Perceived Competition in Networks*

Olivier Bochet[†] Mathieu Faure[‡] Yan Long[§] Yves Zenou[¶]

August 2, 2021

Abstract

We consider an aggregative game of competition in which agents have an imperfect knowledge about the set of agents they are in competition with. We model agent's perceived competitors by a network in which each agent is assumed to only have information on the activities of their direct neighbors. In this framework, a natural equilibrium concept emerges, the *peer-consistent equilibrium* (PCE). In a PCE, each agent chooses an action level that maximizes her subjective perceived utility and the action levels of all individuals in the network must be consistent. We decompose the network into communities and completely characterize peer-consistent equilibria by identifying which sets of agents can be active in equilibrium. An agent is active if she either belongs to a strong community or if few agents are aware of her existence. We show that there is a unique stable PCE. We provide a behavioral foundation of eigenvector centrality, since, in any stable PCE, agents' action levels are proportional to their eigenvector centrality in the network. We illustrate our results with two well-known models: Tullock contest function and Cournot competition.

Keywords: Aggregative game, incomplete network knowledge, peer-consistent equilibrium, eigenvector centrality, policies.

JEL Classification: C72, D85, Z13.

---

*We are very grateful to the editor Sylvain Chassang, two anonymous referees, Pierpaolo Battigalli, Sebastian Bervoets, Adam Brandenburger, Frédéric Deroïan, Itai Fainmesser, Andrea Galeotti, Sanjeev Goyal, Matt O. Jackson, Hervé Moulin, Fabrizio Panebianco, Paolo Pin, Michael Reiter, Evan Sadler, William Thomson, Mich Tvede, John Weymark, Myrna Wooders, Jingyi Xue, and Junjie Zhou for helpful discussions and suggestions. We would also like to thank the network group of New York University Abu Dhabi and seminar and conference participants at Singapore Management University, University of Glasgow, Maastricht University, ZEW Mannheim, the 18th SAET Conference on Current Trends in Economics, and the Cambridge-INET Networks Webinar Series. Olivier Bochet acknowledges financial support from Tamkeen under the NYU Abu Dhabi Research Institute Award for project CG005.

[†]Division of Social Science, New York University Abu Dhabi; Center for Behavioral Institutional Design (C-BID), New York University Abu Dhabi. Email: olivier.bochet@nyu.edu.

[‡]Aix-Marseille University, France. Email: mathieu.faure@univ-amu.fr.

[§]Huazhong University of Science and Technology, China. Email: ylong@hust.edu.cn.

[¶]Monash University, Australia, and CEPR. E-mail: yves.zenou@monash.edu.

1 Introduction

Competition in economics is usually viewed as "reciprocal." That is, if agent i considers agent j as a competitor, then agent j will also consider agent i as a competitor. However, in the real world, agents may *perceive* other agents as competitors when the reverse is not necessarily true. In other words, the network is *directed*. Moreover, many real-world networks are *weakly connected*, that is, there may not exist a path between every pair of agents. This implies that agents may not be aware of others located further away in the network. Therefore, even though competition is *global*, in many cases agents only care about their *local* competitors. For example, when estimating demand and thus making investment decisions, firms may only account for their *perceived* local competitors, even though, in reality, they are competing with a larger set of firms. However, because competition is *global*, their *local* perception is flawed from the outset. They make investments (actions) and then discover that their perceived demand (or resource) is not correct. They will then change their beliefs about their demand and adjust their investments. This pattern will continue until an equilibrium is reached.

In this paper, we study a standard aggregative game in which agents compete in terms of actions. However, they only perceive a subset of agents as competitors and are not aware of the others. For example, if we consider the standard Tullock contest game, each agent believes that their compete for some (perceived) ressource with the agents she is aware of while, in reality, they compete for a larger ressource with the whole set of agents. Similarly, in the standard Cournot oligopoly competition with homogenous product, each firm estimates that the product price is based on the belief that they are only in competition with their direct neighbors. In reality, the price is determined by the quantities produced by all firms in the network.

In this framework, a natural equilibrium concept emerges, the *peer-consistent equilibrium* (PCE), which captures both the agents' local sightedness—each agent chooses an action level that maximizes her *perceived* utility—and the fact that the action levels of all individuals in the network are consistent in equilibrium. Indeed, at a PCE, individual i's perceived subjective utility is equal to her objective payoff. Therefore, although individuals start with a *wrong* perception of their competitors, this wrong perception induces an interaction pattern that eventually leads to a *correct* perception of their competitors at equilibrium. In other words, their wrong perception is peer-consistent. For example, in the contest game, the local perception of agents in terms of resources has to be consistent with the "real" resources observed in equilibrium. Similarly, in the Cournot competition game, the firms' local perception of product price has to be consistent with the "real" price in equilibrium.

Our second contribution is to show that peer-consistent equilibria provide a behavioral foundation of the eigenvector centrality measure.[1] More precisely, we prove that the action level of each agent at a PCE is proportional to her *eigenvector centrality*; this de-

[1] Some papers have also provided an axiomatic foundation of eigenvector centrality; see e.g., Palacios-Huerta and Volij (2004); Dequiedt and Zenou (2017); Bloch et al. (2019).

termines the *intensive margin* of resource demand at peer-consistent equilibria (i.e., how much action an active agent exerts). This result is very general and holds beyond strongly connected networks, for which eigenvector centrality is usually defined. Other papers have provided a microfoundation of eigenvector centrality. For example, Golub and Jackson (2010, 2012) develop models on DeGroot updating in which eigenvector centrality is the right way to characterize an agent's influence. However, this arises from a heuristic learning process rather than behavior in a game. Banerjee et al. (2013) provide a microfoundation of eigenvector centrality by showing that it is the limit of diffusion centrality. In Sadler (2020), a theorem shows that there exists a network game of strategic complements with an equilibrium in which actions are ordered according to eigenvector centrality. Our model is different in the sense that it provides a behavioral foundation for eigenvector centrality measure based on an aggregative game and PCE. Moreover, in all these models, the network is assumed to be strongly connected and all agents exert strictly positive action t in equilibrium. Our model solves for a more general framework in which the network is weakly connected.[2] In our unique stable equilibrium, some agents may exert zero effort and the eigenvector centrality remains well defined. In essence, we provide a behavioral foundation of eigenvector centrality for any weakly connected directed network.

We then explore the role of the network's architecture in determining the *extensive margin* of resource demand at peer-consistent equilibria, that is, which agent is active and which agent is not. A group of at least two agents forms a "community" if it is a strongly connected component of the network. Our third contribution, therefore, is to break down the network into communities where, in each community, all agents are either active or inactive. We show that there are typically multiple peer-consistent equilibria. Some agents may be active at a given PCE, but inactive at another.

The multiplicity of peer-consistent equilibria is interesting as it underlines the behavioral richness of our equilibrium concept. Yet, one may want to make more precise predictions for any given network. To address this question, we study the stability of the peer-consistent equilibria with respect to a very natural dynamic. Indeed, at each period of time, each agent best replies to their "local" utility observed in the previous period and the efforts of their direct neighbors until an equilibrium is reached in which the perceived local utility is equal to the "real" or objective utility. Our fourth contribution is to provide a very simple and intuitive characterization of the stable PCE. We show that a community is active at a *stable* PCE if it is "aware" of the largest and densest community in the whole network. As we will see, this implies that there is always a *unique* stable PCE. In this PCE, some agents are active, but typically not all.

More generally, by considering weakly connected instead of strongly connected networks, we are able to explain why some agents may be inactive in equilibrium. This is due to the fact that some agents are not aware of the activity of other agents in the network. In other words, they believe that competition is "local" whereas it is, in fact, "global." We show that the stable peer-consistent equilibria are the long-run outcome of repeated interactions where agents constantly adjust their beliefs on the resource available. Hence,

[2] Clearly, a strongly connected network is a particular case.

some agents will end up being inactive because the better informed agents make increasing efforts. This is one of our key results that cannot be obtained in network games with perfect information about the network and/or a strongly-connected network.

Next, we study the policy implications of our model. We first examine the impact of adding a directed link between two agents. We show that the agent, who is at the source of the link, is the one who obtains the highest benefit from the link addition. Further, we show that adding a link may decrease the number of active agents in the network. We then study the key-player policy (Ballester et al., 2010) and highlight another counter-intuitive result. By removing an agent in the network, we may make several inactive agents (for example, in terms of criminal effort) active. Finally, we show that, by merging two different connected networks (i.e., social mixing), the total activity is higher than the sum of total activity in each disconnected network.

In the final part of the paper, we provide a clean application of the assumptions and predictions of our model. We consider education and how "perceived" competitors at school lead to different levels of education efforts that are proportional to the eigenvector centrality of each student in the network. Indeed, in many schools, students are assessed on a regular basis and their ranking in the classroom matters for their final evaluation as it translates into a national or state test score. In particular, each time they obtain a grade, they try to estimate their ranking in the classroom but only know the test scores of their direct friends and/or their direct (perceived) competitors. Then, these students adjust their education effort to their perceived ranking. They do so until they obtain their final national or state ranking. We provide evidence that, in the education context, friendship networks are largely directed and local and that the eigenevector centrality of each student is a good predictor of their test scores.

Contribution to the literature

Our paper contributes to the games-on-network literature.[3] In many situations in which networks matter, agents make both binary decisions (*extensive margin*) and quantity decisions (*intensive margin*). Consider, for example, crime. First, an individual has to decide whether to become a criminal (active or not active); this is a binary decision (extensive margin). Then, if she becomes a criminal, she must decide how many crimes to commit (intensive margin). The literature on network games has mostly focused on the intensive margin by assuming that actions are continuous (Jackson and Zenou, 2015). There are, however, some papers that have considered network games with discrete actions (extensive margin); see, for example, Morris (2000), Brock and Durlauf (2001), and Leister et al. (2021). We believe that this is the one of the first papers[4] to consider both extensive and intensive margins. We show that the extensive margin (i.e., who is active in the network) is *community based*, that is, agents belonging to the same community are either all active

[3]For overviews, see Jackson (2008), Jackson and Zenou (2015), Bramoullé et al. (2016), and Jackson et al. (2017).

[4]Other papers (e.g., Calvó-Armengol and Zenou (2004); Bramoullé and Kranton (2007)) have considered both but without being able to provide a general characterization of the equilibria. Moreover, these models are usually plagued by multiple equilibria.

or all inactive, and depends on the density of the community, whereas the intensive margin is *individual based* and solely determined by the position in the network: the higher the individual eigenvector centrality, the higher the effort and, thus, the higher the individual share of the resource in the network.

As with our model, there are also network games that focus on imperfect information about the network and introduce new equilibrium concepts related to our PCE. In particular, McBride (2006), Lipnowski and Sadler (2019) and Battigalli et al. (2020) consider *self-confirming* and *peer-confirming equilibria*. Lipnowski and Sadler (2019) apply self-confirming equilibria (SCE) and rationalizable SCE to games where feedback about the actions of others is described by a network topology: agents observe only the actions of their peers (i.e., neighbors), but their payoffs may depend on everybody's actions and are not observed ex-post. The main difference to our PCE is that Lipnowski and Sadler (2019) allow agents to make conjecture about agents who are not their neighbors;[5] in our model, we assume that agents do not even know these agents exist. The peer-confirming equilbrium concept of Lipnowski and Sadler (2019) is such that adding links in the network restricts the set of permissible profiles/conjectures and thus the set of equilibria.[6] This is not true in our model.[7] Moreover, our concept of peer-consistent equilibrium (PCE) is equivalent to a specific kind of the self-confirming equilibrium (SCE) developed by Battigalli et al. (2020) when the game is written in such a way that the feedback agents receive is made explicit. In fact, our equilibrium concept (PCE) is a refinement of SCE whereby agents wrongly believe that they compete for a local rather than a global resource.

Our equilibrium characterization in terms of communities also relates to other network models that also partition agents into endogenous community structures, including risk sharing (Ambrus et al., 2014), interaction between market and community (Gagnon and Goyal, 2017), behavioral communities (Jackson and Storms, 2019), information resale and intermediation (Manea, 2021), and technology adoption (Leister et al., 2021). However, the driving forces and policy implications are very different. In particular, all these papers assume a perfect knowledge of the network and use standard equilibrium concepts.

Our paper also contributes to the literature on aggregative games (Jensen, 2018). In this literature, usually the network is not explicitly modeled and agents are assumed to know with certainty their competitors. Tullock contest game is one of our applications; thus we also contribute to the literature on conflicts,[8] especially the more recent literature on conflicts in networks.[9] In this literature, the structure of local conflicts is modeled as

[5]Indeed, a strong assumption that is implicit in the definition of peer-confirming equilibrium in Lipnowski and Sadler (2019) is that players know the network structure.

[6]When the network is complete, the set of peer-confirming equilibria coincides with the set of Nash equilibria. For the empty network, peer-confirming equilibria coincide with rationalizable equilibria. Increasing the number of links reduces the number of equilibria. In contrast, the set of PCE may very well increase when links are added.

[7]McBride (2006) applies self-confirming equilibrium (SCE) to games of network formation with asymmetric information in which agents only observe the private information of other linked agents. We instead assume that the network is exogenous and that actions are continuous.

[8]See Corchón (2007), Konrad (2009), and Jensen (2016) for overviews.

[9]See e.g., Goyal and Vigier (2014), Franke and Öztürk (2015), Hiller (2017), König et al. (2017),

a network in which rivals invest in conflict-specific technology to attack their respective neighbors. This literature assumes that the network is undirected (which is a particular case of our network) and that agents know the network, and solves the model using standard Nash equilibrium concept. Further, these studies usually do not provide a general characterization of all possible equilibria.

Finally, our model contributes to the general literature on competition in Industrial Organization (IO). We believe that this is the first model that introduces the concept of perceived competition in this literature and models it through a network.[10]

The rest of the paper unfolds as follows. In the next section, we describe our model and introduce our new concept of equilibrium. Section 3 provides a general characterization of all peer-consistent equilibria (PCEs) for any network by first introducing the concept of communities, as well as an ordering between them, then by providing the exact condition under which each equilibrium exists, and, finally, by determining the unique stable PCE. In Section 4, we study the policy implications of our model, while, in Section 5, we investigate the economic implications of our results. Section 6 offers concluding remarks.

In Appendix A, we provide some useful results on nonnegative matrices and propose a definition of eigenvector centrality in weakly-connected (directed) networks. All the proofs of the results in the main text can be found in Appendix B. We provide additional results in Appendix D. Finally, Appendix E deals with the case when the network is not generic.

2 The model

2.1 Aggregative games

Consider a finite set of agents, denoted by $N = \{1, 2, \cdots, n\}$. Each agent $i \in N$ exerts some action[11] $x_i \in \mathbb{R}_+$. We consider an aggregative game (Jensen, 2018) in which every player i's payoff is a function of the player's own action x_i and the *aggregate* of all players' actions $\sum_{j \in N} x_j$. For each agent i, the total cost of exerting action x_i is equal to cx_i, where $c > 0$ is the (constant) marginal cost of action. Let $\pi_i : \mathbb{R}_+^n \to \mathbb{R}$ be agent i's payoff function. It is given by:

$$\pi_i(\mathbf{x}) = v_i(\mathbf{x}) - cx_i = x_i \, f\,(X) - cx_i, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ denotes the vector of actions chosen by agents and $X = \sum_{i \in N} x_i$ is the sum of all agents' actions in the network. Function f is a non-increasing, differentiable and continuous map, with $f :]0, +\infty[\to \mathbb{R}^+$ such that $\lim_{x \to 0^+} f(x) > c$, $\lim_{x \to +\infty} f(x) < c$.

Kovenock and Roberson (2018), Xu et al. (2019). For overviews, see Kovenock and Roberson (2012) and Dziubiński et al. (2016).

[10]Observe that our model of perceived competition and our equilibrium characterization results in terms of eigenvector centrality are not valid for all contest and IO and, thus, aggregative games. As this will become clear below, they are only true for a class of aggregative games such as, for example, the Tullock contest function and the Cournot oligopoly competition.

[11]In this paper, we use the terms "action" and "effort" interchangeably.

In addition, we assume that the application $x \in]0, +\infty[\mapsto xf\left(\frac{x+z}{W}\right)$ is quasi-concave, for any $W > 0$ and $z \geq 0$. We also assume that $\lim_{x\to 0^+} xf(x) = l \in [0, +\infty[$.[12]

We assume that the function f is *non-increasing*. This implies that $\frac{\partial \pi_i(\mathbf{x})}{\partial x_j} \leq 0$, meaning that we only focus on competition, that is, the higher is the effort of the other agents, the lower is my utility. For example, in the case of Cournot competition, the higher is the production of goods from other firms, the lower is the price of the product, and thus the lower is the profit of firms. Similarly, for the Tullock conflict model, the higher is the effort of other agents, the lower is the share of resources each agent obtains and, thus, the lower is her utility. Observe, however, that this does not imply that it is a game with strategic complements or strategic substitutes, or, equivalently, that $\pi_i(\mathbf{x})$ has increasing or decreasing differences. Indeed, for $j \neq i$, we have:

$$\frac{\partial^2 \pi_i(\mathbf{x})}{\partial x_i \partial x_j} = f'(X) + x_i f''(X). \tag{2}$$

For Cournot competition (see below), $f''(X) = 0$, and we have a game with strategic substitutes. For the Tullock contest function, we have $f''(X) \geq 0$ and it is a game with neither strategic substitutes nor strategic complements.

2.2 Networks: Locally-sighted individuals

We embed the aggregative game played by agents into a network, by assuming that agents are *locally-sighted*. Agents are only aware of those to whom they are directly linked, but otherwise do not know anything else about the network. In other words, local-sightedness implies that each individual only perceives resources and interactions of her *local environment* (i.e., of her direct links).

Formally, given the set of agents N, a (*directed*) *network* is a pair $(N, \mathbf{G})$ where $\mathbf{G}$ is an $n \times n$ adjacency matrix, with entry $g_{ij} \in \{0, 1\}$. For each pair $i, j \in N$, agent i is linked to j if and only if $g_{ij} = 1$. Since the perception of a link is not necessarily reciprocal, it is possible that $g_{ji} = 0$, meaning the network is *directed*. We therefore allow for the situation in which an individual is considered as a neighbor (contender) of another, but not vice versa.[13] To be precise, for each $i \in N$, let $\mathcal{N}_i = \{j \in N : g_{ij} = 1\}$ be the (directed) *neighborhood* of agent i. This will become clearer when we introduce the notion of "perceived competition."

There is a *(directed) path* from individual i to individual j in the network if there is a sequence $\{j_1, j_2, \cdots, j_m\} \subseteq N$ with $j_1 = i$, $j_m = j$ and such that $g_{j_\ell j_{\ell+1}} = 1$ for each $\ell \in \{1, ..., m-1\}$. In this case, agent i is said to be connected to j through a path. In

[12] Note that we do not restrict the map f to be properly defined in 0, and we also handle the case where f goes to infinity at zero. As we will see, an important example where this occurs is the linear Tullock contest. However we need to make sure that $\mathbf{x} = \mathbf{0}$ is not an equilibrium. We thus assume that the limit when x goes to 0 from the right exists.

[13] A network is *undirected* if, for each pair i, $j \in N$, $g_{ij} = g_{ji}$. Note that undirected networks are special cases of directed networks.

order to indicate that such a path exists between i and j, we use the notation $i \rightrightarrows j$. A (directed) *cycle* is a (directed) path from a certain agent $i \in N$ to herself.

Definition 1. *Let $(N, \mathbf{G})$ be a directed network.*

- *$(N, \mathbf{G})$ is **weakly connected** if the underlying undirected graph (i.e., ignoring the directions of edges) is connected. Accordingly, a directed network is **disconnected** if it is not weakly connected.*

- *$(N, \mathbf{G})$ is **semi-connected** if, for any pair of agent $i, j \in N$, there is a path from i to j (i.e., $i \rightrightarrows j$) or a path from j to i (i.e., $j \rightrightarrows i$).*

- *$(N, \mathbf{G})$ is **strongly connected** if each node can reach every other node by a path, that is, for any pair of agents $i, j \in N$, there is a path from i to j (i.e., $i \rightrightarrows j$).*

- *$(N, \mathbf{G})$ satisfies **no-isolation** if, for each $i \in N$, $\mathcal{N}_i \neq \emptyset$.*

Throughout the paper, we consider weakly connected networks satisfying the no-isolation property. Note that strongly-connected networks are semi-connected and also satisfy no-isolation. In turn, semi-connected networks are weakly connected. Note also that a weakly-connected network that satisfies no-isolation necessarily contains at least one directed cycle.

Let W_i be a shifter that captures the *misperceived intensity of the competition faced* by agent i. We assume that W_i is only based on agent i's *local environment* (i.e., agent i as well as her neighbors, $\{i\} \cup \mathcal{N}_i$). In other words, individual and aggregate efforts are not observed in the network, *except* at the neighborhood level, i.e., each agent i only observes her own effort as well as the efforts of her direct neighbors.

Given an effort profile $\mathbf{x} \in \mathbb{R}^n_+$, for each $i \in N$, let $\mathbf{x}_{-i}$ be the effort sub-profile of agents $j \in \mathcal{N}_i$. We can write the *perceived utility* of agent i as

$$u_i(x_i, \mathbf{x}_{-i}; W_i) = x_i \, f\left(\frac{x_i + \sum_{j \in N} g_{ij} x_j}{W_i}\right) - c x_i. \tag{3}$$

Note the importance of W_i in the above definition. Indeed, in the situation where agents observe the *entire vector of actions in the network*, the perceived utility of agents would coincide with their objective payoffs and $W_i = \frac{x_i + \sum_{j \in N} g_{ij} x_j}{X}$. When agents only observe the actions of their neighbors, W_i could take any values coherent with what is happening in agent i's neighborhood.

When we introduce below the definition of our solution concept, *Peer-Consistent Equilibrium (PCE)*, it will become clear why we define the perceived utility as in (3).

Remark 1. *Apart from their position in the network, all agents are identical.*

The aim of our paper is to study how the individual's network position affects the effort and outcome of each agent. This is why we assume that all agents are ex-ante identical, i.e., $c_i = c$ for all $i \in N$. In other words, the only source of heterogeneity stems from the agents' network position and, thus, the set of agents they perceive as competitors. Differences in *perceived competition* are therefore the main source of heterogeneity.

2.3 Two important applications

Our aggregative game with utility function (1) is relatively general and can be applied to many aggregative games such as Tullock contest games, Cournot competition, tournaments, teamwork games, etc. (Alos-Ferrer and Ania, 2005; Jensen, 2010). Here we provide two main applications: Tullock contest games and Cournot competition.

