omega}(r;\hat{\theta})dr = \frac{1}{\overline{r}-\underline{r}}\int_{\underline{r}}^{\overline{r}} \min\left\{\check{\Omega}(r;\hat{\theta}), \check{\Omega}(\check{r};\hat{\theta})\right\} dr.$$

Then $Q \leq q^r \leq \check{Q}(\hat{x}_r(1,q^r);\overline{r}) = \frac{1}{\overline{r}-\underline{r}}\int_{\underline{r}}^{\overline{r}} \check{\Omega}(r;\hat{x}_r(1,q^r))dr \leq \frac{1}{\overline{r}-\underline{r}}\int_{\underline{r}}^{\overline{r}} \check{\Omega}(r;\hat{\theta})dr = \check{Q}(\hat{\theta};\overline{r})$. So there exists $r_+(\hat{\theta},Q) \leq \overline{r}$ such that $Q = \check{Q}(\hat{\theta}; r_+(\hat{\theta},Q))$.

Define $W(\hat{\theta},Q) \triangleq \check{\Omega}(r_+(\hat{\theta},Q);\hat{\theta})$. By construction, $\lim_{r\to\overline{r}-} \Omega\left(r; W(\hat{\theta},Q), Q, 0\right) = W(\hat{\theta},Q)$, and $W(\hat{\theta},Q)$ is continuous and strictly increasing in Q. Its continuity in $\hat{\theta}$ is obvious. Next, we show that it is strictly increasing in $\hat{\theta}$. Suppose $\hat{\theta} < \hat{\theta}'$. Then

$$\frac{1}{\overline{r}-\underline{r}}\int_{\underline{r}}^{\overline{r}} \min\left\{\check{\Omega}(r;\hat{\theta}), W(\hat{\theta},Q)\right\} dr = \frac{1}{\overline{r}-\underline{r}}\int_{\underline{r}}^{\overline{r}} \min\left\{\check{\Omega}(r;\hat{\theta}), \check{\Omega}(r_+(\hat{\theta},Q);\hat{\theta})\right\} dr$$
$$= \check{Q}(\hat{\theta}; r_+(\hat{\theta},Q)) = Q = \check{Q}(\hat{\theta}'; r_+(\hat{\theta}',Q)) = \frac{1}{\overline{r}-\underline{r}}\int_{\underline{r}}^{\overline{r}} \min\left\{\check{\Omega}(r;\hat{\theta}'), W(\hat{\theta}',Q)\right\} dr.$$

Since $\check{\Omega}(r;\hat{\theta}) \geq \check{\Omega}(r;\hat{\theta}')$ and the inequality holds for a positive measure of r, we obtain $W(\hat{\theta},Q) < W(\hat{\theta}',Q)$.

Finally, we show that the $\tilde{W}$ in $[0,1]$ that satisfies $\hat{x}_r\left(\tilde{W},Q\right) = \hat{\theta}$ is unique, which is $W(\hat{\theta},Q)$. Note that there are two possible $\tilde{W}$ such that $\hat{x}_r\left(\tilde{W},Q\right) = \hat{\theta}$: $W(\hat{\theta},Q)$ or $W(\hat{\theta},Q)+\overline{m}$, where $\overline{m}$ solves $\hat{\theta} = \mathbb{E}c + \overline{r}\frac{A(W(\hat{\theta},Q)+\overline{m})-A(W(\hat{\theta},Q))}{\overline{m}}$. Since $W(\hat{\theta},Q) < W(\hat{x}_r(1,q^r), q^r) \leq 1$, we only need to show $W(\hat{\theta},Q)+\overline{m} > 1$. If $\hat{\theta} < \hat{x}_r(1,q^r) \leq \mathbb{E}c + \overline{r}a(1)$, then

$$a(1) \geq \frac{\hat{x}_r(1,q^r) - \mathbb{E}c}{\overline{r}} > \frac{\hat{\theta}-\mathbb{E}c}{\overline{r}} = \frac{A(W(\hat{\theta},Q)+\overline{m}) - A(W(\hat{\theta},Q))}{\overline{m}} > a(W(\hat{\theta},Q)+\overline{m}),$$

