Social Value of Information in Networked Economies

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June 7, 2020

Abstract

This paper studies the welfare implications of information dissemination in networked economies. First, I study how the equilibrium use of information depends on network structures. The key result shows that a centrality measure reminiscent of Katz (1953) and Bonacich (1987) predicts the relative sensitivity of agents' equilibrium actions to private and public information. I then use this result to study how equilibrium payoffs vary with the underlying information structure. The main result relates the topology of the network to the distributional effects of information dissemination. In particular, public information can have a negative value for less central agents while having a positive value for more central ones. Moreover, in economies featuring significant heterogeneity in agents' centralities, the aggregate welfare effect of public information can be negative.

Keywords: Coordination, network game, value of information, beauty contest.

*Email: KotaMurayama2013@u.northwestern.edu. I am grateful to my advisors Asher Wolinsky, Alessandro Pavan, and Jeffrey Ely for their continuous encouragement and guidance. I also thank Piotr Dworczak, Yingni Guo, Bruno Strulovici, Harry Pei, Ong Pinchuan, Toomas Hinnosaar, Takashi Ui, Faruk Gul, Arjada Bardi, Eddie Dekel, Matthew Jackson, Ben Golub, Tomastz Strzalecki, Hideshi Itoh, Kohei Kawamura, Daisuke Hirata, Shintaro Miura, and seminar and conference participants.

1 Introduction

Many socio-economic interactions occur in networks in which different agents are affected by and respond to the actions of other agents differently. For example, a retailer may be affected by a price cut by nearby retailers more so than by distant retailers. Likewise, the utility that a user derives from adopting a new technology may be higher when the same technology is adopted by close friends or co-workers rather than by more remote users. Policy advocates may be more concerned with support from politicians from their own party than from other parties.

Furthermore, most of such network interactions occur under incomplete information. In the examples above, retailers may face uncertainty about product demand, users may face uncertainty about the quality of a new technology, and politicians may face uncertainty about policy they promote.

In this paper, I develop a framework to investigate how the equilibrium use of information depends on the network of payoff interdependencies and how the equilibrium payoffs depend on the interaction between the payoff network and the information available to agents.

More specifically, I study an economy in which agents have incentives to align their actions not only with an unknown state but also with the actions of their neighbors. Importantly, I allow for a rich network structure of payoff interdependencies. Formally, I capture the whole interaction structure by an adjacency matrix where the (i, j)-th entry measures how much agent i benefits from coordinating with agent j. To facilitate a closed form equilibrium characterization and clear comparative statics, I assume a quadratic payoff function and a Gaussian information structure with public and private signals.

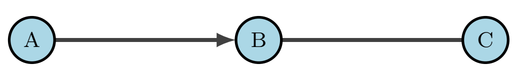
The first result shows how the equilibrium use of information varies with the network structure. A key observation is that public information becomes more useful when agents have stronger coordination motives, when their neighbors have stronger coordination motives, when the neighbors of their neighbors have stronger coordination motives, and so on. For instance, if Ann benefits from using the same product as Bob, Ann finds public information useful to coordinate with Bob. Moreover, if Bob benefits from using the same product as Carol, Ann finds public information more useful because she knows Bob will rely on public information to coordinate with Carol. Thus, Ann has an indirect coordination motive with Carol which positively contributes to the sensitivity of Ann's action to public information.

I show that there is a unique equilibrium, and in that equilibrium, the sensitivity of each agent's action to public (resp. private) information is positive (resp. negative) affine in the agent's *total coordination motive*. Importantly, in economies where different agents care about the actions of others differently, such coordination motive is heterogeneous and related to a centrality measure developed by Katz (1953) and Bonacich (1987).[1] More precisely, an agent's total coordination motive is a discounted sum of his direct and indirect coordination motives with the indirect coordination motives discounted more.

The equilibrium discount factor is determined by the relative precision of private information. Thus, indirect coordination motives have more impact on the equilibrium use of information when public information is relatively imprecise (or equivalently, when private information is relatively precise). The reason is that, when public information is imprecise, it is an effective coordination device only when neighbors also have strong coordination motives; otherwise they do not respond much to it. Conversely, when public information is precise, their neighbors rely on it even without strong coordination motives, and hence agents can effectively coordinate through it independent of their neighbors' coordination motives.

The main result relates the topology of the network to the desirability of information dissemination. I show that public information can have a strong distributional effect: while agents with a larger total coordination motive than their neighbors can benefit from more precise public information, those with a smaller total coordination motive than their neighbors can suffer. As a result, when there is large heterogeneity in agents' total coordination motives, the aggregate welfare effect of public information can be negative. Importantly, in the economies I consider, public information is always beneficial when agents have the same total coordination motive. Hence, heterogeneity in total coordination motives is necessary for the detrimental effect of public information to occur.