2.3.1 Linear (Tullock) contest games

There is a given resource, available in a fixed amount $V \in \mathbb{R}_+ = [0, \infty)$, to be shared between the n agents. The agents play a *contest game*, described as follows. Each agent $i \in N$ exerts some action (effort) $x_i \in \mathbb{R}_+$ before the resource V is distributed. An action profile $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ determines, for each agent i, her share of the resource V using the following *proportional rule*:

$$v_i(\mathbf{x}) = \begin{cases} \frac{x_i}{X} V & \text{if } X > 0, \\ \frac{1}{n} V & \text{if } X = 0. \end{cases} \tag{4}$$

Equation (4) corresponds to the well-known "Tullock contest function" from the contest literature (Skaperdas, 1996; Kovenock and Roberson, 2012).[14] One important difference is that we do not interpret $\frac{x_i}{X}$ as the *probability* of agent i getting V, but as the *percentage* of resource V that agent i can obtain, given her and the other agents' effort choices. We assume that the resource V is given exogenously and that the sharing rule (4) is symmetric and takes a proportional form. This means that the utility function of each agent i is still given by (1) but with

$$f(y) = \frac{V}{y}. \tag{5}$$

Note that the map f has the properties of an aggregative game (see Section 2.1).[15] Given the exogenous resource V, W_i stands for agent i as resources *perceived* to be "owned" by the local environment of agent i, that is, by agent i as well as her neighbors, $\{i\} \cup \mathcal{N}_i$. In other words, the resources V available in the economy as well as the efforts of the n agents in the network are *not* observed by any agent in the network; each agent i only observes W_i, her perceived resources, and $\sum_{j \in N} g_{ij} x_j$, the efforts of her direct links. Thus, each individual $i \in N$ competes for W_i, with agents in $\mathcal{N}_i$ (i.e., her neighbors). Given W_i, the *perceived utility* of agent i is then equal to

$$u_i(x_i, \mathbf{x}_{-i}; W_i) = \begin{cases} \frac{x_i V}{x_i + \sum_{j \in N} g_{ij} x_j} W_i - c x_i & \text{if } x_i + \sum_{j \in N} g_{ij} x_j > 0, \\ \frac{V}{1 + |\mathcal{N}_i|} W_i & \text{if } x_i + \sum_{j \in N} g_{ij} x_j = 0. \end{cases} \tag{6}$$

[14] The theoretical foundations of the Tullock contest function are well established. In particular, the Tullock contest function can be derived using a stochastic, axiomatic, optimally-derived, and microfounded approach (Skaperdas, 1996; Jia, 2008; Jia et al., 2013).

[15] We have that $\lim_{x \to 0^+} f(x) = +\infty$, $\lim_{x \to +\infty} f(x) = 0$ and $x f\left(\frac{x+z}{W}\right) = \frac{Wx}{x+z} V$ is strictly concave in x. Moreover $\lim_{x \to 0^+} x f(x) = V$.

2.3.2 Cournot competition

Consider a standard homogeneous good Cournot oligopoly game on a network with n firms competing in quantities. The profit function for each firm i is given by

$$\pi_i(\mathbf{x}) = (p - c)\, x_i, \tag{7}$$

where

$$p = (\bar{\alpha} - X)_+ \tag{8}$$

We assume that $\overline{\alpha} > c$. In (7), x_i denotes the quantity produced by firm i while p is the price of the product. This means that the profit function of each firm i is still given by (1) but with

$$f\,(y) = (\bar{\alpha} - y)_+\,. \tag{9}$$

Again, the map f satisfies the assumptions required in Section 2.1.[16] Firms only observe the quantities produced by their neighbors in the directed network $(N, \mathbf{G})$. In other terms, they observe $\sum_j g_{ij}x_j$. Given W_i, the *perceived utility* of firm i is then equal to:

$$u_i(x_i, \mathbf{x}_{-i}; W_i) = \left[\bar{\alpha} - \frac{x_i + \sum_j g_{ij}x_j}{W_i}\right] x_i - cx_i \tag{10}$$

Indeed, firm i believes that the price is an affine transformation in the total demand, which is correct. However, she does not observe the actual demand $X := \sum_j x_j$, but only the demand in her neighborhood $x_i + \sum_j g_{ij}x_j$. When observing that the realized price she faces is not her perceived price (the one she perceived from her local demand), she does not know that it is because there are other firms producing the same product in the network. Instead, she believes it is because the perceived slope affecting the price is incorrect.

Observe that, as in the standard Cournot model, there is precisely one price p for the homogeneous good defined in (8), which is not subject to interpretation. This means that firms do not "misperceive" the actual price, because they observe it. However, they "misperceive" how this price emerges because they have a wrong perception of the slope affecting this price.

Remark 2. *Our result can be extended to a Cournot model with a general non-linear demand function, that is, $p = (\bar{\alpha} - h\,(X))_+$. See Appendix C.*

2.4 Peer-consistent equilibrium

The critical assumption of our model is that, in order to choose an effort level, each individual i considers the competition in her neighborhood $\{i\} \cup \mathcal{N}_i$ while, in reality, she is in competition for all agents in the network. The following equilibrium concept captures this idea.

[16]That is, $f(0) = \overline{\alpha} > c$, $\lim_{y\to+\infty} f(y) = 0 < c$ and $x \mapsto x\left(\overline{\alpha} - \frac{x+z}{W}\right)_+$ is quasiconcave. Finally $\lim_{x\to 0^+} xf(x) = 0$.

Definition 2. *A **Peer-Consistent Equilibrium** (PCE) is a vector* $\mathbf{x}^* \in \mathbb{R}^n_+$ *such that,*

(i) *for each* $i \in N$ *and each* $x_i \in \mathbb{R}_+$,

$$u_i(x_i^*, \mathbf{x}_{-i}^*; W_i) \geq u_i(x_i, \mathbf{x}_{-i}^*; W_i),$$

(ii) *for each* $i \in N$,

$$W_i = \frac{x_i^* + \sum_{j \in N} g_{ij} x_j^*}{X^*}.$$

Condition (i) states that, given her perceived competition (total resource share in her neighborhood) and a vector of actions $\mathbf{x}$, each individual i chooses an effort that maximizes her *perceived utility.* Each agent i takes W_i as given, and chooses the action x_i that maximizes $u_i(x_i, \mathbf{x}_{-i}, W_i)$. Note, however, the subtle part of condition (i): taking W_i as given, each agent i is only best responding to the choice of actions of agents in her neighborhood.

Condition (ii) states that the effort levels of *all* individuals in the network will, in turn, determine the benefits W_i obtained by each agent i. In fact, W_i is defined such that $u_i\left(x_i^*, \mathbf{x}_{-i}^*; W_i\right) = \pi_i(\mathbf{x}^*)$. This is a *consistency* requirement imposed in equilibrium. Indeed, at a peer-consistent equilibrium, $u_i\left(x_i^*, \mathbf{x}_{-i}^*; W_i\right)$ individual i's perceived *subjective* utility has to be equal to $\pi_i(\mathbf{x}^*)$, her *objective* payoff function in the underlying aggregative game. This has to be true for all i and, thus, all neighborhoods. Therefore, although individuals initially start with a wrong perception of whom they are in competition with, this wrong perception induces an interaction pattern that eventually leads to a correct perception at equilibrium. This is why we call it a "peer-consistent equilibrium."

Remark 3. *Peer-consistent equilibria and Nash equilibria coincide if and only if the network is complete, in which case the unique PCE is the Nash equilibrium of the aggregative game.*

Indeed, a peer-consistent equilibrium of our aggregative game on a *complete network* is simply a Nash equilibrium on the same game, since all agents observe the whole network and the notion of local-sightedness disappears. As soon as at least one link is missing, the coincidence disappears: at least one agent is not aware of the existence of some other agents. Observe that the PCE is neither a refinement of the concept of Nash equilibrium, nor a superset (such as correlated equilibria or the concept of peer-confirming equilibria defined in Lipnowski and Sadler (2019)). Rather, it is the outcome of a decentralized optimization problem where each agent must choose the action that maximizes their perceived utility, and where each perceived utility must be ex-post consistent with the realized outcome.

Remark 4. *We only consider unweighted networks, that is,* $g_{ij} \in \{0, 1\}$. *In our paper, the network captures the perception of the competition that each agent faces. Thus, it can only take two values, 0 and 1, since* $g_{ij} = 1$ *means that agent* i *perceives* j *a competitor while* $g_{ij} = 0$ *means that she does not perceive* j *as a competitor.*[17]

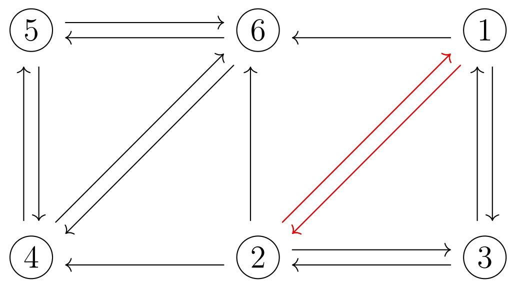
[17]This is an important remark because weakly connected networks are non-generic in the space of

2.5 Peer-consistent equilibrium: An illustration

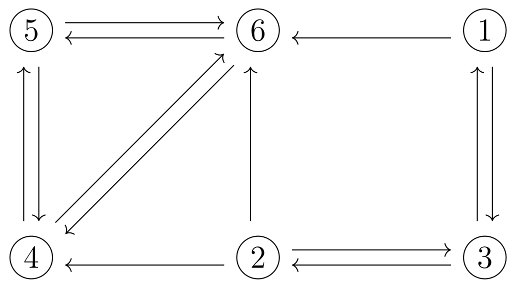
To understand our peer-consistent equilibrium concept (Definition 2) and the role played by the consistency requirement, let us consider the following two-part example in which we compare the set of peer-consistent equilibria in two closely related networks in the linear (Tullock) contest games (Section 2.3.1) where the utility of each agent i given by (6). This will be part of our leading examples in this paper.

Example 1. Consider the two networks displayed in Figure 1. In order to illustrate our definition of a peer-consistent equilibrium (PCE), let us focus on agents 1 and 6.

Figure 1: Two similar networks with different densities



(a) A dense network



(b) A less dense network

Same local competitors, same perceived utility: First, as in Definition 2(i), each agent i maximizes her utility (6) by taking W_i as given. Clearly, in both networks displayed in Figures 1(a) and 1(b), agent 6 has the same perceived utility function, namely

weighted directed graphs. Thus, the case for studying them so carefully rests on links being binary rather than weighted. In our case, it makes sense since the network only captures the "perception" that each agent has about agents. It is not a "physical" network such as roads, freeways, bridges or a financial network where links are bank loans.

$\frac{x_6}{x_4+x_5+x_6}W_6 - cx_6$. Indeed, agent 6's perception of her own environment does not change across the two networks. On the other hand, agent 1's perceived utility function switches from $\frac{x_1}{x_1+x_2+x_3+x_6}W_1 - cx_1$ to $\frac{x_1}{x_1+x_3+x_6}W_1 - cx_1$, which implies that agent 1 will *decrease* her effort when the link from 1 to 2 is removed. At the same time, one may think that agent 6's equilibrium level of effort would be unchanged, since the perception of her environment is unchanged across the two networks. This intuition is wrong, as we show next.

Same perceived utility, different equilibrium efforts: Now, by imposing the consistency requirement of Definition 2(*ii*), we can show that agent 6 will produce *more effort* in the network of Figure 1(b) compared to that of Figure 1(a), even though she faces exactly the same competitors (and thus the same perceived utility), namely 4 and 5. This is due to the fact that agent 1's set of perceived competitors shrinks, triggering a decrease in agent 1's effort. This decrease, in turn, implies that, in the network of Figure 1(b), there are more resources left to grab for agent 6 as a by-product of the *consistency requirement* of the PCE. $\diamond$

The aim of this example was to illustrate the concept of PCE. However, to rigorously explain why, in the networks in Figures 1(a) and 1(b), the efforts of agents are different, we need to understand the underlying dynamic that leads to the PCE in each network. Since the dynamic analysis is performed in Section 3.5, we postpone this discussion to Section 5.2.2 below.

3 General analysis: Peer-consistent equilibria

This section aims to present a complete analysis of peer-consistent equilibria in weakly-connected networks. We first provide a general algebraic characterization in Section 3.1, for which we show that PCE provides a microfoundation of eigenvector centrality. The characterization provided so far being implicit, Sections 3.3 and 3.4 are devoted to providing an alternative characterization; we introduce the concept of community and show that carefully ordering the agent by community allows the easy identification of all PCEs in any network. Finally, we show in Section 3.5 that the set of PCEs can be refined to a unique *stable* PCE. Importantly, this particular PCE provides a microfoundation of eigenvector centrality in the case where the network is no longer strongly connected.

3.1 General characterization of peer-consistent equilibria

In this section, we show that an effort profile is a peer-consistent equilibrium if and only if it is a (properly normalized) nonnegative *eigenvector* of $\mathbf{G}$.

Theorem 1. *Let* $(N, \mathbf{G})$ *be a weakly-connected network and consider the aggregative game where the utility of each agent* i *is given by* (1)*. Then,* $\mathbf{x}^*$ *is a peer-consistent equilibrium if and only if*

$$\mathbf{G}\mathbf{x}^* = \left(\frac{c - f(X^*) - f'(X^*)X^*}{f(X^*) - c}\right)\mathbf{x}^*, \quad \text{and} \quad \mathbf{x}^* \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}. \tag{11}$$

Moreover there is at least one PCE.

Theorem 1 provides a *microfoundation of eigenvector centrality* in such games on networks. It shows that, for any weakly-connected network, at any PCE, the effort of each agent is proportional to her eigenvector centrality.[18] This result is driven by the consistency requirement (*ii*) of Definition 2 and also to the utility function (1) of the aggregative game. This is a new result, complementing that of Ballester et al. (2006), who show that, for any network, in a game with strategic complementarities, for each agent who chooses effort that maximizes a linear-quadratic utility function, her equilibrium effort is equal to her Katz-Bonacich centrality. Here, we show that, if each agent chooses effort that maximizes utility (1) based on the aggregative game, then, at any PCE, her effort will be proportional to her eigenvector centrality.

Observe that there may be more than one PCE. Importantly, it is worth mentioning that there typically are peer-consistent equilibria in which some components are equal to zero, and hence some agents are *inactive* at equilibrium.[19] In the particular case of strongly connected networks, Equation (11) has a unique non-negative solution:

Remark 5. *In any strongly-connected networks, there is a unique peer-consistent equilibrium* $\mathbf{x}^*$*. In addition, for each* $i \in N$*,* $x_i^* > 0$*.*

Let us go back to the general weakly-connected framework. Given a peer-consistent equilibrium $\mathbf{x}^*$, let $N_+(\mathbf{x}^*)$ be the set of agents who are *active* at equilibrium $\mathbf{x}^*$. That is, $N_+(\mathbf{x}^*) = \{i \in N : x_i^* > 0\}$. Observe that, if $\mathbf{x}^*$ is a PCE, then its set of active agents is *closed*:[20]

$$i \rightrightarrows j \text{ and } j \in N_+(\mathbf{x}^*) \implies i \in N_+(\mathbf{x}^*). \tag{12}$$

In other words, if there is a path from i to j and j is active at the PCE $\mathbf{x}^*$, then i is active also.

For any network $(N, \mathbf{G})$ and any subset of agents $M \subseteq N$, let $\mathbf{G}_M$ denote the restriction of matrix $\mathbf{G}$ to M. If $\mathcal{N}_i = \emptyset$, then agent i is irrelevant to the equilibrium analysis. Indeed, agent i's effort is zero in any PCE and $\mathbf{x}^*$ is a PCE for network $(N, \mathbf{G})$ if and only if $\mathbf{x}^*_{-i}$ is a PCE for the network $(N \setminus \{i\}, \mathbf{G}_{N\setminus\{i\}})$. Consequently, for the remainder of this paper, we will always assume that the network satisfies the *no-isolation* property (Definition 1).

[18]Eigenvector centrality is usually defined for strongly-connected networks. Indeed, in this case, it is a well-defined measure of centrality captured by the Perron-Frobenius vector associated with the adjacency matrix (Jackson, 2008). In Section A.2 of Appendix A, we provide a more general definition of eigenvector centrality for networks that are not necessarily strongly connected.

[19]It will be clear from the analysis that follows that weakly-connected networks generally admit several PCEs. It is possible that, in each of these equilibria, some agents are inactive –of course, these are not necessarily the same agents across the different PCEs.

[20]We prove this formally in Lemma B6 in Appendix B.

3.2 Microfoundation of eigenvector centrality: Some appplications

We would like to illustrate our characterization results for our two main applications of aggregative games: Linear (Tullock) contest games and Cournot competition.

3.2.1 Microfoundation of eigenvector centrality: Linear (Tullock) contest games

Let us illustrate how (11) emerges in the Linear (Tullock) contest games described in Section 2.3.1. First, given W_i, each agent i chooses her effort x_i^* that maximizes her perceived utility (6). This leads to:

$$\frac{W_i \sum_j g_{ij} x_j^*}{\left(x_i^* + \sum_j g_{ij} x_j^*\right)^2} V = c. \tag{13}$$

Combining (13) with consistency condition $W_i = \frac{x_i^* + \sum_j g_{ij} x_j^*}{X^*}$, we obtain:

$$\frac{\sum_j g_{ij} x_j^*}{x_i^* + \sum_j g_{ij} x_j^*} \frac{V}{X^*} = c.$$

By solving this equation, we get:

$$\sum_j g_{ij} x_j^* = \left(\frac{cX^*}{V - cX^*}\right) x_i^*,$$

In matrix form, for $\mathbf{x}^* \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, we obtain:

$$\mathbf{G}\mathbf{x}^* = \frac{cX^*}{V - cX^*} \mathbf{x}^*. \tag{14}$$

3.2.2 Microfoundation of eigenvector centrality: Cournot competition

Let us perform the same exercise for the standard homogeneous good Cournot oligopoly game of Section 2.3.2. First, given W_i, each firm i chooses a quantity x_i^* that maximizes her perceived utility (10). This leads to:

$$\bar{\alpha} - \frac{x_i^* + \sum_j g_{ij} x_j^*}{W_i} - \frac{x_i^*}{W_i} = c. \tag{15}$$

Combining (13) with consistency condition $W_i = \frac{x_i^* + \sum_j g_{ij} x_j^*}{X^*}$, we obtain:

$$\bar{\alpha} - X^* - \frac{x_i^* X^*}{x_i^* + \sum_j g_{ij} x_j^*} = c.$$

By solving this equation, we get:

$$\sum_j g_{ij} x_j^* = \left(\frac{2X^* - \bar{\alpha} + c}{\bar{\alpha} - c - X^*}\right) x_i^*,$$

In matrix form, for $\mathbf{x}^* \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, we obtain:

$$\mathbf{G}\mathbf{x}^* = \left(\frac{2X^* - \bar{\alpha} + c}{\bar{\alpha} - c - X^*}\right)\mathbf{x}^*. \tag{16}$$

3.3 Communities

Let us start with some important definitions.

Definition 3. *Given $M \subset N$, $(M, \mathbf{G}_M)$ is a* strongly-connected component *of $(N, \mathbf{G})$ if:*

(i) it is a strongly-connected sub-network;

(ii) for each $I \subset N \setminus M$, $(M \cup I, \mathbf{G}_{M \cup I})$ is not a strongly-connected network.

A strongly-connected component with at least two agents is called a community.[21]

Let $\mathcal{C}(\mathbf{G})$ denote the set of communities in $(N, \mathbf{G})$:

$$\mathcal{C}(\mathbf{G}) := \{M \subset N : (M, \mathbf{G}_M) \text{ is a community of } (N, \mathbf{G})\}.$$

An element of $\mathcal{C}(\mathbf{G})$ is a subset of agents of cardinal at least two, such that the corresponding sub-network is strongly connected. Note that under the no-isolation property, $\mathcal{C}(\mathbf{G})$ is never empty. Indeed, a cycle in the network always exists and communities are disjoints.[22]

To illustrate this definition, consider the network in Figure 1(a) with $N = \{1, 2, 3, 4, 5, 6\}$ and define $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$. We can see that both $(M_1, \mathbf{G}_{M_1})$ and $(M_2, \mathbf{G}_{M_2})$ are strongly-connected components: each is a strongly-connected sub-network, and it is not possible to enlarge any of these two sub-networks to form a larger strongly-connected network. Hence, $\mathcal{C}(\mathbf{G}) = \{M_1, M_2\}$. Note that the set of communities is unchanged in the network of Figure 1(b).

Definition 4. *Let $M \subset N$. Agent $i \in N$ is an **adjunct** to the sub-network $(M, \mathbf{G}_M)$ of $(N, \mathbf{G})$ if i is connected to some agent $j \in M$ through a path. The **adjunct set** of M, denoted by $\bar{M}$, is therefore defined as the set of all agents that are adjuncts to M, that is:*

$$\bar{M} = \{i \in N : \exists j \in M \text{ with } i \rightrightarrows j\}.$$

Given $M, M' \in \mathcal{C}(\mathbf{G})$, we say that M' is adjunct to M if $M' \subset \bar{M}$.

[21] By a slight abuse of language, we use the terminology "community" for both the subset of agents involved and the corresponding sub-network.

[22] More precisely, strongly-connected components (including communities, as well as singletons) form a partition of the set of agents.

Note that the definition of the adjunct set of a community M is inclusive in the sense that M is also part of the adjunct set. In the network of Figure 1(a), $\bar{M}_1 = M_1 = \{1, 2, 3\}$ and $\bar{M}_2 = M_2 \cup M_1 = N = \{1, 2, 3, 4, 5, 6\}$.

Interestingly, this definition of adjunct sets induces a partial ordering $\succeq$ on the set of communities $\mathcal{C}(\mathbf{G})$.[23] The binary relation $\succeq$ is defined as follows:

$$M' \succeq M \text{ if and only if } M' \subset \bar{M}.^{24} \tag{17}$$

In other terms, $M' \succeq M$ if there exists a path from M' to M. As usual, if $M' \succeq M$ and $M' \neq M$, we write $M' \succ M$. A community M is $\succeq$-*maximal* if no community M' exists such that $M' \succ M$. That is, there is no M' that *is aware* of M. If not, we say that M is *hidden* from M'. Maximal elemevnts with respect to this partial ordering are communities of which few agents are aware. More precisely, only "isolated" agents (i.e., agents who are connected to other agents) are aware of them. As we will see, this will allow these particular communities to grab a large share of resources within their neighborhood. This "advantage" can be intuitively captured by making the following observation[25]: given a PCE $\mathbf{x}^*$, we have

$$M \subset N_+(\mathbf{x}^*) \Rightarrow M' \subset N_+(\mathbf{x}^*), \;\; \forall M' \succeq M.$$

Let us now illustrate the concept of communities and the $\succeq$-ordering.