which implies $W(\hat{\theta},Q)+\overline{m} > 1$. If $\hat{x}_r(1,q^r) > \mathbb{E}c + \overline{r}a(1)$, then $W(\hat{x}_r(1,q^r), q^r) < 1$ and $\hat{x}_r(1,q^r) = \mathbb{E}c + \overline{r}\frac{A(1)-A(W(\hat{x}_r(1,q^r),q^r))}{1-W(\hat{x}_r(1,q^r),q^r)}$. If $W(\hat{\theta},Q)+\overline{m} \leq 1$, then

$$\frac{A(1)-A(W(\hat{x}_r(1,q^r),q^r))}{1-W(\hat{x}_r(1,q^r),q^r)} = \frac{\hat{x}_r(1,q^r)-\mathbb{E}c}{\overline{r}}$$

$$> \frac{\hat{\theta} - \mathbb{E}c}{\overline{r}} = \frac{A(W(\hat{\theta}, Q) + \overline{m}) - A(W(\hat{\theta}, Q))}{\overline{m}} \geq \frac{A(1) - A(W(\hat{\theta}, Q))}{1 - W(\hat{\theta}, Q))},$$

which implies $W(\hat{x}_r(1, q^r), q^r) \leq W(\hat{\theta}, Q)$. Contradiction! So, there exists a unique $\tilde{W} \leq 1$, which is $W = \tilde{W}(\hat{\theta}, Q)$, such that $\hat{x}_r(W, Q) = \hat{\theta}$.

Proof of Lemma 6

The derivative of $\hat{x}_c(W, q^c)$ w.r.t. W is

$$\frac{\partial \hat{x}_c(W, q^c)}{\partial W} = \frac{1}{W^2} \left\{ q^c(\overline{c} - \underline{c}) - \mathbb{E}r \cdot A(W) + \mathbb{E}r \cdot W \cdot a(W) \right\}.$$

Since $a(\cdot)$ is strictly decreasing, $\mathbb{E}r \cdot A(W) - \mathbb{E}r \cdot W \cdot a(W)$ is strictly increasing in W. So, $\forall W_1 < W_2$, $\left.\frac{\partial \hat{x}_c(W,q^c)}{\partial W}\right|_{W=W_2} \geq 0 \Rightarrow \left.\frac{\partial \hat{x}_c(W,q^c)}{\partial W}\right|_{W=W_1} > 0$.

If $\overline{c} + \mathbb{E}r \cdot a(1) \geq \hat{x}_c(1, q^c) = \mathbb{E}c + \mathbb{E}r \cdot A(1)$, then

$$q^c(\overline{c} - \underline{c}) - \mathbb{E}r \cdot A(1) + \mathbb{E}r \cdot 1 \cdot a(1) \geq 0 \Leftrightarrow \left.\frac{\partial \hat{x}_c(W, q^c)}{\partial W}\right|_{W=1} \geq 0 \Rightarrow \left.\frac{\partial \hat{x}_c(W, q^c)}{\partial W}\right|_{W<1} > 0,$$

so $\hat{x}_c(W, q^c)$ is continuous and strictly increasing in W over $[q^c, 1]$.

If $\overline{c} + \mathbb{E}r \cdot a(1) < \hat{x}_c(1, q^c) = \mathbb{E}c + \mathbb{E}r \cdot A(1)$, then $\left.\frac{\partial \hat{x}_c(W,q^c)}{\partial W}\right|_{W=1} < 0$, so in the neighborhood of $W = 1$, $\hat{x}_c(W, q^c) > \hat{x}_c(1, q^c)$. On the other hand,

$$\hat{x}_c(q^c, q^c) = \underline{c} + \mathbb{E}r \cdot \frac{A(q^c)}{q^c} < \overline{c} + \mathbb{E}r \cdot \frac{A(1) - A(q^c)}{1 - q^c} \Rightarrow \hat{x}_c(q^c, q^c) < \hat{x}_c(1, q^c).$$