To gain intuition, consider a three-agent network where Ann has a weak coordination motive with Bob, and Bob and Carol have a strong coordination motive with each other. First, suppose that there is no public information, and Ann and Bob have a perfectly



informative private signal but Carol does not.[2] Then, Ann can completely align her action to the state variable and to Bob's action since they are perfectly correlated. Next, suppose that all agents have access to noisy public information in addition to the private information. Carol responds to the public information since she only had access only to noisy private information. In turn, Bob responds to it in order to coordinate with Carol, and Bob's action becomes imperfectly correlated with the state variable. Ann also

[1]In fact, the total coordination motive is an affine transformation of the Katz-Bonacich centrality. My use of terminology intends to avoid confusion when the Katz-Bonacich centrality is defined on directed networks.

[2] Note that this information structure does not satisfy the assumptions in the main analysis where the precision of each agent's private signal is assumed to be same. I use it only for illustration purposes.

responds to the public information in order to coordinate with Bob but not as much as Bob does. The reason is that Bob has a stronger coordination motive than Ann and noisy public information is less valuable for Ann than for Bob. Consequently, Ann chooses to incur a coordination loss with Bob, and her payoff is decreased from the case where there is no public information.

More generally, when agents have imprecise public information and it becomes more precise, the marginal increase in their sensitivity to public information is larger for those who have a larger total coordination motive than for those with a smaller total coordination motive. As a result, if the difference in total coordination motives is sufficiently large, more precise public information increases the coordination loss of agents with a smaller total coordination motive than their neighbors. Ultimately, this difference decreases the formers' payoff.

The above situation can be an example of the negative aggregate effect by duplicating Ann in the network. I compare agents' equilibrium use of information with an efficient use of information that maximizes the utilitarian welfare. It turns out that Bob and Carol overuse the public signal when their coordination motives are sufficiently strong, while Ann always underuses it relative to their unique efficient action. Thus, from the social planner's perspective, public information has a detrimental effect when the cost of inducing Bob and Carol's overuse of public information outweighs the benefit of mitigating Ann's underuse.

Private information can also have a strong distributional effect. Consider a two-agent network where Ann wants to coordinate with Bob, but Bob has no coordination motive. Suppose also that Ann does not care about the product quality while Bob does. In this

example, more precise private information clearly benefits Bob but may hurt Ann. When there is no private information, both of them respond to public information and Ann incurs no coordination loss. When private information becomes a bit more precise, Bob responds to it, and hence Ann incurs some coordination loss.[3] More generally, agents with a smaller total coordination motive increase their sensitivity to private information more than those with a larger total coordination motive do when it is imprecise. As a result, more precise private information can hurt agents who have a larger total coordination motive than their neighbors.[4]

I identify a simple geometric feature of networks that is responsible for the detrimental

[3]The same conclusion follows as long as Ann has a sufficiently strong coordination motive.

[4]As in the case of public information, heterogeneity in total coordination motives is necessary to have the detrimental effect of private information in my environment.

effects of public information. A group of agents is called *cohesive* if each member of the group has a strong coordination motive with at least one of the other members. Agents in a cohesive group have a large total coordination motive because of their cyclic coordination motives within the group. In the three-agent network example, Bob and Carol formed a cohesive group and Ann, who was outside of the group, suffered from more precise public information. I show that public information can have a negative value for agents outside of a cohesive group who have a not so large coordination motive with the agents in the cohesive group.

So far, I have focused on the effects of information dissemination on the equilibrium payoffs and on the utilitarian welfare. The value of information under different objective functions can also be studied using my equilibrium characterization. For instance, I consider an outside authority who cares only about how close the actual product qualities are to the target quality, and does not care about how agents coordinate to use similar products.[5] In this setting, I show that public information can be detrimental to the authority if the sum of the squared total coordination motives is sufficiently large.

As a corollary, it follows that public information is necessarily beneficial if agents have no indirect coordination motive. For instance, it can never be detrimental in networks where there is a salient user who tries to align his action only with the unknown product quality, and there are other fringe users who try to align their action with the unknown product quality and the salient user's action. Then, even when the fringe users engage in coordination, their behavior is well disciplined by the salient user's behavior toward the ideal action for the authority. In contrast, when there are at least two salient users in the networks, public information can be detrimental since the group of salient users can be highly cohesive and their action can be very sensitive to public information.

In the paper, I also consider an optimal dissemination problem in which the outside authority can choose both how precise the information is and who observes it. At the optimum, she releases her information with maximum precision given the optimal subset of agents who observe the information. The optimal subset of agents is chosen to minimize the sum of the squared total coordination motives defined on the *subnetwork* induced by them. For instance, in the star networks, information is disseminated to either all agents or all agents except the center agent, depending on agents' degree of coordination motives.