Example 1. Consider the network $(N, \mathbf{G})$ in Figure 1(a) with $N = \{1, 2, 3, 4, 5, 6\}$. As we saw above, there are two communities: $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$. Since there is a link from 1 to 6, for instance, we have $M_1 \succ M_2$, and, clearly, the community M_1 is $\succeq$*-maximal.*

Example 2. Consider now the network $(N, \mathbf{G})$ displayed in Figure 2 with $N = \{1, 2, \cdots, 10\}$. There are three communities in this network: $M_1 = \{2, 3, 4\}$, $M_2 = \{5, 6\}$, and $M_3 = \{7, 8, 9, 10\}$. To check that these three connected subnetworks satisfy *(ii)* in Definition 3, note that there is no path from M_3 to either M_1 or M_2, and that there is no path from M_2 to M_1. However, there is a link from 3 to 5, so that $M_1 \succ M_2$. There is also a link from 5 to 10, so that $M_2 \succ M_3$. Finally, we obtain $M_1 \succ M_2 \succ M_3$. Clearly, the community M_1 is $\succeq$*-maximal.*

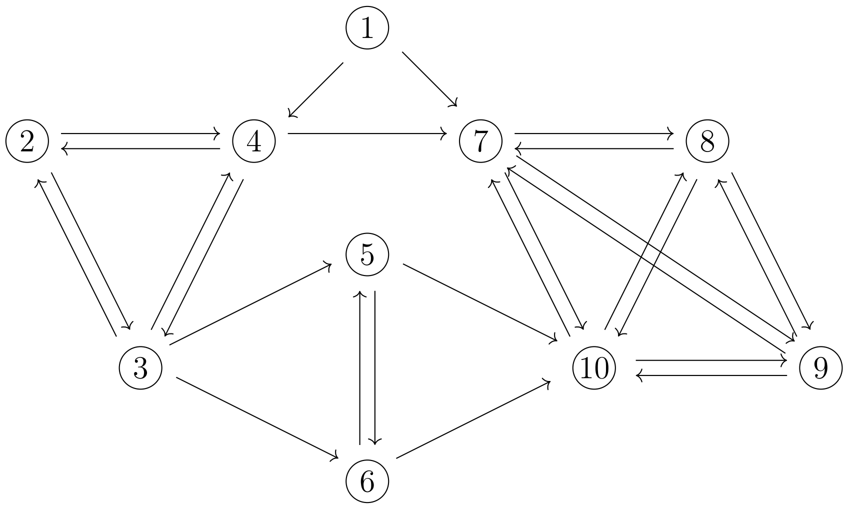
In both examples, the ordering $\succeq$ is *complete* (every pair of communities is comparable as per the $\succeq$ ordering). This is not always the case, as we see in the following example, in which some of the communities are not comparable according to $\succeq$.

3.4 Characterization of peer-consistent equilibria

Thus far, we have shown that any weakly-connected network $(N, \mathbf{G})$ can be associated with an ordering in the set of strongly-connected components. In this section, we characterize

[23] A *partial ordering* is a reflexive, antisymmetric, and transitive binary relation.
[24] Note that it is possible that for $i, j \in \bar{M}$, there is no path from i to j or from j to i.
[25] It is a direct consequence of the fact that $N_+(\mathbf{x}^*)$ is closed.

Figure 2: Network structure in Example 2



active agents at a PCE in terms of their position in the network, along with the "density" of the community they belong to.

Definition 5. *For each community* $(M, \mathbf{G}_M)$*, we refer to its adjunct set* $\bar{M}$ *as a* candidate set with root M*. A peer-consistent equilibrium* $\mathbf{x}^*$ *of* $(N, \mathbf{G})$ *such that* $N_+(\mathbf{x}^*) = \bar{M}$ *for some community* M *is called an* equilibrium with root M*. We refer to any equilibrium that admits a root as a* simple equilibrium.

A *candidate set* is a set of agents that could naturally be the set of active players at equilibrium. Indeed, if there is one agent i who is active in M, then all agents in its adjunct set $\bar{M}$ will be active. This is because all agents who are path-connected to i are necessarily active, since the set of active agents at a peer-consistent equilibrium is closed (see (12)). We obtain the following key result:

Proposition 1. *There is at most one peer-consistent equilibrium* $\mathbf{x}^*$ *with root* M*. It exists if and only if*

$$\rho(\mathbf{G}_M) > \max\left\{\rho(\mathbf{G}_{M'}) : \ M' \in \mathcal{C}(\mathbf{G}), \ M' \succ M\right\}. \tag{18}$$

In particular, for any $\succeq$*-maximal* M*, there always exists an equilibrium with root* M*.*

This proposition is simple but very powerful in terms of characterizing the simple equilibria. Proposition 1 states that, for any community M, there is *at most* one PCE with root M. This provides a necessary and sufficient condition in terms of the largest-eigenvalue comparisons for this equilibrium to exist. This condition is automatically satisfied for maximal communities, because the set $\{M' \in \mathcal{C}(\mathbf{G}) : M' \succ M\}$ is empty. For non-maximal elements of $\mathcal{C}(\mathbf{G})$, however, ascertaining if a PCE with root M exists requires checking the non-trivial inequality (18).

Let us now illustrate Proposition 1 for the linear (Tullock) contest games (Section 2.3.1) with the following examples.[26]

Example 1: Peer-consistent equilibria

• Consider the network $(N, \mathbf{G})$ in Figure 1(a) with $N = \{1, 2, \cdots, 6\}$. As we saw above, there are two strongly-connected components: $M_1 := \{1, 2, 3\}$ and $M_2 := \{4, 5, 6\}$, with $M_1 \succ M_2$.

(a) Equilibrium with root $\boldsymbol{M_1}$: since M_1 is $\succeq$-maximal, there is an equilibrium with root M_1, where only agents $1, 2$, and 3 are active.

(b) No equilibrium with root $\boldsymbol{M_2}$: Since $M_1 \succ M_2$, and $\rho(\mathbf{G}_{M_2}) = \rho(\mathbf{G}_{M_1}) = 2$, there is *no* equilibrium with root M_2.

As a result, there is a unique peer-consistent equilibrium such that $x_1^* = x_2^* = x_3^* = \frac{2V}{9c}$ and $x_4^* = x_5^* = x_6^* = 0$.

• Consider now the network $(N, \mathbf{G})$ displayed in Figure 1(b) with $N = \{1, 2, \cdots, 6\}$, a variation of Figure 1(a) in which we *deleted* the directed edges between agents 1 and 2, so that the strongly-connected component M_1 is now less dense and, thus, its spectral radius is smaller. The partial order is unchanged compared to the network of Figure 1(a), and thus we have the same two candidates:

(a) Equilibrium with root $\boldsymbol{M_1}$: Again, M_1 is $\succeq$-maximal, so that there is an equilibrium with root $M_1 = \{1, 2, 3\}$, where only agents $1, 2$ and 3 are active, with $x_1^* = x_2^* \sim 0.1715$, $x_3^* \sim 0.2426$, and $x_4^* = x_5^* = x_6^* = 0$.

(b) Equilibrium with root $\boldsymbol{M_2}$: Since $\rho(\mathbf{G}_{M_2}) = 2 > \sqrt{2} = \rho(\mathbf{G}_{M_1})$, there is an equilibrium with root M_2, where all agents in the network are active. At such an equilibrium, note that efforts are not symmetric. Indeed, $x_1^* = \frac{V}{9c}$, $x_2^* = \frac{7V}{45c}$, $x_3^* = \frac{2V}{15c}$, and $x_i^* = \frac{4V}{45c}$ for each $i \in \{4, 5, 6\}$.

There are now *two* peer-consistent equilibria. Therefore, by removing the links between agents 1 and 2, we enlarged the set of peer-consistent equilibria from a *unique* equilibrium to *two* PCEs. $\diamond$

Example 2: Peer-consistent equilibria

Consider the network displayed in Figure 2 with $M_1 = \{2, 3, 4\}$, $M_2 = \{5, 6\}$, and $M_3 = \{7, 8, 9, 10\}$. Despite the existence of three strongly-connected components, let us use Proposition 1 to show that there are (only) two peer-consistent equilibria. Each community's subnetwork is complete, so that $\rho(\mathbf{G}_{M_2}) = 1 < \rho(\mathbf{G}_{M_1}) = 2 < \rho(\mathbf{G}_{M_3}) = 3$. Importantly, we know that $M_1 \succ M_2 \succ M_3$, so that M_1 is $\succeq$-maximal.

[26] We can perform the same exercise for Cournot Competition (Section 2.3.2) with utility function (10). It is easily verified that, for the network in Figure 1(a), there is a unique PCE where only firms 1, 2, 3 are active with $x_1^* = x_2^* = x_3^* = (\bar{\alpha} - c)/4$ and $X^* = 3(\bar{\alpha} - c)/4$. In the network in Figure 1(b), there are two PCE. The first one is such that only firms 1, 2, 3 are active with $x_1^* = x_2^* = (\bar{\alpha} - c) / \left[2\left(\sqrt{2} + 1\right)\right]$ and $x_3^* = (\bar{\alpha} - c) / \left[\sqrt{2}\left(\sqrt{2} + 1\right)\right]$, with $X^* = (\bar{\alpha} - c)/\sqrt{2}$. The other PCE is such that all firms are active with $\mathbf{x}^* = \frac{(\bar{\alpha} - c)}{40}(5, 7, 6, 4, 4, 4)$ and $X^* = 3(\bar{\alpha} - c)/4$.

(a) Equilibrium with root M_1: Since M_1 is $\succeq$-maximal, there is an equilibrium with root M_1 (with set of active agents $\bar{M}_1 = \{1, 2, 3, 4\}$) which, using Theorem 1, is given by:

$$x_1^* = \frac{2V}{21c},\ x_i^* = \frac{4V}{21c}\text{for } i \in \{2, 3, 4\},\ x_j^* = 0 \text{ for } j \in \{5, 6, 7, 8, 9, 10\}.$$

The equilibrium payoffs are $u(x_1^*) = \frac{1}{21}V,\ u(x_i^*) = \frac{2}{21}V$ for $i = 2, 3, 4$.

(b) No equilibrium with root M_2: Since $M_1 \succ M_2$ and $\rho(\mathbf{G}_{M_2}) < \rho(\mathbf{G}_{M_1})$, there is no equilibrium with root M_2.

(c) Equilibrium with root M_3: Since $\rho(\mathbf{G}_{M_3}) > \max\{\rho(\mathbf{G}_{M_1}), \rho(\mathbf{G}_{M_2})\}$, there is an equilibrium with root M_3 such that the set of active individuals is $\bar{M}_3 = N$. This equilibrium is given by:

$$x_1^* = \frac{21V}{364c},\ x_i^* = \frac{18V}{364c}\text{for } i \in \{2, 5, 6\},\ x_j^* = \frac{27V}{364c} \text{ for } j \in \{3, 4\},\ \text{and}\ \ x_k^* = \frac{36V}{364c},\ \text{ for } k \geq 7.$$

The equilibrium payoffs are:

$$u(x_1^*) = \frac{7V}{364},\ u(x_i^*) = \frac{6V}{364},\ \text{ for } i \in \{2, 5, 6\},\ u(x_j^*) = \frac{9V}{364} \text{ for } j \in \{3, 4\},\ u(x_k^*) = \frac{12V}{364} \text{ for } k \geq 7.$$

In summary, in the network in Figure 2, there are two PCEs: one in which only agents 1, 2, and 3 are active, and one in which all agents are active. $\diamond$

Consider the two networks in Figures 1(a) and (b) (Example 1). Nobody is aware of community $M_1 = \{1, 2, 3\}$, that is, nobody in the network perceives them as competitors. Thus, in both cases, there is a PCE in which only agents 1, 2, and 3 are active. However, agents in M_1 are aware of the community $M_2 = \{4, 5, 6\}$, since agent 1 perceives agent 6 as a competitor, and agent 2 perceives both agents 4 and 6 as competitors. Moreover, community M_1 is less dense in Figure 1(b) than in Figure 1(a). As a result, there is an additional PCE in which all agents are active in Figure 1(b). Importantly, this additional PCE (in which all agents are active) is not a symmetric equilibrium: while agents 4, 5, and 6 all exert the same effort, differences in terms of perceived competition of agents in the community M_1 lead agent 1 and 2 to exert different levels of effort.

Consider now the network displayed in Figure 2 (Example 2). Aside from the isolated agent 1, no one is aware of the community $\{2, 3, 4\}$. In other words, only agent 1 perceives agents 2, 3, and 4 as her competitor; nobody else in the network does so. Thus, there is an equilibrium where the only active agents in the network are $\{1, 2, 3, 4\}$. On the contrary, all agents in the network can reach community $M_3 = \{7, 8, 9, 10\}$, either directly or through a path; thus, there cannot be an equilibrium where only agents in M_3 are active.

More generally, Proposition 1 provides us with a simple way of assessing whether a given community can be active or not at equilibrium. A community M can be active at a PCE if (i) the communities that are aware of M are less densely connected than M; (ii) if M is aware of other communities, then they are less densely connected than M.

In order to complete our characterization of peer-consistent equilibria, let us now consider an interesting superset of the set of semi-connected networks.

Definition 6. *A weakly-connected network* $(N, \mathbf{G})$ *is* ***generic*** *if for any distinct* $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = \rho$ *implies that* $\max\{\rho(\mathbf{G}_M): M \succ M_1 \text{ or } M \succ M_2\} \geq \rho$.

Hence, generic networks are such that, for any two distinct communities with the same spectral radius, a community must exist whose spectral radius is at least as large, and which is aware of one of them. In other words, (i) we exclude weakly-connected networks for which two PCEs with different roots have the same spectral radius, but (ii) we allow the existence of two communities with the same spectral radius if one of them is not part of a PCE. In particular, we exclude networks in which two $\succeq$-maximal communities have the same spectral radius. For example, the network displayed in Figure E1 in Appendix E.2 is not generic because the two $\succeq$-maximal communities $M_1 = \{2, 3\}$ and $M_2 = \{4, 5\}$ have the same spectral radius (i.e., $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = 1$). On the other hand, the networks in Figures 1(a) and (b) (Example 1) and in Figure 2 (Example 2) are generic. Observe, in particular, that in the network displayed in Figure 1(a), the communities $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$ have the same spectral radius (i.e., $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = 2$). However, because M_2 is not the root of a PCE, this network is generic.

If $\mathbf{G}$ is a semi-connected network, then the strongly-connected components of the network are totally ordered, which implies that SemiCN $\subset$ LGN.[27] The following inclusions summarize the relative strength of all four notions of connectedness considered here (Definitions 1 and 6): Strongly Connected (StrCN), Semi-Connected (SemiCN), Generic (GN) and Weakly Connected (WCN):

$$\text{StrCN} \subset \text{SemiCN} \subset \text{GN} \subset \text{WCN}.$$

We show that, in generic networks, peer-consistent equilibria are always simple, meaning that Proposition 1 provides a full characterization of the set of equilibria in these networks.

Corollary 1. *Let* $(N, \mathbf{G})$ *be a generic network, and let* $\mathbf{x}^*$ *be a peer-consistent equilibrium of* $(N, \mathbf{G})$*. Then,* $\mathbf{x}^*$ *is a simple equilibrium. Moreover, equilibrium efforts are proportional to eigenvector centrality in the sub-network of active players.*

The last statement of Corollary 1 must be understood as follows: if $\mathbf{x}^*$ is a PCE, then the effort of active agents is proportional to their eigenvector centrality *in the sub-network to which they belong.* It is important to understand that this result does not say anything about the eigenvector centrality of agents in the whole network, since inactive agents are not taken into account. A direct consequence of Corollary 1 is that there is a *finite* number of equilibria, because, for any strongly-connected component M of the network, there is at most one PCE with root M. Actually, the set of peer-consistent equilibria is finite if and only if the network is generic. This is stated formally in Proposition D1 in Appendix D.1.

[27] In Section D.1 of the Appendix, we provide a more precise statement for the case where the network is semi-connected (Corollary D2).

When the network is *not* generic, there may exist *non-simple equilibria*, that is, equilibria such that the set of active agents is not a candidate set, but instead a union of candidate sets. In this case, the set of peer-consistent equilibria is *infinite*. In Section E of the Appendix, we consider this case and show that we can still describe the set of peer-consistent equilibria in a simple way (see Proposition E6).

In several of our examples, we have seen multiple peer-consistent equilibria (e.g., the networks in Figure 1(b) and in Figure 2 had two PCEs). Multiplicity of peer-consistent equilibria is a salient feature of our model. Further, and most importantly, in many such equilibria, some agents end up being inactive.[28]

Remark 6 (Equilibrium payoffs)**.** *Pick any peer-consistent equilibrium* $\mathbf{x}^*$ *with root* M*. Then, the payoff of each active agent* i *is given by*

$$\pi_i(\mathbf{x}^*) = (f(X^*) - c)x_i^*, \tag{19}$$

where $\pi_i(\mathbf{x}^*) = u_i\left(x_i^*, \mathbf{x}_{-i}^*; W_i\right)$*. Moreover, the sum of utilities of active agents at* $\mathbf{x}^*$ *is given by:*

$$\sum_i \pi_i(\mathbf{x}^*) = (f(X^*) - c)\, X^* = \frac{-(X^*)^2 f'(X^*)}{(\rho(\mathbf{G}) + 1)}, \tag{20}$$

since $(f(X^*) - c)(\rho(\mathbf{G}) + 1) = -X^* f'(X^*)$

This remark shows that the equilibrium utility of each active agent is proportional to her equilibrium effort and thus the utility of active agents is proportional to their eigenvector centrality in the sub-network of active agents. This only informs us about the relative utility of active agents but does not tell us anything on the aggregate utility. However, for the Tullock context function, we have:

$$\sum_i \pi_i(x_i^*) = \frac{V}{\rho(\mathbf{G}) + 1},$$

while, for Cournot competition, we obtain:

$$\sum_i \pi_i(x_i^*) = \frac{(X^*)^2}{\rho(\mathbf{G}) + 1} = (\bar{\alpha} - c)\frac{\rho(\mathbf{G}) + 1}{(\rho(\mathbf{G}) + 2)^2}.$$

In both examples (Tullock and Cournot), we can write the aggregate utility as an explicit function of $\rho(\mathbf{G})$, the spectral radius of the network associated to the equilibrium. Hence,

[28] Observe that we have assumed that all agents were ex-ante identical and their only heterogeneity stemmed from their network position. If we relax this assumption and allow for agents to have different costs of effort (i.e., $c = c_i$ for agent i), then the ordering on communities introduced in the previous section will be exactly the same, but the link between spectral radius and PCE (Proposition 1) will no longer hold true. There will be a trade-off between belonging to a densely-connected community and the cost of effort. Similarly, if we assume a more general sharing rule than the one defined in (4), the ordering will be unaltered and deliver the same result, but Proposition 1 will be affected. This is because the $\succeq$ ordering does not rely on any parameter of the model, only on the network topology.

in these examples, the aggregate utility decreases when the network becomes denser. In the general case, as can be seen in equation (20), we cannot make such a statement because X^* depends on $\rho(\mathbf{G})$.

3.5 Stability and eigenvector centrality

This section is devoted to refining the set of equilibria by characterizing those peer-consistent equilibria that are stable. Such an approach has two fundamental objectives. First, it allows us to identify which equilibria are robust to perturbations and provides a *dynamic microfoundation* to the concept of peer-consistent equilibrium. Second, refining the set of equilibria is necessary if one wants to extend the eigenvector centrality microfoundation to general networks. Indeed, as noted above, the efforts of active agents are proportional to their eigenvector centrality in the sub-network of active players. However, this raises a natural question: *Is there a link between eigenvector centrality in the whole network and PCE?* The answer is positive and we show that, in any generic network, exactly one PCE is proportional to the eigenvector centrality of the whole network, and it is precisely this PCE that we identify as the stable one.

As usual, stability of equilibria is defined through a meaningful dynamical system, the rest point of which is the equilibria we want to consider. A stable equilibrium is then defined as a stable rest point of the dynamics, that is, a rest point to which, starting from conditions close enough to it, the system asymptotically stabilizes. For this purpose, we introduce *perceived best-response dynamics*. This captures the idea that agents smoothly adapt their actions in the direction of their *best possible action*, given the information available to them.

3.5.1 The perceived best-response dynamics

We now present the continuous-time dynamics to which we characterize stability. Even though it is very close—in terms of interpretation—to the classical continuous-time best-response dynamics,[29] we explain how it is related to a simple discrete-time model.

Consider a discrete-time sequence of effort profiles in which, after observing their neighbors' effort level as well as the local resources in the previous period, agents adapt their effort levels at each period of time. Specifically, before choosing her effort level at period t, agent i observes the effort of her neighbors $\mathbf{x}_{-i}^{t-1}$ as well as the realized local parameter W_i^{t-1} at period $t-1$. She can then compute her optimal effort level with respect to quantity W_i^{t-1} by maximizing the map[30]

$$b_i \in [0, +\infty[\mapsto b_i \cdot f\left(\frac{b_i + (\mathbf{G}\mathbf{x}^{t-1})_i}{W_i^{t-1}}\right) - cb_i.$$

[29]See Fisher (1961), Gilboa and Matsui (1991), Matsui (1992), and, more recently, Bramoullé et al. (2014) and Bervoets and Faure (2019).

[30]Observe that $(\mathbf{G}\mathbf{x})_i = \sum_{j \in \mathcal{N}_i} x_j^*$. We use this more compact notation whenever it is convenient.

We denote by $Br_i(\mathbf{x}^{t-1})$ this maximizer. Since $W_i^{t-1} = \frac{x_i^{t-1}+(\mathbf{G}\mathbf{x}^{t-1})_i}{X^{t-1}}$, the map $Br_i(\cdot)$ is given by

$$Br_i(\mathbf{x}) = Argmax_{b_i \geq 0} \, b_i \cdot f\left(\frac{X(b_i + (\mathbf{G}\mathbf{x})_i)}{x_i + (\mathbf{G}\mathbf{x})_i}\right) - cb_i.$$

Tullock model. In the linear Tullock contest model, we can compute explicitely the perceived best-response map:[31]

$$Br_i(\mathbf{x}) = \max\left\{-(\mathbf{G}\mathbf{x})_i + \left(\frac{V}{cX}(\mathbf{G}\mathbf{x})_i\,(x_i + (\mathbf{G}\mathbf{x})_i)\right)^{1/2}, 0\right\}. \tag{21}$$

Cournot competition. In the Cournot model, we can also compute explicitely the perceived best-response map:

$$Br_i(\mathbf{x}) = \frac{1}{2}\max\left\{-(\mathbf{G}\mathbf{x})_i + (\bar{\alpha} - c)\frac{x_i + (\mathbf{G}\mathbf{x})_i}{X}, 0\right\}. \tag{22}$$

Agent i chooses an effort level equal to a convex combination of her last effort level and the perceived best response based on what she observed at the last time period:

$$x_i^t = (1-\epsilon)x_i^{t-1} + \epsilon Br_i(\mathbf{x}^{t-1}) \tag{23}$$

When ϵ is small, the sequence generated by (23) is related to the solution curves of the continuous-time system

$$\dot{\mathbf{x}}(t) = \mathbf{B}(\mathbf{x}(t)), \tag{24}$$

where $B_i(\mathbf{x}) = -x_i + Br_i(\mathbf{x}),\ i = 1, ..., N$. Indeed, system (23) is a so-called *Cauchy-Euler scheme*, designed to approximate the solutions of (24) by choosing a small ϵ. In other words, system (24) can be interpreted as a smooth *limit* version of (23).