Therefore, there must exist $\tilde{W} \in (q^c, 1)$ such that $\hat{x}_c\left(\tilde{W}, q^c\right) = \hat{x}_c(1, q^c)$. Then

$$\frac{q^c \cdot \underline{c} + \left(\tilde{W} - q^c\right) \cdot \overline{c}}{\tilde{W}} + \mathbb{E}r \cdot \frac{A\left(\tilde{W}\right)}{\tilde{W}} = \mathbb{E}c + \mathbb{E}r \cdot A(1)$$
$$\Rightarrow q^c(\overline{c} - \underline{c}) - \mathbb{E}r \cdot A(\tilde{W}) = -\mathbb{E}r \frac{\tilde{W}}{1 - \tilde{W}}[A(1) - A(\tilde{W})],$$

so

$$\left.\frac{\partial \hat{x}_c(W, q^c)}{\partial W}\right|_{W=\tilde{W}} = \frac{1}{\tilde{W}^2} \left\{ q^c(\overline{c} - \underline{c}) - \mathbb{E}r \cdot A(\tilde{W}) + \mathbb{E}r \cdot \tilde{W} \cdot a(\tilde{W}) \right\}$$

$$= \frac{1}{\tilde{W}^2}\left\{-\mathbb{E}r\frac{\tilde{W}}{1-\tilde{W}}[A(1)-A(\tilde{W})]+\mathbb{E}r\cdot\tilde{W}\cdot a(\tilde{W})\right\} = \frac{\mathbb{E}r}{\tilde{W}}\left\{-\frac{A(1)-A(\tilde{W})}{1-\tilde{W}}+a(\tilde{W})\right\} \geq 0.$$

Thus, $\hat{x}_c(W, q^c)$ is continuous and strictly increasing in W over $[q^c, \tilde{W}]$. Also, $\forall W \in [\tilde{W}, 1]$, $\hat{x}_c(W, q^c) \geq \hat{x}_c(1, q^c)$. Suppose it does not hold for $\tilde{W}' \in (\tilde{W}, 1)$. Then there must exist $\tilde{W}'' \in (\tilde{W}', 1)$ such that $\hat{x}_c\left(\tilde{W}'', q^c\right) = \hat{x}_c(1, q^c)$. Following the above analysis, we know that $\hat{x}_c(W, q^c)$ is continuous and strictly increasing in W over $[q^c, \tilde{W}'']$. This implies $\hat{x}_c\left(\tilde{W}', q^c\right) > \hat{x}_c\left(\tilde{W}, q^c\right)$. Contradiction! Therefore, for $\hat{\theta} < \hat{x}_c(1, q^c)$, $\hat{x}_c(W, q^c) = \hat{\theta}$ has a unique solution.

C.3 Proofs of Propositions in the Internet Appendix

Proof of Proposition 13

Consider $\hat{x}_{z,h}$ for $z \in P_{max}$. By Proposition 3, there exist $\left\{\Delta_{(i,j),(z,h)}\right\}_{i\leq n, j\leq t_i}$ such that

$$\begin{aligned}\hat{x}_{z,h} &= c_{z,h} + r_z \int_0^1 a_\kappa\left(\sum_{i=1}^{n}\sum_{j=1}^{t_i} w_{i,j}\Phi\left(\Phi^{-1}(x)+\Delta_{(i,j),(z,h)}\right)\right)dx \\ &\geq c_{z,h} + r_z\int_0^1 a_\kappa\left(\sum_{i\notin P_{max}} w_i + \sum_{i\in P_{max}}\sum_{j=1}^{t_i} w_{i,j}\Phi\left(\Phi^{-1}(x)+\Delta_{(i,j),(z,h)}\right)\right)dx.\end{aligned}$$

So,

$$A_\kappa(W) - A_\kappa\left(\sum_{z\notin P_{max}} w_z\right) \leq \sum_{z\in P_{max}}\sum_{h=1}^{t_z}\frac{\hat{x}_{z,h}-c_{z,h}}{r_z}w_{z,h}$$

$$\leq \sum_{z\in P_{max}}\sum_{h=1}^{t_z}\frac{\max_{i\in P_{max}}\{\hat{x}_{i,j}\}-c_{z,h}}{r_z}w_{z,h} = \sum_{z\in P_{max}}\frac{\max_{i\in P_{max}}\{\hat{x}_{i,j}\}-c_z}{r_z}w_z.$$

Thus, $\max_{i\in P_{max}}\{\hat{x}_{i,j}\} \geq \frac{1}{\sum_{z\in P_{max}}\frac{w_z}{r_z}}\left[\sum_{z\in P_{max}}\frac{c_z}{r_z}w_z + A_\kappa(W) - A_\kappa\left(\sum_{z\notin P_{max}} w_z\right)\right] = \hat{x}_n$.