Choosing the appropriate state space, the stationary points of this ordinary differential equation are precisely the peer-consistent equilibria of our problem. We now consider the stability notion to be naturally associated to the dynamics (24). Stability for a given PCE $\mathbf{x}^*$ means that the solutions of (24) starting from initial conditions close enough to $\mathbf{x}^*$ converge back to $\mathbf{x}^*$. Formally:

Definition 7. *A peer-consistent equilibrium* $\mathbf{x}^*$ *is said to be* ***asymptotically stable*** *for (24) if there exists an open neighborhood* U *of* $\mathbf{x}^*$ *such that*

$$\lim_{t \to +\infty} \sup_{\mathbf{x}_0 \in U \cap \mathbf{S}} \|\phi(\mathbf{x}_0, t) - \mathbf{x}^*\| = 0,$$

where $\mathbf{S}$, defined in (B.5) in Section B.1.3 of the Appendix, contains all the relevant states of the problem we consider, and $(\phi(\mathbf{x}, t))_{\mathbf{x} \in \mathbf{S}, t \geq 0}$ is the semi-flow associated to (24) on $\mathbf{S}$. Specifically, $\phi(\mathbf{x}, t)$ is equal to the position of the (unique) solution of (24) starting at $\mathbf{x}$.

Definition 7 states that a PCE $\mathbf{x}^*$ is asymptotically stable if it *uniformly* attracts all solutions starting in an open neighborhood of itself. This is a standard concept of stability used in economics (Benaïm and Hirsch, 1999; Weibull, 2003), and in network games in particular (Bramoullé et al., 2016; Bervoets and Faure, 2019).

[31] This holds if and only if the action profile $\mathbf{x}$ is such that $(\mathbf{G}\mathbf{x})_i = 0 \Rightarrow x_i = 0$.

3.5.2 Stable PCE: A simple characterization

We now characterize the PCEs that are asymptotically stable with respect to the best-response dynamics (24). It turns out that being asymptotically stable depends entirely on the sub-network of active players in this PCE, in a very simple and intuitive way. Let $(N, \mathbf{G})$ be a generic network.[32] Given a PCE $\mathbf{x}^*$, we call $\rho(\mathbf{x}^*)$ the largest eigenvalue of the sub-network $(N_+(\mathbf{x}^*), \mathbf{G}_{N_+(\mathbf{x}^*)})$.

Theorem 2. *Let $(N, \mathbf{G})$ be a generic network. Then, there is a unique asymptotically stable equilibrium $\mathbf{x}^*$ such that $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$. Moreover, agents' effort levels at the stable PCE are proportional to their eigenvector centrality in the whole network $(N, \mathbf{G})$.*

The intuition behind the characterization in terms of largest eigenvalues is as follows. Since the network is generic, there is exactly one PCE for which the largest eigenvalue of the set of active players is equal to $\rho(\mathbf{G})$. We must show that it is the only asymptotically stable equilibrium. Suppose that $\mathbf{x}^*$ is a PCE such that $\rho(\mathbf{x}^*)$ is strictly smaller than $\rho(\mathbf{G})$. Then, one can find a community M in which agents are inactive at $\mathbf{x}^*$, while having $\rho(\mathbf{G}_M) = \rho(\mathbf{G})$. Now, suppose that we slightly perturb $\mathbf{x}^*$ so that, instead of playing zero, agents in M play $\epsilon \mathbf{u}_i$, where $\mathbf{u}$ is the normalized positive eigenvector associated with $\rho(\mathbf{G})$. Since, for agents in M, this initial condition is associated with an eigenvalue that is strictly larger than the eigenvalue associated with $\mathbf{x}^*$, the agents in M will want to increase their effort and not come back to zero. Thus, it is clear that $\mathbf{x}^*$ cannot be stable.[33] We conclude the proof by showing that the (unique) PCE for which $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$ is stable using standard methods. The last part of the theorem directly follows from the definition of eigenvector centrality.

Theorem 2 provides a simple and efficient analytic method for checking which PCEs are stable by looking for communities with the highest spectral radii. First, consider Example 1 with the networks displayed in Figure 1(a) and Figure 1(b) with $N = \{1, 2, \cdots, 6\}$. There are two communities: $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$, with $M_1 \succ M_2$ in both networks. The only difference between these two networks is that the one in Figure 1(a) has two extra links between agents 1 and 2 compared to the network in Figure 1(b). This is an important difference because the largest eigenvalue of the $\succeq$-maximal community, M_1, changes: it is equal to 2 in Figure 1(a), whereas it is equal to $\sqrt{2}$ in Figure 1(b). In Figure 1(a), there is a unique equilibrium that is clearly stable, in which only agents 1, 2, and 3 are active. In Figure 1(b), we have seen that there were two PCEs, one with root $M_1 = \{1, 2, 3\}$ and one with root $M_2 = \{4, 5, 6\}$. Since $\rho(\mathbf{G}_{M_1}) = \sqrt{2} < \rho(\mathbf{G}_{M_2}) = 2 = \rho(\mathbf{G})$, there is a unique stable PCE for which all agents are active. Thus, disconnecting agents 1 and 2 has a dramatic impact on the stable peer-consistent equilibria. The fact that the $\succeq$-maximal community in Figure 1(b) is less dense than in Figure 1(a) prevents agents 1, 2, and 3 from

[32] Our main result (Theorem 2) holds under the less restrictive assumption that $(N, \mathbf{G})$ has a *unique dominant component*, as properly defined in condition (UDC) in Section A.2 of Appendix A. In fact, eigenvector centrality is well defined if and only if the network satisfies the condition (UDC).

[33] For ease of presentation, *asymptotically stable* PCEs are referred to as *stable* PCEs.

capturing the entire resource V and thus obliges them to share V with the other players in the PCE.

Second, consider Example 2 with the network depicted in Figure 2. We have seen that there were two PCE with roots $M_1 = \{1, 2, 3\}$ and $M_3 = \{6, 7, 8, 9\}$, respectively. Since $\rho(\mathbf{G}_{M_1}) = 2 < \rho(\mathbf{G}_{M_3}) = 3 = \rho(\mathbf{G})$, the only stable PCE is the equilibrium with root M_3, where all agents are active. Note that in both examples where there exists a peer-consistent equilibrium $\mathbf{x}^*$ with $N_+(x^*) = N$, $\mathbf{x}^*$ must be the stable equilibrium. This is actually always true:

Corollary 2. *Pick a generic network. If there exists a peer-consistent equilibrium* $\mathbf{x}^*$ *with* $N_+(\mathbf{x}^*) = N$*, then* $\mathbf{x}^*$ *is the asymptotically stable PCE.*

In summary, for any (generic) network, we can determine the unique stable peer-consistent equilibrium. First, we establish the $\succeq$-ordering as defined in Section 3.3. Second, we determine the different peer-consistent equilibria by checking, for each community, if its spectral radius is strictly greater than that of the communities that dominate it as per the $\succeq$-ordering (Proposition 1). For each PCE, we can ascertain the effort of each agent, which is equal to her eigenvector centrality (Theorem 1) in the set of active agents. Finally, the unique stable peer-consistent equilibrium in the network is the PCE for which the corresponding root has the same largest eigenvalue as the whole network (Theorem 2).

4 Policy interventions

4.1 Adding links

We now consider the policy implications of our model. We start with the simplest intervention: *Given a network and its unique stable peer-consistent equilibrium, what would happen if we added a link between two agents?*

Consider networks with a unique dominant component. We only focus on *stable* peer-consistent equilibria, that is, equilibria for which the largest eigenvalue of the root is equal to that of the whole network (Theorem 2). We examine whether adding a link from individual i to individual j has an impact on individual efforts. If we do not make additional assumptions on the payoff structure, adding links does not have a clear impact on the aggregate equilibrium effort.[34] Hence we only obtain results on relative efforts in the general model:

Proposition 2. *Pick a generic network* $(N, \mathbf{G})$ *with* $\mathbf{x}^*$ *being the asymptotically stable peer-consistent equilibrium. Suppose that* $i, j \in N_+(\mathbf{x}^*)$*, and* $g_{ij} = 0$*. Let* $\widehat{\mathbf{G}}$ *be the network obtained from* $\mathbf{G}$ *by adding a link from* i *to* j*. Then,* $\widehat{\mathbf{G}}$ *admits an asymptotically stable peer-consistent equilibrium* $\widehat{\mathbf{x}}^*$ *that has the following properties:*

[34] Observe that, in the general case, X^* is increasing with $\rho(\mathbf{G})$ if $(f(X) - c)(-f'(X) - Xf''(X)) + X(f'(X))^2 > 0$. A sufficient condition is that $Xf'(X)$ is non-increasing. However, it is not necessary. For example, in the linear Tullock model, $Xf'(X) = -V/X$. Nevertheless, $(f(X) - c)(-f'(X) - Xf''(X)) + X(f'(X))^2) = cV/X^2 > 0$

(*i*) $N_+(\widehat{\mathbf{x}}^*) \subseteq N_+(\mathbf{x}^*)$,

(*ii*) *for any* $k \in N$, *we have* $\frac{\widehat{x}_i^*}{x_i^*} > \frac{\widehat{x}_k^*}{x_k^*}$.

Since both players i and j initially belong to the set of active agents, there is no reason why adding a link between them should induce a positive effort from an initially inactive agent. Indeed, the spectral radius of the subgraph of inactive agents remains the same while the spectral radius of the set of initially active agents can only increase. In other words, $N_+(\widehat{\mathbf{x}}^*) \subseteq N_+(\mathbf{x}^*)$. This is part (*i*). Additionally, agent i becomes more central relatively to other agents when adding a link from i to j. This implies that the relative effort increase is maximal for agent i. This is captured by part (*ii*) of the proposition. Note that, if $f(\cdot)$ is such that X^* increases with the spectral radius of the graph, then part (*ii*) of Proposition 2 directly implies that $\hat{x}_i^* > x_i^*$.

4.2 Key players

Another possible intervention involves removing one agent as well as all links from the network. This is known as the *key-player* policy (Zenou, 2016) and is particularly relevant in the crime application (Ballester et al., 2006, 2010) but also in the conflict application (König et al., 2017; Amarasinghe et al., 2020), because governments want to target these individuals (the key players) in order to reduce total activity X (total crime or total conflict).

Because there is no clear relationship between network density (captured by the spectral radius $\rho(\mathbf{G})$) and total equilibrium effort X^* (see Remark 6), it is difficult to obtain general results of the key player policy. However, in specific cases, such as the Tullock contest function model,[35] we can derive some results. Proposition D2 in Appendix D.2.1 shows that, in the Tullock model, when removing a player, total effort will never increase. This is because the largest eigenvalue either stays the same or is reduced; the latter decreases total effort. However, the distribution of efforts may be greatly altered, as shown in the following example.

Example 3. Key players and the spread of efforts across local neighborhoods for the (linear) Tullock context function
Consider the network displayed in Figure 1(a) (Example 1). We have shown that there is a unique stable peer-consistent equilibrium where the only active agents belong to $\succeq$-maximal community $M_1 = \{1, 2, 3\}$ with $x_1^* = x_2^* = x_3^* = \frac{2V}{9c}$ and thus the total effort is $X^* = \frac{2V}{3c}$.

Let us now remove the active agent 1 from the network as well as all of her links. It is easily verified that the unique stable PCE $\mathbf{x}^{[-1]*}$ is such that *now* $\{2, 3, 4, 5, 6\} \subseteq N_+(\mathbf{x}^*)$, even though the total effort remains the same at $\frac{2V}{3c}$. Indeed, by removing agent 1, the spectral radius of $M_1 = \{1, 2, 3\}$ decreases from 2 to 1 and becomes strictly smaller than the spectral radius of $M_2 = \{4, 5, 6\}$, which is equal to 2. As a result, the only stable PCE

[35] The same is true for the Cournot Competition model.

is now such that agents 2, 3, 4, 5, and 6 are active. Removing an agent can thus have the *counter-productive effect* of making inactive agents active. In the standard key-player policy (Zenou, 2016), this is not possible since total effort always decreases as *all* agents reduce their individual effort. $\diamond$

4.3 Social mixing

We conclude this section with a brief look at the issue of *social mixing.* To address this issue, we need to depart slightly from our initial model in which there was one (generic) network. Suppose, instead, that we start with two disconnected (generic) networks $(N^1, \mathbf{G}^1)$ and $(N^2, \mathbf{G}^2)$, each of which has a unique stable PCE. As above, we can obtain results only for specific cases. We consider here social mixing in the Tullock contest function model.[36] Think of social mixing as starting with two fully segregated neighborhoods, each endowed with their own resources V^1 and V^2. The key question for the planner is whether merging these two neighborhoods (social mixing) into a connected network $(N, \mathbf{G})$, with $N = N^1 + N^2$, $V = V^1 + V^2$, leads to an increase in total activity and resources.

Proposition D3 in Appendix D.2.2 shows that the total effort in any new stable PCE of the connected network $(N, \mathbf{G})$ is higher than the sum of total efforts in each disconnected neighborhood. Hence, linking the two neighborhoods is beneficial to aggregate effort. On the other hand, the distribution of resources between agents in N^1 and N^2 is less clear. Indeed, distribution in the new equilibrium depends on the specific connections that are formed between the two groups. It is therefore possible to have some agents who are worse off following the mixing of the two neighborhoods.

5 Economic and empirical implications of our model

5.1 The concept of perceived competition

We would now like to illustrate our results and to highlight our concept of “perceived” competition. Understanding how community ordering works in combination with community densities is crucial to detect who is active in a network as well as how much effort agents exert when they are active. However, these two questions need to be answered separately.

Let us first focus on the question: *Who is active in a network?* Note that, given a community M, either all agents in M are active in the stable equilibrium, or none of them are. In the former case, we will say that the community is active (at the stable equilibrium). For a given community, being active or not will be determined by a combination of the two following ingredients: (i) its relative position with respect to other communities in the network (indeed, the relation $\succeq$ translates into an advantage in terms of competition); and (ii) its density, in terms of the largest eigenvalue of the corresponding sub-network.

Indeed, according to Proposition 1 and Theorem 2, in order to be active at the stable equilibrium, a community must satisfy (at least) *one of the two* following conditions:

[36] Similar results can be obtained for the Cournot competition model.

(a) it must exhibit the largest spectral radius among all communities and be "hidden" from all other communities, if any, that have the same property;[37]

(b) it must be aware of all communities with the largest spectral radius.[38]

For obvious reasons, being "denser" makes it more likely for a community to satisfy condition (a), while having a better relative position as per $\succeq$-ordering makes condition (b) more likely to hold.

Once the set of active players is established, we can turn to the second question: *How active is an agent among the set of active agents?* Here, the answer is simpler, since the effort level of an active player is determined by her relative position in the sub-network of active players, which is fully captured by her eigenvector centrality. Consequently, the more aware of other active players an agent is, the more active she is. Also, the more "hidden" from other active agents she is, the more active she is. However, this does not mean that removing links from other active agents will necessarily increase her effort level because, in doing so, it might be that the community is no longer dense enough, rendering this community inactive in equilibrium.

5.2 Dynamic competition

In this section, we illustrate how the dynamics of our game works and how it converges to the unique stable PCE for the Tullock contest function.[39]

5.2.1 Dynamic competition: An illustration for the (linear) Tullock contest function

Consider an economy with two sectors A and B that are in competition for a fixed amount of resources V. To illustrate this situation, consider the network displayed in Figure 1(a) in which sector A has three firms (firms 1, 2 and 3), whereas sector B also has three firms (firms 4, 5 and 6). Given this network structure, firms 4, 5, and 6 are not aware that they are in competition with firms 1, 2, and 3, but firms 1 and 2 are aware that they are in competition with firm 6. Each firm $i = 1, 2, 3, 4, 5, 6$ has to decide on a quantity effort x_i (i.e., how much to produce or to invest). The total resource V will be distributed according to the sharing rule (4) (Tullock contest function), meaning it is proportional to the (quantity) effort of each firm. At the end of the year, V is shared according to this rule and all firms observe the (quantity) efforts. In other words, each sector $s = A, B$ obtains $V/2$ of the total resources. From the viewpoint of firms from sector B, i.e., $M_1^2 = \{4, 5, 6\}$, which believe that they are in competition only among themselves, they perceive a revenue of $V/2$, which they will share between them; thus, each firm perceives that it will obtain $V/6$. On the contrary, the firms from sector A, i.e., $M_1^1 = \{1, 2, 3\}$, believe that they are

[37]This means that a path must not exist from another community with the same spectral radius.

[38]This means a path exists from this community to every community with the largest spectral radius.

[39]A similar exercise can be done for the Cournot competition model.

in competition with not only the other firms from sector A but also firms from sector B. In particular, since firms 1 and 2 perceive that they are in competition with three other firms (two from sector A and firm 6 from sector B), both perceive that their resources are equal to $4V/6$, since they know that each firm has received $V/6$. Since $4V/6 > 3V/6$, compared to the firms from sector B, firms 1 and 2 will exert more (quantity) effort in the following year to obtain a larger share of V. This pattern continues and reinforces itself over time, so that firms 1 and 2 make more and more effort, which induces firm 3 (which has no information about the firms from sector B) to also increase its effort. On the contrary, firms 4, 5, 6, which only observe the other firms from sector B, see their share of V decreasing over time without knowing why. Indeed, they believe that there are fewer and fewer resources in the economy over time. After some time, firms 4, 5, and 6 will end up making no (quantity) effort (and thus exit the market), and all the resources will go to firms 1, 2, and 3. This is the unique stable PCE.

Consider, now, the network displayed in Figure 1(b) where the links between firms 1 and 2 have been removed. Firms 1 and 2 perceive that they are in competition with only two firms for firm 1, and only three firms for firm 2. The perceived resources of firm 1 are equal to $V/2$, which is different from those of firms 4, 5, 6, which also perceive, like firm 1, that they have two competitors. Indeed, firms 4, 5, and 6 perceive that their resources are equal to $2V/5$. Thus, in the following year, firms 1, 2, and 3 (sector A) and firms 4, 5, and 6 (sector B) will exert different levels of effort but these efforts will all be positive. This will persist over time, so the only stable PCE is such that all six firms in this network will be active even though they exert different quantity efforts.

Technically, as illustrated by this example, to calculate the agents' efforts at the PCE, one needs to understand the underlying dynamic process in which, at each period of time t, each agent i best replies to the *observed* W_i^{t-1}. See equation (23) or (24). This process converges to a unique fixed point when $W_i = \frac{x_i^* + \sum_{j \in \mathcal{N}_i} x_j^*}{\sum_{j \in N} x_j^*}$, for each agent i. This unique fixed point is the unique stable PCE of this game.

5.2.2 Dynamic competition: Numerical simulations for the (linear) Tullock contest function

To understand the underlying dynamics of our model, consider the network displayed in Figure 1(a). As stated above, each agent i best replies to the observed W_i^{t-1}. When $\epsilon = 0.1$, this is described by the following equation (see (23)):

$$x_i^t = \frac{9}{10} x_i^{t-1} + \frac{1}{10} \epsilon B r_i(\mathbf{x}^{t-1}), \tag{25}$$

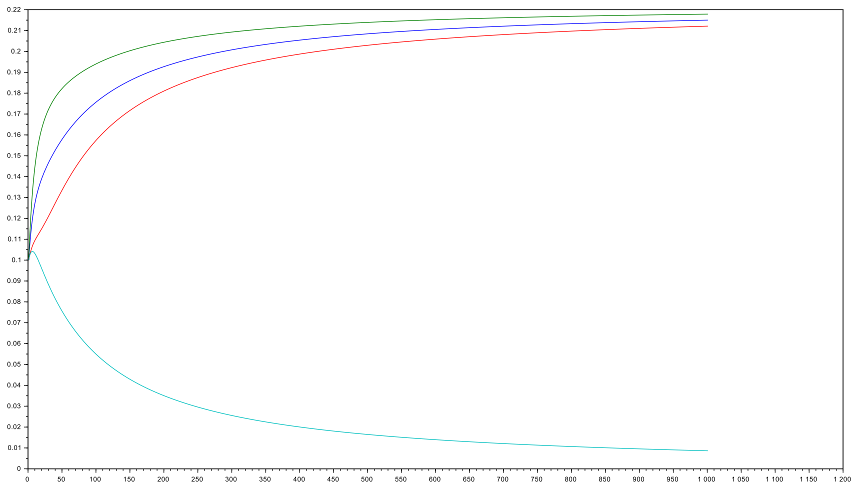
Indeed, at time t, each agent i does not know V or the efforts of all agents in the network. She only observes W_i^{t-1}, her own payoff from the previous period, and $(\mathbf{Gx})_i = \sum_j g_{ij} x_j^*$, her neighbors' effort.

By taking the initial conditions $x_i^0 = 0.1$ for all $i = 1, \ldots, 6$, we obtain Figure 3.[40] The

[40]For simplicity, in all numerical simulations, we take $V/c = 1$.

green, blue, and red curves correspond to the effort of agents 2, 1, and 3, respectively, while the cyan curve corresponds to that of agent 4, 5, or 6. By using (25), agents best respond to their neighbors' efforts and their perceived resources up to the point when this dynamic process converges to the unique stable PCE in which agents 1, 2, and 3 make $2V/9c = 2/9$ effort while agents 4, 5, and 6 exert zero effort.[41] At this equilibrium, $W_i = \frac{x_i^* + \sum_{j \in \mathcal{N}_i} x_j^*}{\sum_{j \in N} x_j^*}$, for each agent $i \in N$. In other words, perceived and real resources are equal. Observe that agents 4, 5, and 6 start with the wrong perception that there are large local resources W_i^0; over time, they observe that these resources decrease and thus reduce their effort until it reaches zero, since there are no resources left to grab.

Figure 3: Convergence to the unique stable PCE in the network of Figure 1(a)



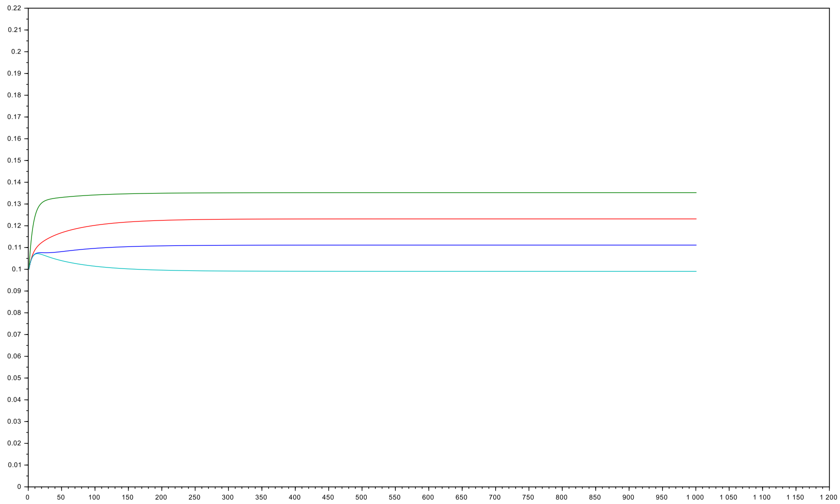
If we now turn to the network in Figure 1(b) and consider the same dynamic system given by (25), starting with the same initial conditions, we obtain Figure 4. Here, agents adjust their effort and reach the interior equilibrium in which $x_1^* = 1/9$ (blue), $x_2^* = 7/45$ (green), $x_3^* = 2/15$ (red), and $x_i^* = 4/45$ for $i = 4, 5, 6$ (cyan). Agents 4, 5, and 6 do not end up making zero effort because W_i^0; their perception of local resources at $t = 0$ is not too far from the reality and thus they slightly change their effort over time.

These two examples illustrate the fact that local perceived resources vary over time, which makes agents change their effort in order to best reply to what they observe (that is, their local resources and their neighbors' effort from the previous period) at each period of time.

In Section 3.4, we showed that, in the network of Figure 1(b), there were two PCEs: one in which all agents are active and one in which only agents 1, 2, and 3 are active, with

[41] This is independent of the initial conditions.

Figure 4: Convergence to the unique stable PCE in the network of Figure 1(b)



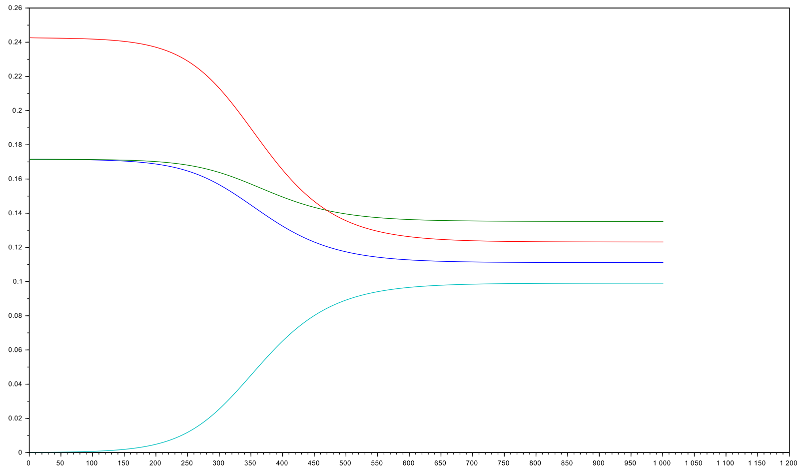
$x_1^* = x_2^* \sim 0.1715$ and $x_3^* \sim 0.2426$. We showed that the latter was asymptotically *unstable*. To understand this, let us start with initial conditions very close to this equilibrium, that is, $x_1^0 = x_2^0 = 0.1715$, $x_3^0 = 0.2426$, and $x_i^0 = 0.001$. Figure 5 shows the dynamics of the system and the convergence to the unique stable PCE in which all agents are active. This clearly illustrates that the equilibrium in which agents 4, 5, and 6 are inactive is unstable. Moreover, since the initial conditions are very close from another equilibrium, it takes some time for agents to adjust their effort and to converge back to the unique stable PCE discussed above. The fact that this equilibrium is unstable does not change the fact that the drift is very small around it and, thus, the dynamical system moves very slowly.

5.3 Evidence on PCE: Education in schools

5.3.1 Evidence on the mechanisms of the PCE

Let us now show that PCE is conceptually compelling. Indeed, PCE is based on the idea that there is some form of ignorance from the player, that is, players will never know the behavior of the set of all players N. In particular, PCE assumes long-lived misperceptions that are robust and that do not stand up to much scrutiny by the players. For example, an agent's perceived and actual marginal utilities differ. In a PCE, this implies that the players are *not* able to falsify their misperceptions with a minimal amount of experimentation. Furthermore, in terms of dynamics, PCE assumes that players do not update their neighbors $\mathcal{N}_i$ but do update W_i. In this section, we would like to provide an example where indeed competitors persistently misperceive who their opponents are and

Figure 5: Convergence to the unique stable PCE in the network of Figure 1(b) when starting from the unstable PCE



act accordingly and do not change their neighbors over time.

Consider education in schools. Students have different (internal) assessments during the year on each subject they take. These assessments take place on a regular basis. For each internal assessment, each student obtains a grade. However, she does not know her exact ranking (her utility) in the classroom because all the grades in the classroom are not disclosed by the teacher. She may ask her friends or friend of friend or anybody that she perceives as her direct competitors about their grades to have a sense of her ranking in the classroom. This is not necessarily reciprocal. For example, if I ask my friend which grade her own friend got, it is not necessarily true that this person will ask our common friend about my grade. Then, depending of her ranking in a particular subject, each student will obtain a "perceived" utility (since this is not her real or objective ranking) and will decide upon her education effort x_i, that is, how much hours she will spend working on this subject.

This application to education displays two key features that are consistent with the definition of a PCE. First, even with experimentation, each student will never know the grades of *all* her competitors (classmates) in the classroom, especially in developing countries where classroom size can be as large as 100.[42] Thus, each student will not know her exact ranking and, therefore, her "objective" utility. She will only know her "perceived" utility by comparing herself to her "perceived" competitors (i.e., neighbors). This means that there is some form of ignorance from each student because she will never know the

[42] For example, in Bangladesh, Hahn et al. (2020) and Islam et al. (2021) document that each school has only one class for each grade, a single teacher, and a large class size (40 students on average), with some classrooms having as much as 100 students.

grades of all students in her classroom and therefore the behavior of all students N in the classroom. This would be even more true for external assessments. Indeed, in many countries, at the end of year 12, students take a final exam on all their subjects and obtain a national score (SAT score in the US, ATAR in Australia, etc.). This score, which provides a ranking of each student in the country or state, allows the student to get into university programs. Clearly, it is impossible to know in advance this ranking (even with experimentation) and each student makes education effort based on their perceived competitors, mostly their friends and students they interact with.

Second, in terms of dynamics, in the context of education, it is reasonable to assume that the set of student i's neighbors $\mathcal{N}_i$ is the same over time, since it is easier to ask again the same students (who may be your friends or friends of friends) their grades because they already gave you their grades in the previous exam.

5.3.2 Does the PCE organize the data better than other concepts?

Let us now provide some convincing evidence that PCE can help organize real world observations.

One of the unique predictions of PCE is provided in Theorem 2, which says that, at the unique stable PCE, each student makes an (education) effort proportional to her *eigenvector centrality* in the whole network. Thus, even the "inactive" students will be the students with very low education effort. In other words, we can rank the education effort of all students by their eigenvecor centrality in the network. This prediction is unique to our concept of equilibrium, the PCE, since it is due to consistency requirement of Definition 2(*ii*). In particular, this prediction cannot be obtained in the models mentioned in the Introduction that focus on imperfect information about the network with new equilibrium concepts (self-confirming equilibrium and peer-confirming equilibrium) related to our PCE (McBride, 2006; Lipnowski and Sadler, 2019; Battigalli et al., 2020).

Let us provide some evidence of this. Islam et al. (2021) collect the networks of students in Bangladesh and study their education outcomes (both in cognitive skills, i.e., test scores, and non-cognitive skills) using some field experiments. They show that friendships are not always reciprocated (i.e., directed networks)[43] and that the *eigenvector centrality of each student is a strong predictor of their education outcomes*, in particular, their test scores (see their Table 8).[44]

Observe that the fact that, in Islam et al. (2021), the education effort of each student is shown to be proportional to her eigenvector centrality proves that students cannot experiment in order to discover their "objective" utility (i.e., their "real" or "objective" ranking in

[43]This is a standard empirical result in friendship networks where it is documented that about 50% to 60% of friendship relationships are not reciprocated (see e.g., Calvó-Armengol et al. (2009); Huitsing et al. (2012); Almaatouq et al. (2016); Algan et al. (2020)), in particular in the education context (Calvó-Armengol et al., 2009; Algan et al., 2020).

[44]Eigenvector centrality has also been showed to be relevant in other contexts. For example, investigating microfinance diffusion in 43 villages in India, Banerjee et al. (2013) find that the eigenvector centrality and diffusion centrality of the first contacted individuals (i.e., the set of original injection points in a village) are the only significant predictors of the eventual diffusion.

the classroom), as predicted by the PCE. Indeed, if students could experiment and discover their "real" ranking and thus maximize their "objective" rather than their "perceived" utility, then the equilibrium would be Nash and not PCE. As shown in the paper, in this case, agents would then know all their competitors, which is equivalent to say that the network would be complete (Remark 3). This would imply that all agents would have the same eigenvector centrality and, thus, the latter could not be a predictor of education effort (or test scores).

6 Conclusion

In this paper, we consider an aggregative game of competition in which agents have an imperfect knowledge of their competitors. We model this imperfect knowledge by a network by assuming that each agent only has information on the activities of their direct neighbors, that is, their *perceived* competitors. We develop a new concept of equilibrium, which we refer to as peer-consistent equilibrium (PCE). Each agent chooses an effort level that maximizes her perceived utility. However, at the PCE, effort levels of all agents have to be consistent: for each agent, her *perceived subjective* utility and resource has to be equal to her *objective* payoff and resource.

We first show that, at any PCE, the effort of an active agent is proportional to her eigenvector centrality. This is true for any network. We then introduce the concept of community: within each community, all agents have the same propensity to exert positive effort. We construct an ordering of the communities in terms of active agents. Agents in the better ranked communities are more likely to be active because few agents are "aware" of them, and agents in these communities can therefore grab a significant amount of resources within their neighborhood. Then, we determine all peer-consistent equilibria by comparing the spectral radius of these communities and that of their adjunct set (i.e., agents that can reach them through a path) in the whole network. We show that, to be active in equilibrium, one needs to belong either to a $\succeq$-maximal community (because few agents are aware of these individuals) or to communities of large size. Finally, we demonstrate that there is a unique stable PCE in each network. This PCE corresponds to the community that has the largest spectral radius in the network. Depending on the network structure, at the unique stable PCE, either all or only a subset of agents are active. We illustrate all our results with two well-known applications of aggregative games: Tullock contest function (rent-seeking games) and Cournot oligopoly competition.

Lastly, we study the policy implications of our model. We show that adding a link can reduce the number of active agents in the network because it creates a new path that makes some agents more likely to be reached; in turn, this may lower their status in terms of community. We also study the key-player policy and show that, by removing an agent from the network, we may make several inactive agents active. Further, we examine social mixing by merging two different disconnected networks, highlighting that total activity is higher than the sum of total activity in each network.

In many real-world situations, agents are not aware of the full set of agents with whom they are in competition and thus only take into account their "perceived" local competitors when deciding which actions to take. In this paper, we shed some light on this issue by foregrounding the importance of individual network position and the community to which each agent belongs. More generally, we believe that the concept of "perceived" competition is important to understand and explain many situations in which competition is not perceived as reciprocal and agents only care about their local competitors, even though competition is global.

References

Algan, Y., Dalvit, N., Do, Q.-A., Le Chapelain, A., and Zenou, Y. (2020). Friendship networks and political opinions: A natural experiment among future French politicians. Unpublished manuscript, Monash University.

Almaatouq, A., Radaelli, L., Pentland, A., and Shmueli, E. (2016). Are you your friends friend? Poor perception of friendship ties limits the ability to promote behavioral change. *PloS One*, 11(3):e0151588.

Alos-Ferrer, C. and Ania, A. (2005). The evolutionary logic of perfectly competitive behaviour. *Economic Theory*, 26:497–516.

Amarasinghe, A., Raschky, P., Zenou, Y., and Zhou, J. (2020). Conflicts in spatial networks. CEPR Discussion Paper No. 14300.

Ambrus, A., Mobius, M., and Szeidl, A. (2014). Consumption risk-sharing in social networks. *American Economic Review*, 104(1):149–82.

Ballester, C., Calvó-Armengol, A., and Zenou, Y. (2006). Who's who in networks. Wanted: The key player. *Econometrica*, 74(5):1403–1417.

Ballester, C., Calvó-Armengol, A., and Zenou, Y. (2010). Delinquent networks. *Journal of the European Economic Association*, 8:34–61.

Banerjee, A., Chandrasekhar, A., Duflo, E., and Jackson, M. (2013). The diffusion of microfinance. *Science*, 341:(6144).

Battigalli, P., Panebianco, F., and Pin, P. (2020). Learning and selfconfirming equilibria in network games. Unpublished manuscript, Bocconi University.

Benaïm, M. and Hirsch, M. W. (1999). Learning processes, mixed equilibria and dynamical systems arising from fictitious play in perturbed games. *Games and Economic Behavior*, 29:36–72.

Bervoets, S. and Faure, M. (2019). Stability in games with continua of equilibria. *Journal of Economic Theory*, 179:131–162.

Bloch, F., Jackson, M., and Tebaldi, P. (2019). Centrality measures in networks. Unpublished manuscript, Stanford University.

Bonacich, P. (1972). Factoring and weighting approaches to status scores and clique identification. *Journal of Mathematical Sociology*, 2(1):113–120.

Bramoullé, Y. and Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, 135(1):478–494.

Bramoullé, Y., Kranton, R., and D'Amours, M. (2014). Strategic Interaction and Networks. *American Economic Review*, 104(3):898–930.

Bramoullé, Y., Rogers, B., and Galeotti, A. (2016). *The Oxford Handbook of the Economics of Networks*. Oxford: Oxford University Press.

Brock, W. and Durlauf, S. (2001). Discrete choice with social interactions. *Review of Economic Studies*, 68:235–260.

Calvó-Armengol, A., Patacchini, E., and Zenou, Y. (2009). Peer effects and social networks in education. *Review of Economic Studies*, 76:1239–1267.

Calvó-Armengol, A. and Zenou, Y. (2004). Social networks and crime decisions. The role of social structure in facilitating delinquent behavior. *International Economic Review*, 45:939–958.

Corchón, L. C. (2007). The theory of contests: a survey. *Review of Economic Design*, 11(2):69–100.

Dequiedt, V. and Zenou, Y. (2017). Local and consistent centrality measures in networks. *Mathematical Social Sciences*, 88:28–36.

Dziubiński, M., Goyal, S., and Vigier, A. (2016). Conflict and networks. In: Y. Bramoullé, B.W. Rogers and A. Galeotti (Eds.), *The Oxford Handbook of the Economics of Networks,* Oxford: Oxford University Press.

Fisher, F. M. (1961). The stability of the cournot oligopoly solution: The effects of speeds of adjustment and increasing marginal costs. *Review of Economic Studies*, 28(2):125–135.

Franke, J. and Öztürk, T. (2015). Conflict networks. *Journal of Public Economics*, 126:104–113.

Gagnon, J. and Goyal, S. (2017). Networks, markets, and inequality. *American Economic Review*, 107(1):1–30.

Gilboa, I. and Matsui, A. (1991). Social stability and equilibrium. *Econometrica*, 59(3):859–867.

Golub, B. and Jackson, M. O. (2010). Naïve learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics*, 2(1):112–149.

Golub, B. and Jackson, M. O. (2012). How homophily affects the speed of learning and best-response dynamics. *Quarterly Journal of Economics*, 127(3):1287–1338.

Goyal, S. and Vigier, A. (2014). Attack, defence and contagion in networks. *The Review of Economics Studies*, 81(3):1518–1542.

Hahn, Y., Islam, A., Patacchini, E., and Zenou, Y. (2020). Friendship and female education: Evidence from a field experiment in bangladeshi primary schools. *Economic Journal*, 130:740–764.

Hiller, T. (2017). Friends and enemies: A model of signed network formation. *Theoretical Economics*, 12:1057–1087.

Hu, S. and Qi, L. (2016). A necessary and sufficient condition for existence of a positive perron vector. *SIAM Journal of Matrix Analysis and Applications*, 37(4):1747–1770.

Huitsing, G., Van Duijn, M., Snijders, T., Wang, P., Sainio, M., Salmivalli, C., and Veenstra, R. (2012). Univariate and multivariate models of positive and negative networks: Liking, disliking, and bully-victim relationships. *Social Networks*, 34(4):645–657.

Islam, A., Vlassopoulos, M., Zenou, Y., and Zhang, X. (2021). Centrality-based spillover effects. CEPR Discussion Paper No. 16321. Available at http://ssrn.com/abstract=3880728.

Jackson, M., Rogers, B., and Zenou, Y. (2017). The economic consequences of social network structure. *Journal of Economic Literature*, 55:49–95.

Jackson, M. O. (2008). *Social and Economics Networks*. Princeton: Princeton University Press.

Jackson, M. O. and Storms, E. (2019). Behavioral communities and the atomic structure of networks. Unpublished manuscript, Stanford University.

Jackson, M. O. and Zenou, Y. (2015). Games on networks. In: P. Young and S. Zamir (Eds.), *Handbook of Game Theory, vol. 4,* Amsterdam: Elsevier, pages 91–157.

Jensen, M. (2010). Aggregative games and best-reply potentials. *Economic Theory*, 43:45–66.

Jensen, M. (2018). Aggregative games. In: L.C. Corchón and M.A. Marini (Eds.), *Handbook of Game Theory and Industrial Organization,* London: Edward Alger.

Jensen, M. K. (2016). Existence, uniqueness, and comparative statics in contests. In: P.V. Mouche and F. Quartieri (Eds.)*, Equilibrium Theory for Cournot Oligopolies and Related Games,* Berlin: Springer, pages 233–244.

Jia, H. (2008). A stochastic derivation of the ratio form of contest success functions. *Public Choice*, 135:125–130.

Jia, H., Skaperdas, S., and Vaidya, S. (2013). Contest functions: Theoretical foundations and issues in estimation. *International Journal of Industrial Organization*, 31:211–222.

König, M. D., Rohner, D., Thoenig, M., and Zilibotti, F. (2017). Networks in conflict: Theory and evidence from the great war of Africa. *Econometrica*, 85:1093–1132.

Konrad, K. A. (2009). *Strategy and Dynamics in Contests.* Oxford: Oxford University Press.

Kovenock, D. and Roberson, B. (2012). Conflicts with multiple battlefields. In: Michelle R. Garfinkel and Stergios Skaperdas (Eds.)*, The Oxford Handbook of the Economics of Peace and Conflict,* Oxford: Oxford University Press, pages 503–531.

Kovenock, D. and Roberson, B. (2018). The optimal defense of networks of targets. *Economic Inquiry*, 56:2195–2211.

Leister, M., Zenou, Y., and Zhou, J. (2021). Social connectedness and local contagion. *Review of Economic Studies,* forthcoming.

Lipnowski, E. and Sadler, E. (2019). Peer-confirming equilibrium. *Econometrica*, 87(2):567–591.

Manea, M. (2021). Bottleneck links, essential intermediaries, and competing paths of diffusion in networks. *Theoretical Economics*, 16:1017–1053.

Matsui, A. (1992). Best response dynamics and socially stable strategies. *Journal of Economic Theory*, 57(2):343–362.

McBride, M. (2006). Imperfect monitoring in communication networks. *Journal of Economic Theory*, 126:97–119.

Morris, S. (2000). Contagion. *Review of Economic Studies*, 67(1):57–78.

Palacios-Huerta, I. and Volij, O. (2004). The measurement of intellectual influence. *Econometrica*, 72:963–977.

Rothblum, U. (2014). Nonnegative matrices and stochastic matrices. In: L. Hogben (Ed.), *Handbook of Linear Algebra, Second Edition,* Chap. 10. CRC Press, pages 1–26.

Sadler, E. (2020). Ordinal centrality. Unpublished manuscript, Columbia University.

Skaperdas, S. (1996). Contest success functions. *Economic Theory*, 7(2):283–290.

Weibull, J. (2003). *Evolutionary Game Theory.* Cambridge: MIT Press.

Xu, J., Zenou, Y., and Zhou, J. (2019). Networks in conflict: Theory and applications. CEPR Discussion Paper No. 13647.

Zenou, Y. (2016). Key players. In: Y. Bramoullé, B.W. Rogers and A. Galeotti (Eds.), *The Oxford Handbook of the Economics of Networks,* Oxford: Oxford University Press, pages 244–274.

Appendix

A Non-negative matrices and eigenvector centrality

A.1 The Frobenius normal form

A matrix is called **nonnegative** if all its elements are nonnegative. Here we consider only nonnegative square matrices of order n, i.e., matrices that have n rows and n columns. A nonnegative matrix A is called **irreducible** if the associated directed graph is strongly connected. For convenience any one-by-one matrix is regarded as irreducible.

Lemma A1. *(Perron-Frobenius Theorem) Let $\mathbf{A}$ be an irreducible matrix. Then*

(i) *$\mathbf{A}$ has a positive eigenvalue $\rho(\mathbf{A})$ such that the value of $\rho(\mathbf{A})$ is not less than the absolute value of any other eigenvalue of $\mathbf{A}$;*

(ii) *the eigenvalue $\rho(\mathbf{A})$is simple, and corresponds to a positive eigenvector $\mathbf{x}(\mathbf{A})$;*

(iii) *any non-negative eigenvector is a multiple of $\mathbf{x}(\mathbf{A})$.*

*The vector $\mathbf{x}(\mathbf{A})$ and the number $\rho(\mathbf{A})$ that appear in this lemma are called the **Perron-Frobenius vector** and the **Perron-Frobenius eigenvalue** of $\mathbf{A}$, respectively.*

The following lemma extends some conclusions of the Perron-Frobenius Theorem to non-negative matrices (not necessarily irreducible).

Lemma A2. *Let $\mathbf{A}$ be a nonnegative matrix; then*

a) *$\mathbf{A}$ has a nonnegative eigenvalue $\rho(\mathbf{A})$ such that the value of $\rho(\mathbf{A})$ is not less than the absolute value of any other eigenvalue of $\mathbf{A}$.*

b) *To eigenvalue $\rho(\mathbf{A})$ corresponds a nonnegative eigenvector $\mathbf{x}(\mathbf{A})$.*

c) *If there exists a positive eigenvector, then it is necessarily associated to eigenvalue $\rho(\mathbf{A})$.*

Note that if $\mathbf{x}$ is a non-negative eigenvector of $\mathbf{A}$, $\mathbf{x}$ is not necessarily associated with $\rho(\mathbf{A})$. Also there could exist eigenvectors with both negative and positive entries, associated to $\rho(\mathbf{A})$.

Lemma A3. *Any nonnegative matrix* $\mathbf{A}$ *can be put in an upper-triangular block form as follows:*[1]

$$\mathbf{A} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1r} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2r} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{sr} \\ 0 & ... & ... & 0 & A_{s+1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & A_{r-1} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & A_r \end{bmatrix} \tag{A.1}$$

such that:

(*i*) *each block matrix* A_i *is square and irreducible;*

(*ii*) *for any* $i = 1, ...s$*, there exists* $j \in \{i+1, ..., r\}$ *such that the block matrix* A_{ij} *is not zero.*

This upper triangular block form is known as the **Frobenius normal form**. It is unique up to a permutation. We have $\rho(\mathbf{A}) = \max_{i=1...r} \rho(A_r)$. We call V_i the set of nodes corresponding to the block matrix A_i.

Definition A1. *A nonnegative matrix* $\mathbf{A}$ *is* ***strongly nonnegative*** *if we have*

$$\rho(A_r) = \rho(A_{r-1}) = ... = \rho(A_{s+1}) > \max_{i=1,\dots,s} \{\rho(A_i)\}$$

Obviously, any irreducible matrix is strictly nonegative because the Frobenius normal form then consists of one block. The next results can be found in Rothblum (2014) or Hu and Qi (2016).