Proof of Proposition 14

Let $\tilde{i}$ be the category such that $c_{\tilde{i}} = \min\{c_i : \Delta_{i,n} < +\infty\}$. Consider a disclosure that reveals information about the $\tilde{i}$th category but no information about others, such that the set of the scores whose cutoffs are equal to the maximum all come from the category in P_{max}.

$$\max\{\hat{x}_{i,j}\} = \frac{1}{\sum_{z\in P_{max}, z\neq \tilde{i}} \frac{w_z}{r_z} + \sum_{j=1}^{t_{\tilde{i}}} \frac{w_{\tilde{i},j}}{r_{\tilde{i},j}}} \left[\sum_{z\in P_{max}, z\neq \tilde{i}} \frac{c_z}{r_z} w_z + c_{\tilde{i}} \sum_{j=1}^{t_{\tilde{i}}} \frac{w_{\tilde{i},j}}{r_{\tilde{i},j}} + A_\kappa(W) - A_\kappa\left(\sum_{z\notin P_{max}} w_z \right) \right]$$

$$= \frac{1}{\sum_{z\in P_{max}} \frac{w_z}{r_z} + \left(\sum_{j=1}^{t_{\tilde{i}}} \frac{w_{\tilde{i},j}}{r_{\tilde{i},j}} - \frac{w_{\tilde{i}}}{r_{\tilde{i}}} \right)} \left[\sum_{z\in P_{max}} \frac{c_z}{r_z} w_z + c_{\tilde{i}} \left(\sum_{j=1}^{t_{\tilde{i}}} \frac{w_{\tilde{i},j}}{r_{\tilde{i},j}} - \frac{w_{\tilde{i}}}{r_{\tilde{i}}} \right) + A_\kappa(W) - A_\kappa\left(\sum_{z\notin P_{max}} w_z \right) \right]$$

Note that $\max\{\hat{x}_{i,j}\}$ is a weighted average of $\hat{x}_n$ and $c_{\tilde{i}}$. Since

$$\hat{x}_n = \frac{1}{\sum_{z\in P_{max}} \frac{w_z}{r_z}} \left[\sum_{z\in P_{max}} \frac{c_z}{r_z} w_z + A_\kappa(W) - A_\kappa\left(\sum_{z\notin P_{max}} w_z \right) \right] > c_{\tilde{i}}, \hat{x}^r_{max} < \hat{x}_n.$$

Proof of Claims in Section C.1.2

Consider the disclosure rule $\{(r_i, c_i, w_i)\}_{i=1,2}$. Then $\hat{x}_1 < \hat{x}_2$ iff

$$c_1 + r_1 \frac{A(w_1)}{w_1} < c_2 + r_2 \frac{A(1) - A(w_1)}{1 - w_1} \Leftrightarrow r_1 \left[\beta + \frac{A(w_1)}{w_1} \right] < r_2 \left[\beta + \frac{A(1) - A(w_1)}{1 - w_1} \right];$$

$\hat{x}_1 > \hat{x}_2$ iff

$$c_1 + r_1 \frac{A(1) - A(w_2)}{1 - w_2} > c_2 + r_2 \frac{A(w_2)}{w_2} \Leftrightarrow r_1 \left[\beta + \frac{A(1) - A(w_2)}{1 - w_2} \right] > r_2 \left[\beta + \frac{A(w_2)}{w_2} \right];$$

otherwise, $\hat{x}_1 = \hat{x}_2$.

The average cutoff under $\{(r_i, c_i, w_i)\}_{i=1,2}$ is

$$w_1\hat{x}_1 + w_2\hat{x}_2 = \frac{1}{\frac{w_1}{r_1} + \frac{w_2}{r_2}} \left[\frac{c_1 w_1}{r_1} + \frac{c_2 w_2}{r_2} + A(1) \right] = \mathbb{E}c - \beta \mathbb{E}r + \frac{1}{\frac{w_1}{r_1} + \frac{w_2}{r_2}} [\beta + A(1)].$$

Note that as r_2/r_1 increases, $\frac{1}{\frac{w_1}{r_1} + \frac{w_2}{r_2}}$ decreases.

If $\beta + A(1) = 0$, it is always that

$$\hat{x}_1 = \hat{x}_2 = c_1 + r_1 \cdot A(1) = c_2 + r_2 \cdot A(1) = \mathbb{E}c + \mathbb{E}r \cdot A(1).$$

If $\beta + A(1) > (<) 0$, the average cutoff decreases (increases) with r_2/r_1.