Lemma A4. *A nonnegative matrix* $\mathbf{A}$ *admits a positive eigenvector if and only if* $\mathbf{A}$ *is strongly nonnegative.*

Note that, if $\mathbf{A}$ is an irreducible nonnegative matrix, then the conclusion of Lemma A4 directly implies point (*ii*) of Lemma A1, i.e., the Perron Frobenius Theorem.

We illustrate the Frobenius normal form for network $(N, \mathbf{G})$ displayed in Figure 2, with $N = \{1, 2, \cdots, 10\}$ and with three communities: $M_1 = \{2, 3, 4\}$, $M_2 = \{5, 6\}$, and $M_3 = \{7, 8, 9, 10\}$.

Let $\mathbf{C}(m)$ be the adjacency matrix of the complete m-agents network.[2] Keeping the

[1] Up to a permutation of indices.

[2] That is, $C(m)_{ii} = 0$, $C(m)_{ij} = 1$ for $i \neq j$

indexing of agents as it is, we have

$$\mathbf{G} = \begin{bmatrix} 0&0&0&1&0&0&1&0&0&0\\ 0&0&1&1&0&0&0&0&0&0\\ 0&1&0&1&1&1&0&0&0&0\\ 0&1&1&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0&1&1\\ 0&0&0&0&0&0&1&1&0&1\\ 0&0&0&0&0&0&1&1&1&0 \end{bmatrix} = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & A_2 & A_{23} & A_{24} \\ 0 & 0 & A_3 & A_{34} \\ 0 & 0 & 0 & A_4 \end{bmatrix},$$

where $A_1 = 0$, $A_2 = \mathbf{G}_{M_1} = \mathbf{C}(3)$, $A_3 = \mathbf{G}_{M_2} = \mathbf{C}(2)$, and $A_4 = \mathbf{G}_{M_3} = \mathbf{C}(4)$, while $A_{12} = [0\,0\,1]$, $A_{13} = [0\,0]$, $A_{14} = [1\,0\,0\,0]$, etc. In particular, A_{ij} is distinct from the null matrix, except for A_{13} (there is no link from group 1, i.e., agent 1, to community M_2, i.e., agents $\{5, 6\}$). Consequently we have $s = 3$ and $r = 4$ and $\rho(A_4) = 3$ while $\rho(A_1) = 0, \rho(A_2) = 2$ and $\rho(A_3) = 1$. Hence, $\mathbf{G}$ is strongly nonnegative and thus admits a positive eigenvector. Now, remove agent 9 from this network. Then, the Frobenius normal form has the same structure, except that $\rho(A_4) = 2 = \max_{i=1,\dots,3} \rho(A_i)$. Hence, the matrix is no longer strictly nonnegative and, thus, there is no positive eigenvector.

It might be useful to clarify the relationship between the Frobenius normal form and the $\succeq$-ordering on communities. In the Frobenius normal form of $\mathbf{G}$, any A_i corresponds to the submatrix of a strongly connected component, which can either be a community, or a singleton. Note that, by the no-isolation assumption, A_i cannot be a size one matrix for $i = s + 1, ..., r$; it then necessarily corresponds to a community for these indexes. If $M' \succ M$, then there exists some i, i' such that $i' < i$, $\mathbf{G}_M = A_i$ and $\mathbf{G}_{M'} = A_{i'}$. In other words the indexes in the Frobenius normal form are inversely ordered in accordance with the $\succeq$ ordering.

The Frobenius normal form does not help us to characterize the peer-consistent equilibria (i.e., which agents are active and which are not) but will be very useful for some of our proofs because of Lemma A4, which can be applied to any closed set, as we will see in the proof section.

A.2 Eigenvector centrality in weakly connected networks

Eigenvector centrality has been informally introduced by Bonacich (1972) to measure popularity in friendship networks. Given a weighted network $(N, \mathbf{G})$, it was originally defined as any non-negative vector $\mathbf{e}$ having the property that the centrality of agent i is proportional to the average centrality of her neighbors:

$$\lambda e_i = \sum_j \mathbf{G}_{ij} e_j, \ \forall i. \tag{A.2}$$

In the particular case of strongly connected networks, this vector is well-defined because there is a unique solution to the system (A.2), given by the eigenvector associated to the largest eigenvalue λ of $\mathbf{G}$. More generally, there is a consensus consisting in regarding eigenvector centrality as being the normalized eigenvector associated to the largest eigenvalue of the network (see e.g., Jackson (2008)).[3]

In weakly connected networks, however, eigenvector centrality cannot be defined in the same way because the largest eigenvalue of a weakly connected network is not always simple. For instance, consider the network in Figure E1 in Appendix E, where $\rho(\mathbf{G}) = 1$. The eigenspace associated to $\rho(\mathbf{G})$ is generated by normalized vectors $(1/3, 1/3, 1/3, 0, 0)$ and $(1/3, 0, 0, 1/3, 1/3)$. Hence, any convex combination of these two vectors is a non-negative eigenvector, which means that eigenvector centrality is not defined for this network.

Consequently, we focus on an (arguably large) subset of weakly connected graphs, in which the notion of eigenvector centrality can be naturally extended. A weakly connected network has *a unique dominant component* if

$$\forall M, M' \in \mathcal{C}(\mathbf{G}),\ \ \rho(\mathbf{G}_M) = \rho(\mathbf{G}_{M'}) = \rho(\mathbf{G}) \Rightarrow M \succeq M' \text{ or } \ M' \succeq M. \tag{UDC}$$

Obviously any *generic network* has a unique dominant component. A simple adaptation of the proof of Proposition D1 shows that a weakly connected network admits a unique normalized eigenvector associated to $\rho(\mathbf{G})$ if and only if it has a unique dominant component.

Definition A2 (Eigenvector centrality)**.** *Suppose that $(N, \mathbf{G})$ has a unique dominant component. Then, the eigenvector centrality of agent i is the i-th component of the normalized eigenvector associated to $\rho(\mathbf{G})$.*

In some networks, it may be the case that some agents in the network exhibit a null eigenvector centrality, and one may wonder what it means, and whether or not this definition makes sense when this happens. As we show now, this definition is indeed meaningful, because our definition of eigenvector centrality is robust to any small perturbations of the network, in the following sense:

Lemma A5. *Suppose that $(N, \mathbf{G})$ has a unique dominant component and call $\mathbf{e}$ the normalized eigenvector associated to $\rho(\mathbf{G})$. Let $(\mathbf{G}^n)_n$ be a sequence of irreducible matrices such that $\lim_{n\to+\infty} \mathbf{G}^n_{ij} = \mathbf{G}_{ij}$. Then $\mathbf{e}^n \to \mathbf{e}$, where $\mathbf{e}^n$ is the normalized eigenvector associated to $\rho(\mathbf{G}^n)$.*

In other words, the sequence of centrality measures always converge to the same vector, regardless of *how* $\mathbf{G}^n$ converges to $\mathbf{G}$. The implication of this observation is that eigenvector centrality is unambiguously defined in networks having a unique dominant component.

Observe that the network $(N, \mathbf{G})$ depicted in Figure E1 in Appendix E does not exhibit such a property; thus, defining an eigenvector centrality for such a network would imply making an arbitrary choice. Indeed, it can be shown that, for any $\lambda \in [0, 1]$, one can find

[3] meaning the eigenvector whose components sum to one.

a sequence of strongly connected weighted networks $(N, \mathbf{G}^n)$ such that $\mathbf{e}^n$ converges to $\frac{1}{3}(1, \lambda, \lambda, 1-\lambda, 1-\lambda)$.

B Proofs of all results in the main text

B.1 Proof of results in Section 3

B.1.1 Proof of results in Section 3.1

Proof of Theorem 1. Let $\mathbf{x}^* \in \mathbb{R}^n_+$ be a PCE. We first show that $X^* > 0$. Assume, by contradiction, that $X^* = 0$, i.e. $\mathbf{x}^* = \mathbf{0}$. Then for each $i \in N$, agent i's subjective utility is equal to

$$l := \lim_{x \to 0^+} x f(nx).$$

Now consider the situation where some agent i deviates and exerts some effort $\epsilon > 0$, for ϵ small, while the others $j \neq i$ each exert $x_j = 0$. Then agent i's subjective utility is equal to $\epsilon g\left(\frac{n\epsilon}{1+|\mathcal{N}_i|}\right) - c\epsilon$. However $\epsilon g\left(\frac{n\epsilon}{1+|\mathcal{N}_i|}\right) \sim_{\epsilon \to 0^+} (1+\mathcal{N}_i)l$. Since $1 + |\mathcal{N}_i| \geq 2$, for small enough ϵ, this is a profitable deviation. Therefore, $\mathbf{x}^*$ is not an equilibrium effort vector. Hence $X^* > 0$.

We next show that

$$\mathbf{G}\mathbf{x}^* = \frac{c - f(X^*) - f'(X^*)X^*}{f(X^*) - c}\mathbf{x}^* \tag{B.1}$$

in three steps:

• Suppose first that $(\mathbf{G}\mathbf{x}^*)_i = 0$. We must then show that that $x_i^* = 0$. Assume, by contradiction, that $x_i^* > 0$. Then $W_i > 0$. Since $(\mathbf{G}\mathbf{x}^*)_i = 0$, we have that $u_i(x_i, \mathbf{x}^*_{-i}; W_i) = x_i f\left(\frac{x_i}{W_i}\right) - c x_i$ for any $x_i > 0$, contradicting the fact that x_i^* maximizes $x_i \mapsto u_i(x_i, \mathbf{x}_{-i}, W_i)$.

• Let now i be such that $x_i^* > 0$. We just showed that $(\mathbf{G}\mathbf{x}^*)_i > 0$. Since

$$x_i^* = Argmax_{b_i \geq 0}\, b_i f\left(\frac{b_i + (\mathbf{G}\mathbf{x}^*)_i}{W_i}\right) - c b_i$$

and using the assumptions on f, it necessarily implies that x_i^* satisfies the first-order condition

$$f\left(\frac{x_i^* + (\mathbf{G}\mathbf{x}^*)_i}{W_i}\right) + f'\left(\frac{x_i^* + (\mathbf{G}\mathbf{x}^*)_i}{W_i}\right)\frac{x_i^*}{W_i} = c.$$

Using the consistency condition (ii) in the definition of a peer-consistent equilibrium, this translates to,

$$(\mathbf{G}\mathbf{x}^*)_i = \left(\frac{c - f(X^*) - f'(X^*)X^*}{f(X^*) - c}\right) x_i^*.$$

As a consequence, we obtain that,

$$\frac{c - f(X^*) - f'(X^*)X^*}{f(X^*) - c} > 0.$$

• Finally suppose that we have $(\mathbf{Gx}^*)_i > 0$. We must show that $x_i^* > 0$ and the proof will be complete. By condition (i) of the definition of a PCE,

$$x_i^* = Argmax_{b_i \geq 0}\; b_i f\left(\frac{b_i + (\mathbf{Gx}^*)_i}{W_i}\right) - cb_i.$$

Again, $x_i^* > 0$ if and only if the first order condition

$$(\mathbf{Gx}^*)_i = \left(\frac{c - f(X^*) - f'(X^*)X^*}{f(X^*) - c}\right) x_i^*.$$

admits a positive solution, which is the case since we proved that $\frac{c-f(X^*)-f'(X^*)X^*}{f(X^*)-c}$ is strictly positive.

We now prove the reverse implication. Suppose that $\mathbf{x}^* \in \mathbb{R}_+^n$ is different from zero and satisfies identity (B.1). For each agent i for whom $(\mathbf{Gx}^*)_i > 0$, x_i^* satisfies the first-order condition associated to the maximization problem (i) of the PCE definition, with $W_i = \frac{x_i^* + (\mathbf{Gx}^*)_i}{X^*}$. Meanwhile, for each agent i for whom $(\mathbf{Gx}^*)_i = 0$, $x_i^* = 0$ solves the optimization problem (i) of the PCE definition, with $W_i = 0$. This proves the reverse implication.

In order to conclude the proof, we finally prove existence of a peer-consistent equilibrium. A nonnegative matrix always admits a nonnegative eigenvector $\mathbf{u}$ associated to eigenvalue $\lambda > 0$ such that $\sum_i u_i = 1$, (see Lemma A2 in Appendix A.1). Let $\bar{X}$ be the (unique) positive real number such that $f(\bar{X}) = c$. Note that the map $X \in]0, \bar{X}[\mapsto \frac{c-f(X)-f'(X)X}{f(X)-c}$ can take any value in $[0, \infty[$. Hence there exists $\tilde{X} \in]0, \bar{X}[$ such that $\frac{c-f(\tilde{X})-f'(\tilde{X})\tilde{X}}{f(\tilde{X})-c} = \lambda$. Then $\tilde{X}\mathbf{u}$ is a PCE by construction. □

Lemma B6. *If* $\mathbf{x}$ *is a peer-consistent equilibrium then* $N_+(\mathbf{x})$ *is a closed set of* $(N, \mathbf{G})$*.*

Proof. Let $j \in N_+(\mathbf{x})$ and i be connected to j through a path: there exists $p \in \mathbb{N}^*$ such that $\mathbf{G}_{ij}^p > 0$. By Theorem 1 there exists $\rho > 0$ such that $\mathbf{Gx} = \rho\mathbf{x}$. We then have

$$x_i = \frac{1}{\rho^p}(\mathbf{G}^p\mathbf{x})_i \geq \frac{1}{\rho^p}\mathbf{G}_{ij}^p x_j > 0.$$

This concludes the proof. □

Proof of Remark 5. Since Remark 5 is a special case of Theorem 1, we will prove Proposition 5 as the following corollary of Theorem 1.

Corollary B1. *Let* $(N, \mathbf{G})$ *be a strongly connected network. Then, there exists a unique peer-consistent equilibrium.*

Proof. Suppose that $(N, \mathbf{G})$ is a strongly connected network. Then $\mathbf{G}$ is irreducible and, by Perron-Frobenius Theorem, there exists a positive eigenvector $\mathbf{y}$ associated to $\rho(\mathbf{G})$. Moreover any non-negative eigenvector of $\mathbf{G}$ is a multiple of $\mathbf{y}$. By Theorem 1, $\mathbf{x}^*$ is a PCE if and only if it is a non-negative eigenvector of $\mathbf{G}$, associated to eigenvalue $\frac{c-f(X^*)-f'(X^*)X^*}{f(X^*)-c}$. Hence $\mathbf{x}^*$ is a PCE if and only if $\mathbf{x}^*$ is a multiple of $\mathbf{y}$ and $\rho(\mathbf{G}) = \frac{c-f(X^*)-f'(X^*)X^*}{f(X^*)-c}$. Such a vector exists and is uniquely defined. □

B.1.2 Proof of results in Section 3.4

We start by providing some insights on the relationship between the $\succeq$ ordering and the Frobenius normal form.

Lemma B7. *Let $(N, \mathbf{G})$ be a weakly connected network. Consider its Frobenius normal form (A.1). For any $i = 1, ..., r$ either $|V_i| = 1$ or $V_i \in \mathcal{C}(\mathbf{G})$. As a consequence*

$$\rho(\mathbf{G}) = \max_{i=1,\dots,r} \rho(A_i) = \max_{M \in \mathcal{C}(\mathbf{G})} \rho\left(\mathbf{G}_M\right) \tag{B.2}$$

Proof. Suppose that $|V_i| > 1$. By construction of the Frobenius normal form, (V_i, A_i) is a strongly connected component of $(N, \mathbf{G})$. Hence V_i belongs to the set of communities $\mathcal{C}(\mathbf{G})$. Since $\rho(\mathbf{G}) = \max_{i=1\dots r} \rho(A_i)$ and $\rho(A_i) = 0$ if $|V_i| = 1$ this concludes the proof of (B.2). □

For any closed set $N' \subset N$, note that $\mathcal{C}(\mathbf{G}_{N'}) = \{M \in \mathcal{C}(\mathbf{G}) : \ M \subset N'\}$. Hence we have

$$\rho(\mathbf{G}_{N'}) = \max_{M' \in \mathcal{C}(\mathbf{G}):\, M \subseteq N'} \rho(\mathbf{G}_{M'}) \tag{B.3}$$

Lemma B8. *Suppose that $\mathbf{A}$ is a nonnegative matrix that admits a Frobenius normal form (A.1) with $r = s+1$ and $\rho(A_{s+1}) > \max_{i=1,\dots,s} \{\rho(A_i)\}$. Then $\mathbf{A}$ admits a* ***unique*** *positive eigenvector.*[4]

Proof. We only need to show that, if $\mathbf{x}$ and $\mathbf{y}$ are two positive eigenvector of $\mathbf{A}$ then $\mathbf{x} = \alpha \mathbf{y}$ for some $\alpha > 0$. We can write $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{bmatrix} A' & B \\ 0 & A_{s+1} \end{bmatrix}, \text{ where } A' = \begin{bmatrix} A_1 & A_{12} & \dots & \dots & A_{1s} \\ 0 & A_2 & A_{23} & \dots & A_{2s} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & A_s \end{bmatrix} \text{ and } B = \begin{bmatrix} A_{1s+1} \\ A_{2s+1} \\ \dots \\ \dots \\ A_{ss+1} \end{bmatrix}.$$

Let us write $\mathbf{x}$ as $(\mathbf{x}', \mathbf{x}_{[s+1]})$, according to the decomposition of $\mathbf{A}$ we just wrote and let $\rho := \rho(A_{s+1}) = \rho(\mathbf{A})$. We have

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{x}_{[s+1]} \end{bmatrix} = \rho^{-1} \begin{bmatrix} \mathbf{A}' \cdot \mathbf{x}' + \mathbf{B} \cdot \mathbf{x}_{[s+1]} \\ \mathbf{A}_{s+1} \cdot \mathbf{x}_{[s+1]} \end{bmatrix},$$

[4]Uniqueness is up to multiplication by a constant.

so that, in particular, $(\mathbf{I}-\rho^{-1}\mathbf{A}')\,\mathbf{x}' = \rho^{-1}\mathbf{B}\mathbf{x}_{[s+1]}$. Since $\rho(\mathbf{A}') < \rho$ by construction, the matrix $\mathbf{I}-\rho^{-1}\mathbf{A}'$ is invertible and we have

$$\mathbf{x}' = \rho^{-1}\left(\mathbf{I}-\rho^{-1}\mathbf{A}'\right)^{-1}\mathbf{B}\mathbf{x}_{[s+1]} \tag{B.4}$$

Now the matrix $\mathbf{A}_{s+1}$ being irreducible and $\mathbf{x}_{[s+1]}, \mathbf{y}_{[s+1]}$ both being positive eigenvectors of $\mathbf{A}_{s+1}$ we must have $\mathbf{x}_{[s+1]} = \alpha\mathbf{y}_{[s+1]}$ Since identity (B.4) holds for both $\mathbf{x}$ and $\mathbf{y}$, we obtain that $\mathbf{x}' = \alpha\mathbf{y}'$, concluding the proof. □

Proof of Proposition 1. First note that if $\mathbf{x}$ is a PCE with root M then its restriction to $\bar{M}$ is a positive eigenvector of $\mathbf{G}_{\bar{M}}$. By definition of $\bar{M}$, the matrix $\mathbf{G}_{\bar{M}}$ admits a Frobenius normal form as follows:

$$\mathbf{G}_{\bar{M}} = \begin{bmatrix} A_1 & A_{12} & ... & ... & A_{1s+1} \\ 0 & A_2 & A_{23} & ... & A_{2s+1} \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} \\ 0 & ... & ... & 0 & A_{s+1} \end{bmatrix}, \text{ with } \ A_{s+1} = \mathbf{G}_M.$$

Note that the set $V := \cup_{i=1}^{s} V_i$ is closed and, by definition of $\bar{M}$ we necessarily have $\{M' \in \mathcal{C}(\mathbf{G}) : \ M' \subset V\} = \{M' \in \mathcal{C}(\mathbf{G}) : \ M' \succ M\}$. Hence

$$\rho(\mathbf{G}_V) = \max_{i=1,\dots,s} \rho(A_i) = \max_{M' \in \mathcal{C}(\mathbf{G}):\, M' \succ M} \rho(\mathbf{G}_{M'}).$$

If $\rho(\mathbf{G}_M) > \max_{M' \in \mathcal{C}(\mathbf{G}):M' \succ M} \rho(\mathbf{G}_{M'})$ then

$$\rho(A_{s+1}) = \rho(\mathbf{G}_M) > \max_{M' \in \mathcal{C}(\mathbf{G}):M' \succ M} \rho(\mathbf{G}_{M'}) = \max_{i=1,\dots,s} \rho(A_i),$$

Consequently, we are in the conditions of Lemma B8. Thus $\mathbf{G}_{\bar{M}}$ then admits a unique positive eigenvector $\mathbf{y} = (y_i)_{i \in D_l^k}$, such that $\sum_{i \in D_l^k} y_i = \frac{V}{c}\frac{\rho}{1+\rho}$. Let then $\mathbf{x}$ be defined as $x_i = y_i$ if $i \in \bar{M}$ and $x_i = 0$ if $i \in N \setminus \bar{M}$. By construction, $\mathbf{x}$ is a PCE with root M and there can be no other one.
Now suppose that $\rho(\mathbf{G}_M) \leq \max_{M' \in \mathcal{C}(\mathbf{G}):M' \succ M} \rho(\mathbf{G}_{M'})$. Then

$$\rho(A_{s+1}) = \rho(\mathbf{G}_M) \leq \max_{M' \in \mathcal{C}(\mathbf{G}):M' \succ M} \rho(\mathbf{G}_{M'}) = \max_{i=1,\dots,s} \rho(A_i),$$

meaning that $\mathbf{G}_{\bar{M}}$ admits no positive eigenvector, by Lemma A4. This concludes the proof.
□

Proof of Corollary 1. Suppose that $\mathbf{x}$ is a non simple PCE. Then, by Propositions E4 and E5, we have $N_+(\mathbf{x}) = \cup_{i=1}^{n} \bar{M}_i$, with $n \geq 2$, $M_1, ..., M_n$ being distinct elements of $\mathcal{C}(\mathbf{G})$ and

$$\rho\left(\mathbf{G}_{M_i}\right) = \rho > \max_{M' \in \mathcal{C}(\mathbf{G}):M' \succ M_i} \rho(\mathbf{G}_{M'}), \ \ \forall i = 1, ..., n,$$

which contradicts the fact that $\mathbf{G}$ is generic. □

B.1.3 Proof of the results in Section 3.5

Let us show that the system (24) is well-behaved on the set

$$\mathbf{S} := \{\mathbf{x} \neq \mathbf{0} : x_i \geq 0 \; \forall i, \; f(X) > c\} \tag{B.5}$$

in the sense that, for any initial condition in $\mathbf{S}$, there exists a unique solution $(\mathbf{x}(t))_{t\geq 0}$ which forever remains in $\mathbf{S}$.

Given a PCE $\mathbf{x}^*$, we have $\mathbf{x}^* \in \mathbf{S}$, because the expression $\frac{c-f(X)-Xf'(X)}{f(X)-c}$ is either undefined or negative when $f(X) \leq c$. As a consequence $\mathbf{S}$ contains all the relevant states of the problem we consider. We denote by $(\phi(\mathbf{x}, t))_{\mathbf{x}\in\mathbf{S}, t\geq 0}$ the semi-flow associated to (24) on $\mathbf{S}$. Namely $\phi(\mathbf{x}, t)$ is equal to the position of the (unique) solution of (24) starting in $\mathbf{x}$.

Lemma B9. *System (24) induces a semiflow on* $\mathbf{S}$*.*

Proof. We need to check that the vector field B points inward on the boundary of $\mathbf{S}$. Suppose that $\mathbf{x} \in \mathbf{S}$, with $f(X) = c$. Then $Br_i(x_{-i}) = Argmax_{b_i \geq 0} \; b_i \left(f\left(X \frac{b_i+(\mathbf{Gx})_i}{x_i+(\mathbf{Gx})_i}\right) - c\right)$. Since $f(X) = c$, this map is equal to zero in $b_i = x_i$, negative when $b_i > x_i$ and positive when $b_i < x_i$. Hence $Br_i(x_{-i}) < x_i$ for all i such that $x_i > 0$ and $\dot{X} < 0$ in $\mathbf{x}$. ■

For the Tullock model, on the positively invariant set $\mathbf{S}$, system (24) writes

$$\dot{x}_i(t) = -x_i(t) - (\mathbf{Gx})_i(t) + \left(\frac{V}{cX(t)}(\mathbf{Gx})_i(t)\left(x_i(t) + (\mathbf{Gx})_i(t)\right)\right)^{1/2} \text{ for } i = 1, ..., N.$$

For the Cournot model we have

$$\dot{x}_i(t) = -x_i(t) - \frac{1}{2}(\mathbf{Gx})_i(t) + \frac{1}{2}(\bar{\alpha} - c)\frac{x_i(t) + (\mathbf{Gx})_i(t)}{X(t)} \text{ for } i = 1, ..., N.$$

The following result will be useful to prove that a point is not asymptotically stable. It directly follows from the definition of asymptotic stability.

Lemma B10. *Suppose that there exists an open neighborhood U_0 of $\mathbf{x}^*$ with the property that, for any open neighborhood U of $\mathbf{x}^*$ and any $T > 0$, there exists $\mathbf{x} \in U$ such that $\phi(\mathbf{x}, t) \notin U_0$, for any $t \geq T$. Then $\mathbf{x}^*$ is not asymptotically stable.*

Lemma B11. *Let $\mathbf{x}^*$ be a PCE such that $\rho(\mathbf{x}^*) < \rho(\mathbf{G})$. Then $\mathbf{x}^*$ is not asymptotically stable.*

Proof. Recall that $\mathbf{x}^*$ is an eigenvector of $\mathbf{G}$, associated to eigenvalue $\rho(\mathbf{x}^*)$, given by

$$\rho(\mathbf{x}^*) = \frac{c - f(X^*) - f'(X^*)X^*}{f(X^*) - c}.$$

In what follows, let $\rho^* := \rho(\mathbf{x}^*)$ and $\rho := \rho(\mathbf{G})$.

Let $M^* \in \mathcal{C}$ be the root of $\mathbf{x}^*$. For any $M \succeq M^*$ we necessarily have $\rho(\mathbf{G}_M) < \rho^*$. Let $C := N \setminus \bar{M}^*$. By construction, $\mathbf{G}_C$ is a nonnegative matrix with largest eigenvalue ρ, and we call $\mathbf{u}$ the eigenvector associated to ρ, whose components sum to one.
Define $\mathbf{a}^\epsilon$ as follows:

$$\mathbf{a}_i^\epsilon = \epsilon u_i \; \forall i \in C, \quad \text{and} \quad \mathbf{a}_i^\epsilon = x_i^* \; \forall i \in \bar{M}^*,$$

where ϵ is a positive number. We claim that, for any $i \in C$, $B_i(\mathbf{a}^\epsilon) > 0$. By definition of C, we have $g_{ij} = 0$ for any $i \in C$ and any $j \in \bar{M}^*$. Consequently

$$(\mathbf{G}\mathbf{a}\epsilon)_i = \sum_{j \in C} g_{ij}\mathbf{a}_j^\epsilon = (\mathbf{G}_C\mathbf{a}^\epsilon)_i = \rho\epsilon u_i.$$

Define, for $i \in C$ and $b_i > 0$,

$$H_i^\epsilon(b_i) := g\left(\frac{A^\epsilon(b_i + \rho\epsilon u_i)}{(1+\rho)\epsilon u_i)}\right) + \frac{b_i A^\epsilon}{(1+\rho)\epsilon u_i} g'\left(\frac{A^\epsilon(b_i + \rho\epsilon u_i)}{(1+\rho)\epsilon u_i)}\right) - c.$$

where $A(\epsilon) = \sum_i a(\epsilon)_i = X^* + \epsilon$. Then $Br_i(\mathbf{a}^\epsilon)$ is the unique zero of H_i, and $H_i(b_i) < 0 \; \forall b_i > Br_i(\mathbf{a}^\epsilon)$ (resp. $H_i(b_i) > 0 \; \forall b_i < Br_i(\mathbf{a}^\epsilon)$). We have

$$H_i^\epsilon(a_i^\epsilon) = f(A^\epsilon) + \frac{A^\epsilon}{1+\rho} f'(A^\epsilon) - c.$$

Since $\rho > \rho^* = \frac{c - f(X^*) - f'(X^*)X^*}{f(X^*) - c}$, we have

$$f(X^*) + \frac{X^*}{1+\rho} f'(X^*) - c > 0.$$

Hence, for small enough $\epsilon > 0$, we have $H_i^\epsilon(a_i^\epsilon) > 0$, and thus $\mathbf{a}_i^\epsilon < Br_i(\mathbf{a}^\epsilon)$, i.e. $B_i(\mathbf{a}^\epsilon) > 0$. This concludes the proof that $\mathbf{x}^*$ is not asymptotically stable for dynamics (24).

We now prove the following lemma, which completes the proof of Theorem 2:

Lemma B12. *Let $\mathbf{x}^*$ be a PCE such that $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$. Then $\mathbf{x}^*$ is asymptotically stable.*

Proof. We prove this lemma in the particular case of Tullock contest. The proof is similar in the case of Cournot contest. However, writing a general proof without relying on the explicit formulation of map $f(\cdot)$ would be extremely tedious and lengthy. We believe that illustrating the spirit of the proof on a concrete example is more illminating. If $\mathbf{B}(.)$ in (24) is differentiable in an open neighborhood of a PCE, then a simple sufficient condition for an interior equilibrium to be asymptotically stable is that the eigenvalues of the Jacobian matrix of $B(.)$, evaluated at $\mathbf{x}^*$, have negative real parts. Unfortunately the map $\mathbf{B}$ is not differentiable at a non-interior PCE, and we then cannot use this result. However we can compute the directional derivatives of $\mathbf{B}$ at any PCE: let $\mathbf{u} \neq \mathbf{0}$ be such that $u_i \geq 0 \; \forall i$. Then the directional derivative of $\mathbf{B}$ in $\mathbf{x}^*$ along $\mathbf{u}$, namely the quantity

$$D_{\mathbf{u}}\mathbf{B}(\mathbf{x}^*) := \lim_{h \to 0, h > 0} \frac{\mathbf{B}(\mathbf{x}^* + h\mathbf{u})}{h}$$

exists, and we can compute it: given $h > 0$,

$$B_i(\mathbf{x}^*+h\mathbf{u}) = -(\mathbf{x}_i^*+hu_i+(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)+\left(\frac{V}{c(X^*+hU)}(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)(x_i^*+hu_i+(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)\right)^{1/2}$$

The term in the square root can be written

$$\begin{aligned}
& \frac{V}{cX^*}\left(1-h\frac{U}{X^*}\right)[(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)+h\left[(\mathbf{G}\mathbf{x}^*)_i(u_i+(\mathbf{G}\mathbf{u})_i)+(\mathbf{G}\mathbf{u})_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\right]]+\mathcal{O}(h^2) \\
= \; & \frac{V}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\left(1-h\frac{U}{X^*}\right)\left[1+h\left[\frac{u_i+(\mathbf{G}\mathbf{u})_i}{x_i^*+(\mathbf{G}\mathbf{x}^*)_i}+\frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right]+\mathcal{O}(h^2) \\
= \; & \frac{V}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\left[1+h\left[-\frac{U}{X^*}+\frac{u_i+(\mathbf{G}\mathbf{u})_i}{x_i^*+(\mathbf{G}\mathbf{x}^*)_i}+\frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right]+\mathcal{O}(h^2)
\end{aligned}$$

Observing that $(\frac{V}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i))^{1/2} = x_i^*+(\mathbf{G}\mathbf{x}^*)_i$, the square root of the above quantity is equal to

$$(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\left[1+\frac{h}{2}\left[\frac{-U}{X^*}+\frac{u_i+(\mathbf{G}\mathbf{u})_i}{x_i^*+(\mathbf{G}\mathbf{x}^*)_i}+\frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right]+\mathcal{O}(h^2)$$

Hence, since $(x_i^*+(\mathbf{G}\mathbf{x}^*)_i) = \frac{V}{V-cX^*}x_i^*$, we obtain

$$\begin{aligned}
B_i(\mathbf{x}^*+h\mathbf{u}) &= -(hu_i+h(\mathbf{G}\mathbf{u})_i)+\frac{h}{2}\left[\frac{-UV}{X^*(V-cX^*)}x_i^*+(u_i+(\mathbf{G}\mathbf{u})_i)+\frac{V}{cX^*}(\mathbf{G}\mathbf{u})_i\right]+\mathcal{O}(h^2) \\
&= \frac{h}{2}\left[\frac{-UV}{X^*(V-cX^*)}x_i^*-u_i+\frac{V-cX^*}{cX^*}(\mathbf{G}\mathbf{u})_i\right]+\mathcal{O}(h^2)
\end{aligned}$$

Consequently

$$\lim_{h\to+\infty,h>0}\frac{B_i(\mathbf{x}^*+h\mathbf{u})}{h} = \frac{1}{2}\left[\frac{-UV}{X^*(V-cX^*)}x_i^*-u_i+\frac{V-cX^*}{cX^*}(\mathbf{G}\mathbf{u})_i\right] = \frac{1}{2}\left(\mathbf{D}F(\mathbf{x}^*)\mathbf{u}\right)_i,$$

which proves that

$$D_{\mathbf{u}}\mathbf{B}(\mathbf{x}^*) = \frac{1}{2}\left(-I_N+\frac{1+\rho(\mathbf{x}^*)}{X^*}L(\mathbf{x}^*)+\frac{1}{\rho(\mathbf{x}^*)}\mathbf{G}\right)\cdot\mathbf{u},$$

where $\mathbf{L}(\mathbf{x}^*)$ is the matrix where every column is equal to $\mathbf{x}^*$.

Let $\mathbf{D}(\mathbf{x}^*) := \frac{1}{2}\left(-I_N+\frac{1+\rho(\mathbf{x}^*)}{X^*}L(\mathbf{x}^*)+\frac{1}{\rho(\mathbf{x}^*)}\mathbf{G}\right)$. We first show that all eigenvalues of $\mathbf{D}(\mathbf{x}^*)$ have a negative real part. Suppose that $\mathbf{D}(\mathbf{x}^*)\cdot\mathbf{u} = \lambda\cdot\mathbf{u}$, with $\mathbf{u}\neq 0$. Call $U := \sum_{i\in N}u_i$. Then we have

$$-\mathbf{u}-\frac{1+\rho}{X^*}U\mathbf{x}^*+\frac{1}{\rho}\mathbf{G}\mathbf{u} = 2\lambda\mathbf{u}$$

which gives

$$\left(\mathbf{I}_N-\frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)\mathbf{u} = -\frac{1+\rho}{X^*(1+2\lambda)}U\mathbf{x}^*.$$

Suppose that $Re(\lambda) > 0$ or that λ is pure imaginary. Then $|1+\lambda| > 1$ and the matrix $\mathbf{G}/(\rho(1+2\lambda))$' spectral radius is strictly smaller than one. As a consequence $\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}$ is invertible and

$$\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)^{-1} = \sum_{p=0}^{+\infty} \frac{1}{\rho^p(1+2\lambda)^p}\mathbf{G}^p.$$

Consequently

$$\begin{aligned}
\mathbf{u} &= -\frac{1+\rho}{X^*(1+2\lambda)} U \left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)^{-1} \mathbf{x}^* \\
&= -\frac{1+\rho}{X^*(1+2\lambda)} U \sum_{p=0}^{+\infty} \frac{1}{\rho^p(1+2\lambda)^p}\mathbf{G}^p \mathbf{x}^* \\
&= -\frac{1+\rho}{X^*(1+2\lambda)} U \sum_{p=0}^{+\infty} \frac{1}{(1+2\lambda)^p}\mathbf{x}^* \\
&= -\frac{1+\rho}{2X^*\lambda} U \mathbf{x}^*
\end{aligned}$$

Since $\mathbf{u} \neq 0$, this equality implies that $U \neq 0$ and summing the coordinates of $\mathbf{u}$ we obtain that $2\lambda = -(1+\rho) < 0$, a contradiction.
Suppose now that $\lambda = 0$. Then we have

$$\left(\mathbf{I}_N - \frac{1}{\rho}\mathbf{G}\right)\mathbf{u} = -\frac{1+\rho}{X^*} U \mathbf{x}^*.$$

Suppose that $U \neq 0$. Then, multiplying both sides of the equality by $\sum_{k=0}^{K} \frac{1}{\rho^k}\mathbf{G}^k$, we obtain the identity

$$\left(\mathbf{I}_N - \frac{1}{\rho^{K+1}}\mathbf{G}^{K+1}\right)\mathbf{u} = -\frac{1+\rho}{X^*} U \sum_{k=0}^{K} \frac{1}{\rho^k}\mathbf{G}^k \mathbf{x}^* = -\frac{1+\rho}{X^*} U K \mathbf{x}^*$$

The modulus of the left-hand is bounded above by $2|\mathbf{u}|$, while the modulus of the right-hand side term grows to infinity with K, which is a contradiction. Hence $U = 0$. This means that

$$\mathbf{G}\mathbf{u} = \rho\mathbf{u},$$

i.e. that $\mathbf{u}$ is in fact an eigenvector associated to the largest eigenvalue of $\mathbf{G}$. Since $\sum_i u_i = 0$, this contradicts the fact that $(N, \mathbf{G})$ is a generic network.
We proved that the real part of every eigenvalue of $\mathbf{D}F(x^*)$ is strictly negative.

As we proved above, for $\mathbf{x} \in \mathbf{X}$, we have

$$\mathbf{B}(\mathbf{x}) = \mathbf{D}(\mathbf{x}^*) \cdot (\mathbf{x} - \mathbf{x}^*) + \|\mathbf{x} - \mathbf{x}^*\|^2 g(\|\mathbf{x} - \mathbf{x}^*\|)$$

Denote by $(\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_P, ..., \lambda_P)$ the eigenvalues of $\mathbf{D}(\mathbf{x}^*)$, and call n_p the multiplicity of eigenvalue λ_p. Let us first put $\mathbf{D}(\mathbf{x}^*)$ in its Jordan form:

$$\mathbf{D}F(\mathbf{x}^*) = \mathbf{PJP}^{-1},$$

where $\mathbf{J}$ is diagonal by blocks, i.e.

$$\mathbf{J} = Diag\left(\mathbf{J}_1, ..., \mathbf{J}_P\right) := \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & ... & \mathbf{0} \\ ... & ... & ... & ... \\ \mathbf{0} & ... & \mathbf{0} & \mathbf{J}_P \end{pmatrix}, \quad \text{with } \mathbf{J}_p = \begin{pmatrix} \lambda_p & 1 & 0 & ... & 0 \\ 0 & \lambda_p & 1 & ... & 0 \\ ... & ... & ... & ... & \\ 0 & ... & 0 & \lambda_p & 1 \\ 0 & ... & ... & 0 & \lambda_p \end{pmatrix}$$

Define now $\mathbf{Q} := Diag\left(\mathbf{Q}_1, ..., \mathbf{Q}_P\right)$, with $\mathbf{Q}_p = Diag(1, \epsilon, ..., \epsilon^{n_p - 1})$. We then have

$$\mathbf{Q}_p^{-1}\mathbf{J}_p\mathbf{Q}_p = \begin{pmatrix} \lambda_p & \epsilon & 0 & ... & 0 \\ 0 & \lambda_p & \epsilon & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & \lambda_p & \epsilon \\ 0 & ... & ... & 0 & \lambda_p \end{pmatrix}$$

Thus, defining $\mathbf{R} := \mathbf{PQ}$ we obtain

$$\mathbf{R}^{-1}\mathbf{D}(\mathbf{x}^*)\mathbf{R} = \mathbf{Q}^{-1}\mathbf{JQ} = \mathbf{D}(\lambda) + \epsilon\mathbf{B},$$

where $\mathbf{D}(\lambda)$ is the diagonal matrix filled with the eigenvalues of $\mathbf{D}(\mathbf{x}^*)$.
Now define $V : \mathbf{S} \to \mathbb{R}^+$ as follows:

$$V(\mathbf{x}) := \left|\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*)\right|^2 = \left\langle \mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*) \right\rangle$$

We have

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \left\langle \mathbf{R}^{-1}\dot{\mathbf{x}} \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*) \right\rangle + \left\langle \overline{\mathbf{R}^{-1}}\dot{\mathbf{x}} \mid \mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \right\rangle \\ &= \left\langle (\mathbf{D}(\lambda) + \epsilon\mathbf{B})\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*) \right\rangle + \left\langle (\overline{\mathbf{D}(\lambda)} + \epsilon\overline{\mathbf{B}})\overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*) \mid \mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \right\rangle \\ &+ \|\mathbf{x} - \mathbf{x}^*\|^2 h(\|\mathbf{x} - \mathbf{x}^*\|), \end{aligned}$$

where $h(a) \to_{a \to 0} 0$. Hence we have

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \left\langle (\mathbf{D}(\lambda) + \overline{\mathbf{D}(\lambda)})\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*) \right\rangle + 2\epsilon Re\left(\left\langle \mathbf{B}\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*) \right\rangle \right) \\ &+ \|\mathbf{x} - \mathbf{x}^*\|^2 h(\|\mathbf{x} - \mathbf{x}^*\|) \end{aligned}$$

Let $\alpha := \max_{p=1,...,P} Re(\lambda_p) < 0$. We have

$$\left\langle (\mathbf{D}(\lambda) + \overline{\mathbf{D}(\lambda)})\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x} - \mathbf{x}^*) \right\rangle \leq 2\alpha|\mathbf{R}^{-1}(\mathbf{x} - \mathbf{x}^*)|^2 = 2\alpha V(\mathbf{x}).$$

As a consequence, choosing ϵ small enough and $\mathbf{x}$ close enough of $\mathbf{x}^*$ we obtain that

$$\dot{V}(\mathbf{x}) \leq \alpha V(\mathbf{x}),$$

which proves that $V(\mathbf{x}(t))$ goes to zero exponentially fast, as t goes to infinity, and this concludes the proof. □

B.2 Proofs of results in Section 4

Proof of Proposition 2. Let M be the root of $\mathbf{x}^*$, meaning that $N_+(\mathbf{x}^*) = \bar{M}$, and $\rho := \rho(\mathbf{G}_{\bar{M}}) = \rho(\mathbf{G})$. The network $\hat{\mathbf{G}}$ also has a unique dominant component, $\hat{M}$. Either $\hat{M} = M$[5], or $\hat{M}$ is a community which did not exist in $\mathbf{G}$. When it is the case, we have $i, j \in \hat{M}$, $\hat{M} \subset \bar{M}$ and $\rho(\mathbf{G}_{\hat{M}}) \geq \rho$. Hence there is a unique stable equilibrium $\hat{\mathbf{x}}$ (with root $\hat{M}$) in $\hat{\mathbf{G}}$, $N_+(\hat{\mathbf{x}}) \subset N_+(\mathbf{x})$ and $\hat{\rho} := \rho(\hat{\mathbf{G}}_{\hat{M}}) = \rho(\hat{\mathbf{G}})$.

We now prove that (ii) holds. Note that $\hat{\mathbf{x}}^* \neq \mathbf{x}^*$: suppose by contradiction that $\mathbf{x}^* = \hat{\mathbf{x}}^*$. Let $k \neq i$ with $k \in N^+(\mathbf{x}^*)$. Then $\rho x_k^* = (\mathbf{G}\mathbf{x}^*)_k = (\hat{\mathbf{G}}\mathbf{x}^*)_k = \hat{\rho}\hat{x}_v^*$, implying that $\rho = \hat{\rho}$. Thus $\rho x_i^* = (\mathbf{G}\mathbf{x}^*)_i = (\hat{\mathbf{G}}\mathbf{x}^*)_i - x_j^* = \rho x_i^* - x_j^*$, a contradiction. Consider the following subsets of agents:

$$K_+ := \left\{ k \in \hat{M} : \frac{\hat{x}_k^*}{x_k^*} \geq \frac{\hat{x}_l^*}{x_l^*} \ \forall l \in \hat{M} \right\}, \ K_- := \left\{ k \in \hat{M} : \frac{\hat{x}_k^*}{x_k^*} \leq \frac{\hat{x}_l^*}{x_l^*} \ l \in \hat{M} \right\}.$$

We actually prove a stronger property, namely that $i \in K_+$. Note that if $k \neq i$ and $k \in K_+$ then $\frac{\hat{x}_k^*}{x_k^*} = \frac{\sum_{w \in \mathcal{N}_k} \hat{x}_w^*}{\sum_{w \in \mathcal{N}_k} x_w^*}$. Hence $w \in K_+$ for all $w \in \mathcal{N}_k$. By a recursive argument this implies that, if k is connected to w through a path then $w \in K_+$. The same property also holds for K_-. As a consequence $i \in K_+ \cup K_-$. If this were not the case there would exist two nodes $k_+ \neq i$ and $k_- \neq i$ such that $k_+ \in K_+$ and $k_- \in K_-$, which would imply that elements of M belong to both K_+ and K_-, a contradiction.
Suppose first that we are in the case where $\hat{\rho} > \rho$, and let $k \neq i$. Suppose that $k \in K_+$. Then

$$\frac{1}{\hat{\rho}} = \frac{\hat{x}_k^*}{(\hat{\mathbf{G}}\hat{\mathbf{x}}^*)_k} = \frac{\hat{x}_k^*}{\sum_{w \in \mathcal{N}_k} \hat{x}_w^*} \geq \frac{x_k^*}{\sum_{w \in \mathcal{N}_k} x_w^*} = \frac{x_k^*}{(\mathbf{G}\mathbf{x}^*)_k} = \frac{1}{\rho},$$

a contradiction. Hence $K_+ = \{i\}$.
Suppose now that $\hat{\rho} = \rho$. Showing that $i \in K_+$ is equivalent to showing that $i \notin K_-$. Suppose by contradiction that $i \in K_-$. Then

$$\frac{1}{\rho} = \frac{\hat{x}_i^*}{(\hat{\mathbf{G}}\hat{\mathbf{x}}^*)_i} = \frac{\hat{x}_i^*}{\sum_{w \in \mathcal{N}_i} \hat{x}_w^* + \hat{x}_j^*} < \frac{\hat{x}_i^*}{\sum_{w \in \mathcal{N}_i} \hat{x}_w^*} \leq \frac{x_i^*}{\sum_{w \in \mathcal{N}_i} x_w^*} = \frac{x_i^*}{(\mathbf{G}\mathbf{x}^*)_i} = \frac{1}{\rho},$$

where the strict inequality follows from the fact that $j \in N_+(\hat{\mathbf{x}})$ (see above). This is a contradiction. Thus $i \in K_+$. $\square$

[5]If, for instance there is no path from j to i.

C Cournot with non-linear demand

Consider a standard homogeneous good Cournot oligopoly game on a network with n firms competing in quantities but with a demand that is not given by (8) but by the following non-linear demand:

$$p = (\bar{\alpha} - h(X))_+ ,$$

where $h : [0, +\infty($ is non-decreasing, such that $h(0) = 0$, $\lim_{x \to +\infty} h(x) \geq \overline{\alpha}$, and $x \in]0, +\infty[\mapsto xh\left(\frac{x+z}{W}\right)$ is quasi-concave for any $z \geq 0, W > 0$. Hence, firm i's perceived utility can be written as

$$u_i(x_i, \mathbf{x}_{-i}; W_i) = \left[\bar{\alpha} - h\left(\frac{x_i + \sum_j g_{ij}x_j}{W_i}\right)\right] x_i - cx_i.$$

Again, the map f satisfies the assumptions required in Section 2.1. That is, $f(0) = \overline{\alpha} > c$, $\lim_{y \to +\infty} f(y) = 0 < c$ and $x \mapsto x\left(\overline{\alpha} - h\left(\frac{x+z}{W}\right)\right)_+$ is quasiconcave. Finally, $\lim_{x \to 0^+} xf(x) = 0$.

First, given W_i, each firm i chooses quantity x_i^* that maximizes her perceived utility. This leads to:

$$\bar{\alpha} - h\left(\frac{x_i + \sum_j g_{ij}x_j}{W_i}\right) - \frac{x_i}{W_i} h'\left(\frac{x_i + \sum_j g_{ij}x_j}{W_i}\right) = c.$$

Second, Definition 2(ii) requires that quantity choices are consistent at a PCE by imposing that

$$W_i = \frac{x_i^* + \sum_j g_{ij}x_j^*}{\sum_j x_j^*}.$$

By plugging this value in the FOC above, we obtain:

$$\bar{\alpha} - h(X^*) - \frac{x_i^* X^*}{x_i^* + \sum_j g_{ij}x_j^*} h'(X^*) = c.$$

or equivalently

$$\sum_j g_{ij}x_j^* = \left(\frac{h(X^*) + X^* h'(X^*) - \bar{\alpha} + c}{\bar{\alpha} - c - h(X^*)}\right) x_i^*.$$

In matrix form, we have

$$\mathbf{G}\mathbf{x}^* = \left(\frac{h(X^*) + X^* h'(X^*) - \bar{\alpha} + c}{\bar{\alpha} - c - h(X^*)}\right) \mathbf{x}^*, \quad \text{and} \quad \mathbf{x}^* \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}.$$

It is easily verified that, when $h(X) = X$, we obtain the result with linear-demand (equation (16)).

D Additional results

D.1 Peer-confirming equilibria

Corollary D2. *If the network $(N, \mathbf{G})$ is a semi-connected network then there are at most n peer-consistent equilibria, where n is the number of communities.*

Proof of Corollary D2. If the network is semi-connected then the communities are totally ordered: $M_1 \succ M_2 \succ ... \succ M_n$. Hence the number of PCE is equal to

$$Card \left\{ s = 1, ..., n : \ \rho(\mathbf{G}_{M_s}) > \max_{k=1,...,s-1} \rho(\mathbf{G}_{M_k}) \right\}.$$

$\square$

Proposition D1. *Let $(N, \mathbf{G})$ be a weakly-connected network. The following are equivalent:*

(i) *The set of peer-consistent equilibria is finite.*

(ii) *For any pair $(\mathbf{x}^{1*}, \mathbf{x}^{2*})$ of peer-consistent equilibria, $\rho\left(\mathbf{G}_{N_+(\mathbf{x}^{1*})}\right) \neq \rho\left(\mathbf{G}_{N_+(\mathbf{x}^{2*})}\right)$.*

(iii) *$(N, \mathbf{G})$ is a generic network.*

Proof of Proposition D1. $(i) \Rightarrow (ii)$: suppose that (ii) does not hold. Then there exists two PCE $\mathbf{x}_1, \mathbf{x}_2$ such that $\rho(\mathbf{x}_1) = \rho(\mathbf{x}_2) =: \rho$. For $\lambda \in [0, 1]$ and define $\mathbf{x}^\lambda := \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$. Then $X^\lambda = X_1 = X_2$. Hence

$$\mathbf{G}\mathbf{x}^\lambda = \lambda \mathbf{G}\mathbf{x}_1 + (1-\lambda)\mathbf{G}\mathbf{x}_2 = \lambda\rho\mathbf{x}_1 + (1-\lambda)\rho\mathbf{x}_2 = \frac{cX}{V - cX}\mathbf{x}^\lambda,$$

and $\mathbf{x}^\lambda$ is a PCE. Thus there is a continuum of PCE, contradicting (i).

$(ii) \Rightarrow (i)$: this implication follows from the fact that the set of eigenvalues of subgraphs of $\mathbf{G}$ is finite.

$(ii) \Rightarrow (iii)$: Suppose that (iii) does not hold. Then there exists M_1, M_2 such that $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_1'})$, $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_1} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_1})$ and $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_2} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_2})$. The last two strict inequalities mean that there exists a PCE with root M_1, and a PCE with root M_2, contradicting (ii).

$(iii) \Rightarrow (ii)$: Assume that (ii) does not hold, and let M_1 (resp. M_2) be the root of $\mathbf{x}_1$ (resp. $\mathbf{x}_2$). Being both PCE, it follows that we have $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_1} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_1})$ and $\max_{M' \in \mathcal{C}(\mathbf{G}): M' \succ M_2} \rho(\mathbf{G}_{M'}) < \rho(\mathbf{G}_{M_2})$, contradicting (iii).

Finally we obtain $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and the proof is complete. $\square$

D.2 Policy interventions

D.2.1 Key players

Proposition D2. *Consider the (linear) Tullock contest game. Let* $\mathbf{x}^*$ *be the (unique) asymptotically stable equilibrium of the generic network* $(N, \mathbf{G})$ *and* $\widehat{\mathbf{x}}^*$ *the (unique) asymptotically stable equilibrium of the generic network* $(N\backslash\{i\}, \mathbf{G}_{N\backslash\{i\}})$*. Then,* $\widehat{X}^* \leq X^*$.

Proof of Proposition D2. We have $X^* = \frac{V\rho(\mathbf{G})}{c[1+\rho(\mathbf{G})]}$ and $\widehat{X}^* \leq \frac{V\rho(\mathbf{G}_{N\backslash\{i\}})}{c[1+\rho(\mathbf{G}_{N\backslash\{i\}}]}$. By standard results, $\rho(\mathbf{G}) \geq \rho(\mathbf{G}_{N\backslash\{i\}})$. Hence $\widehat{X}^* \leq X^*$. □

D.2.2 Social mixing

Proposition D3. *Consider the (linear) Tullock contest game. Let* $(N^1, \mathbf{G^1})$ *and* $(N^2, \mathbf{G}^2)$ *be two generic networks endowed with resources equal to* V_1 *and* V_2*, respectively. Let* $\mathbf{x}^{1*}$ *(resp.* $\mathbf{x}^{2*}$*) be the unique stable PCE of* $(N^1, \mathbf{G^1})$ *(resp.* $(N^2, \mathbf{G}^2)$*), with root* M_1 *(resp.* M_2*).Let also* $(N, \mathbf{G})$ *be the network obtained from* $(N^1, \mathbf{G^1})$ *and* $(N^2, \mathbf{G}^2)$ *in which* $N = N^1 \cup N^2$*,* $V = V^1 + V^2$*, with* $g_{ij} = 1$ *and* $g_{k\ell} = 1$ *for some* $(i, \ell) \in M_1$*,* $(j, k) \in M_2$*. Then, there is a unique stable PCE* $\mathbf{x}^*$ *of* $(N, \mathbf{G})$ *satisfying* $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$*, and* $X^* > X^{1*} + X^{2*}$.

Proof of Proposition D3. We have

$$X^1 = \frac{V^1}{c}\frac{\rho(\mathbf{G}^1)}{\rho(\mathbf{G}^1)+1}; \;\; X^2 = \frac{V^2}{c}\frac{\rho(\mathbf{G}^2)}{\rho(\mathbf{G}^2)+1}; \;\; X = \frac{V^1+V^2}{c}\frac{\rho(\mathbf{G})}{\rho(\mathbf{G})+1}$$

We have $\rho(\mathbf{G}) = \rho(M_1 \cup M_1) > \max\{\rho(\mathbf{G}^1), \rho(\mathbf{G}^2)\}$. Hence

$$X^1 + X^2 = \frac{V^1}{c}\frac{\rho(\mathbf{G}^1)}{\rho(\mathbf{G}^1)+1} + \frac{V^2}{c}\frac{\rho(\mathbf{G}^2)}{\rho(\mathbf{G}^2)+1}\frac{V^1+V^2}{c} < \frac{\rho(\mathbf{G})}{\rho(\mathbf{G})+1} = X.$$

□

E Beyond generic graphs

In this section we assume that $\mathbf{G}$ is a weakly connected directed graph satisfying the no isolation assumption. As mentioned in the text, the PCE set can be infinite if we drop the genericity assumption.

E.1 Structure of the equilibrium set

We say that two distinct communities M_1 and M_2 are **disconnected** if neither $M_1 \succ M_2$ nor $M_2 \succ M_1$.

Proposition E4. *Let $\mathbf{x}^*$ be a PCE. Then, there exists a family of pairwise disconnected communities $\{M_i\}_{i=1,\ldots,n}$ such that*

$$N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i. \tag{E.1}$$

Proof of Proposition E4. Since $N_+(\mathbf{x}^*)$ is a closed set of $\mathbf{G}$, we have that $\mathbf{x}^*$ is a positive eigenvector of $\mathbf{G}_{N_+(\mathbf{x})}$, associated to eigenvalue $\rho > 0$. By Lemma A4, that implies that $\mathbf{G}_{N_+(\mathbf{x})}$ is strongly nonnegative, and thus can be written

$$\mathbf{G}_{N_+(\mathbf{x})} = \begin{bmatrix} A_1 & A_{12} & \ldots & \ldots & \ldots & \ldots & \ldots & A_{1r} \\ 0 & A_2 & A_{23} & \ldots & \ldots & \ldots & \ldots & A_{2r} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & A_s & A_{ss+1} & \ldots & \ldots & A_{sr} \\ 0 & \ldots & \ldots & 0 & A_{s+1} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & A_{r-1} & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & A_r \end{bmatrix} \tag{E.2}$$

where $r > s$, $\rho(A_r) = \ldots = \rho(A_{s+1}) = \rho$, and $\rho(A_i) < \rho$ for $i = 1, \ldots, s$. Each A_{s+i} being such that $|V_{s+i}| \geq 2$ for $i = 1, \ldots, r-s$, we have $V_{s+i} \in \mathcal{C}(\mathbf{G})$. Hence, taking $n := r - s$, there exists $M_1, \ldots, M_n \in \mathcal{C}(\mathbf{G})$ such that $A_{s+i} = \mathbf{G}_{M_i}$ for $i = 1, \ldots, n$.
We now show that $N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i$. Since $N_+(\mathbf{x}^*)$ is closed and $M_i \subset N_+(\mathbf{x})$ we have $\bar{M}_i \subset N_+(\mathbf{x}^*)$. Hence $\cup_{i=1}^n \bar{M}_i \subset N_+(\mathbf{x})$. Now pick $j \in N_+(\mathbf{x}^*)$. By property (*ii*) of the Frobenius normal form (see Definition A3), there exists some $i \in \{1, \ldots, n\}$ such that $j \rightrightarrows M_i$, meaning that $j \in \bar{M}_i$. This concludes the proof. $\square$

Proposition E5. *Let $(M_i)_{i=1,\ldots,n}$ be a family of pairwise disconnected communities. There exists a peer-consistent equilibrium (PCE) $\mathbf{x}^*$ with $N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i$ if and only if*

$$\rho(\mathbf{G}_{M_1}) = \ldots = \rho(\mathbf{G}_{M_n}) > \max_{i=1,\ldots,n} \max_{M' \in \mathcal{C}(\mathbf{G}):\, M' \succ M_i} \rho(\mathbf{G}_{M'}). \tag{E.3}$$

Proof of Proposition E5. The Frobenius normal form of $\mathbf{G}_{\cup_{i=1}^n \bar{M}_i}$ can be written as

$$\mathbf{G}_{\cup_{i=1}^n \bar{M}_i} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1s+n} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2s+n} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{ss+n} \\ 0 & ... & ... & 0 & \mathbf{G}_{M_1} & 0 & ... & 0 \\ 0 & ... & ... & .... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & \mathbf{G}_{M_{n-1}} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & \mathbf{G}_{M_n} \end{bmatrix}. \tag{E.4}$$

By Lemma A4, this matrix admits a positive eigenvector (and therefore there exists a PCE $\mathbf{x}^*$ such that $N_+(\mathbf{x}^*) = \cup_{i=1}^n \bar{M}_i$) if and only if

$$\rho\left(\mathbf{G}_{M_1}\right) = ... = \rho\left(\mathbf{G}_{M_n}\right) > \max_{i=1,...,s} \rho(A_i).$$

Note that $V = \cup_{i=1,...,s} V_i$ is a closed set and thus

$$\{M' \in \mathcal{C}(\mathbf{G}) : \ M' \subset V\} = \{M' \in \mathcal{C}(\mathbf{G}) : M' \succ M_i \ \text{ for some } i = 1, ..., n\}.$$

Hence

$$\max_{i=1,...,s} \rho(A_i) = \rho\left(\mathbf{G}_V\right) = \max_{i=1,...,n} \max_{M' \in \mathcal{C}(\mathbf{G}):\, M' > M_i} \rho\left(\mathbf{G}_{M'}\right)$$

This concludes the proof. $\square$

In full generality, even if the set of peer-confirming equilibria is no longer finite, we can still describe it in a simple way; it is always a finite union of convex sets. Recall that the set of simple equilibria is finite: there is at most one PCE with root M, for $M \in \mathcal{C}$. Let $\{\rho_1, ..., \rho_P\}$ be the set of positive eigenvalues of $\mathbf{G}$. The set of simple equilibria can be written as

$$\bigcup_{p=1}^{P} S_p, \quad \text{where } S_p := \{\mathbf{x}^* : \ \mathbf{x}^* \ \text{ is a simple PCE with root } M \text{ such that } \ \rho(\mathbf{G}_M) = \rho_p\},$$

Proposition E6. *Given any network* $\mathbf{G}$ *the set of peer-consistent equilibria can be written as*

$$PCE = \bigcup_{p=1}^{P} \Lambda_p,$$

where Λ_p *is the convex polytope generated by* S_p*:* $\Lambda_p = Conv\left(S_p\right)$.

Proof of Proposition E6. We first show that $\bigcup_{p=1}^{P} \Lambda_p \subset PCE$. It amounts to showing that, if $S_p = \{\mathbf{x}^1, ..., \mathbf{x}^n\}$, and $\lambda_1, ..., \lambda_p$ are nonnegative numbers that sum to one then $\mathbf{x} := \sum_{j=1} \lambda_j \mathbf{x}^j$ is a PCE. We have

$$\mathbf{G}\mathbf{x} = \sum_{j=1}^{n} \lambda_j \mathbf{G}\mathbf{x}^j = \sum_{j=1}^{p} \lambda_j \rho_p \mathbf{x}^j = \rho_p \mathbf{x}.$$

Moreover $X = \sum_i x_i = \sum_i \sum_{j=1}^n \lambda_j x_i^j = \sum_{j=1}^n \lambda_j \sum_i s_i^j = \sum_{j=1}^n \lambda_j \frac{\rho_p}{\rho_p+1} = \frac{\rho_p}{\rho_p+1}$. Hence $\rho_p = \frac{cX}{V-cX}$ and this concludes this implication.

We now turn to the other inculsion. Let $\mathbf{x}$ be a PCE. Then, by Proposition E5, there exists $p \in \{1, ..., P\}$ and a family of pairwise disconnected communities $\{M_i\}_{i=1,...,n}$ such that $N_+(\mathbf{x}) = \cup_{i=1}^n \bar{M}_i$, and $\rho\left(\mathbf{G}_{M_i}\right) = \rho_p > \max_{M' \succ M_i} \rho\left(\mathbf{G}_{M'}\right), \ \forall i = 1, ..., n$. Call $\mathbf{x}^i$ the simple equilibrium with root M_i, for $i = 1, ..., n$. We first define the following objects:

$$\tilde{M}_i := \bar{M}_i \setminus \left(\cup_{j \neq i} \bar{M}_j\right); \ \ \tilde{M} := \cup_{i=1}^n \bar{M}_i \setminus \left(\cup_{i=1}^n \tilde{M}_i\right); \ \ \lambda_i := \frac{\sum_{j \in \tilde{M}_i} x_j}{\sum_{i \in \tilde{M}_i} x_j^i}.$$

Note that, by construction, the family $\left\{\tilde{M}, \tilde{M}_1, ..., \tilde{M}_n\right\}$ constitutes a partition of $\cup_{i=1}^n \bar{M}_i$. Call $\mathbf{A}_i := \mathbf{G}_{\tilde{M}_i}$ and $\mathbf{A} := \mathbf{G}_{\tilde{M}}$. Then we can write

$$\mathbf{G}_{\cup_{i=1}^n \bar{M}_i} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & ... & ... & \mathbf{B}_n \\ 0 & \mathbf{A}_1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & \mathbf{A}_{n-1} & 0 \\ 0 & ... & ... & 0 & \mathbf{A}_n \end{bmatrix}$$

Be aware that this is not a Frobenius normal form because matrices $\mathbf{A}$ and $\mathbf{A}_i$ are in general not irreducible. However we know the following: $\rho(\mathbf{A}_i) = \rho_p$ for $i = 1, ..., n$ and $\rho(\mathbf{A}) < \rho_p$. Moreover, for $j = 1, ..., p$, $\mathbf{x}^i_{|\tilde{M}_i}$ is, by definition, a positive eigenvector of matrix $\mathbf{A}_i$. This is also true for $\mathbf{x}_{|\tilde{M}_i}$. The Frobenius normal form of $\mathbf{A}_i$ verifies the conditions of Lemma B8, (A_{s+1} corresponding here to M_i). As a result $\mathbf{x}_{|\tilde{M}_i}$ and $\mathbf{x}^i_{|\tilde{M}_i}$ are proportionnal:

$$\mathbf{x}_{|\tilde{M}_i} = \alpha_i \mathbf{x}^i_{|\tilde{M}_i}. \tag{E.5}$$

Since $\mathbf{x}_{|\cup_{i=1}^n \bar{M}_i}$ is an eigenvector of $\mathbf{G}_{\cup_{i=1}^n \bar{M}_i}$ associated to ρ_p we have

$$\rho_p \mathbf{x}_{|\tilde{M}} = \mathbf{A}\mathbf{x}_{|\tilde{M}} + \sum_{i=1}^n \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i},$$

and thus, since $\mathbf{I} - \rho_p^{-1}\mathbf{A}$ is invertible,

$$\rho_p \mathbf{x}_{|\tilde{M}} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \sum_{i=1}^n \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \sum_{i=1}^n \alpha_i \mathbf{B}_i \mathbf{x}^i_{|\tilde{M}_i}.$$

On the other hand $\mathbf{x}^i_{|\tilde{M} \cup \tilde{M}_j}$ is an eigenvector of $\mathbf{G}_{|\tilde{M} \cup \tilde{M}_j}$ associated to ρ_p. Hence

$$\rho_p \mathbf{x}^i_{|\tilde{M}} = \mathbf{A}\mathbf{x}^i_{|\tilde{M}} + \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i},$$

that is,

$$\rho_p \mathbf{x}^i_{|\tilde{M}} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \mathbf{B}_i \mathbf{x}_{|\tilde{M}_i}.$$

Finally we get

$$\rho_p \mathbf{x}_{|\tilde{M}} = \sum_{i=1}^{n} \alpha_i \rho_p \mathbf{x}^i_{|\tilde{M}_i},$$

i.e. $\mathbf{x}_{|\tilde{M}} = \sum_{i=1}^{n} \alpha_i \mathbf{x}^i_{|\tilde{M}_i}$. Combining this equality with (E.5) and the fact that $\mathbf{x}^m_{|\tilde{M}_i} = 0$ when $i \neq m$, we obtain that

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{x}^i$$

Now $\mathbf{x}$ and $\mathbf{x}^i$ being all associated to the same eigenvalue ρ_p we necessarily have $X = X^i = \frac{\rho_p}{\rho_p+1}$ for $i = 1, ..., n$. As a result $\sum_{i=1}^{n} \alpha_i = 1$ and this concludes the proof. $\square$

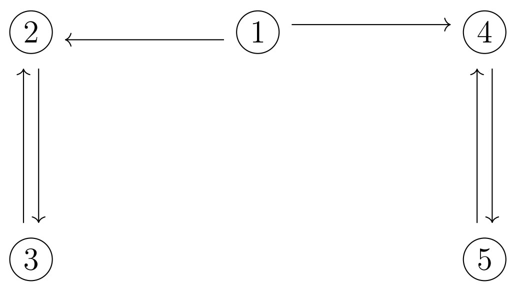
Remark E1. *When* $\mathbf{G}$ *is a generic network, then every component is degenerate, i.e., they reduce to a singleton. In full generality, in a given component, the largest eigenvalue of the subgraph of active players is invariant.*

E.2 Example

We illustrate this in the following example for the linear Tullock contest game.

Example E1. Non-finiteness of equilibria

Figure E1: Infinite set of PCE in a non generic network



Consider the network $(N, \mathbf{G})$ in Figure E1 with $N = \{1, 2, \cdots, 5\}$. Both $M_1 = \{2, 3\}$, and $M_2 = \{4, 5\}$ are $\succ$-maximal communities. Moreover we have $\rho(\mathbf{G}_{M_1}) = \rho(\mathbf{G}_{M_2}) = 1$. Consequently, the set of peer-consistent equilibria is not finite since the network is non generic. More precisely:

$$PCE = \left\{ \frac{V}{12c}(1, \lambda, \lambda, 1-\lambda, 1-\lambda) : \ \lambda \in [0, 1] \right\}.$$

$\diamond$