The remainder of the paper is organized as follows. In Section 1, I review related literature. Section 2 introduces the model, and Section 3 characterizes the unique equilibrium. Section 4 provides the main result on the social value of information. Section 5 studies

[5] In the context of financial markets, Morris and Shin (2002) use this objective function to study the desirability of transparent communication from a central bank to investors. In fact, my analysis here naturally extends theirs to incorporate richer payoff interdependencies among investors.

the value of information under a different objective function. Section 6 discusses modeling assumptions and Section 7 concludes. All proofs are relegated to Appendix C.

1.1 Related literature

This paper contributes to the literature on the value of information in strategic environments. Early studies include the work of Radner (1979), followed by Basar and Ho (1974), Levine and Ponssard (1977), and Vives (1984, 1988). Morris and Shin (2002) and Angeletos and Pavan (2007) generated renewed interest in the topic using flexible and tractable modeling frameworks (see Pavan and Vives (2015) for an overview of the recent literature). The relationships between payoff interdependencies/externalities and the social value of information are further investigated by Angeletos and Pavan (2004), Myatt and Wallace (2015), and Ui and Yoshizawa (2015) among others.

Almost all papers in this literature assume symmetric payoff structures, while the focus of the present paper is on payoff asymmetries.[6] One notable exception is the work of Leister (2019). He studies a model of endogenous information acquisition with the network of payoff interdependencies.[7] In his model, agents receive one completely private signal and there is no public signal. As a result, his welfare analysis, which compares utilitarian welfare under overt and covert information acquisition, is different from mine.

In addition, my paper is related to Myatt and Wallace (2019). While they do not conduct welfare analysis, their equilibrium characterization provides an implication implication that is qualitatively similar to mine. They study a model similar to that of Leister (2019) under more general information structures.[8] Their main result characterizes the equilibrium information acquisition and shows that agents with a higher Katz-Bonacich centrality acquire relatively public signals more than those with a lower centrality. This result may appear similar to mine, but their Katz-Bonacich centrality has a different discount factor which is given by one. The assumption of linear information acquisition costs plays an important role in their representation, and hence their method cannot be applied to my environment.[9]

[6] In related contexts, recent papers by Candogan and Drakopoulos (2019) and Egorov and Sonin (2019) study persuasion on networks.

[7] The models of endogenous information acquisition in symmetric economies include Dewan and Myatt (2008), Hellwig and Veldkamp (2009), Myatt and Wallace (2011), Llosa and Venkateswaran (2013), and Colombo, Femminis, and Pavan (2014).

[8] Calvó-Armengol, de Martí, and Prat (2015), Denti (2017), and Myatt and Wallace (2017) also study endogenous information acquisition in asymmetric economies.

[9] Proposition 1 of Myatt and Wallace (2019) characterizes the equilibrium second-stage action under general information structures. Agents' use of information is shown to be proportional to the *weighted* Katz-Bonacich centralities where the underlying interaction structure is adjusted by the information structure through the weighting vector.

The present paper also contributes to the follow-up debate on the anti-transparency result by Morris and Shin (2002). Morris and Shin (2002) point out potential detrimental effects of public announcement by a central bank. In response, Svensson (2006) argues that their result is a consequence of unrealistic parameter values. Angeletos and Pavan (2007) and Hellwig (2002) argue that it is due to their payoff specification. Colombo and Femminis (2008) allow endogenous acquisition of the private signal and find the optimality of fully transparent communication. James and Lawler (2011, 2012) show that when a central bank can directly affect agents' payoffs as well as public announcement, public information necessarily decreases welfare. My results complement these papers by showing under what interaction structures public information can be detrimental.

There is a vast amount of literature on games on networks.[10] To the best of my knowledge, Ballester, Calvó-Armengol, and Zenou (2006) are the first to use the Katz-Bobacih centrality to represent Nash equilibria. Their result is extended to incomplete information games by de Martí and Zenou (2015), Ui (2016), and Lambert, Martini, and Ostrovsky (2017).[11] The novelty of my equilibrium characterization relative to these papers is to associate the Katz-Bobacih centrality to the use of information, and in particular to the publicity of information.

Finally, while this paper focuses on payoff networks, there is a sizable literature on communication networks. Calvó-Armengol and de Martí (2007, 2009) and Herskovic and Ramos (2018) study a model of endogenous information sharing through networks. Hagenbach and Koessler (2010) and Galeotti, Ghiglino, and Squintani (2013) study strategic information transmission in networks. Galeotti and Goyal (2009) and Galeotti and Rogers (2013) study information diffusion through networks. Incorporating such communication networks into my model seems a promising research direction for future research.

2 Model

2.1 Payoff structure

There are n agents with $n \geq 2$. An individual agent is indexed by $i \in N = \{1, \ldots, n\}$. Agent i's action is a real number $a_i \in \mathbb{R}$. Agent i's payoff is a quadratic function of an action profile $a = (a_i)_{i \in N}$ and a (common) payoff state $\theta \in \mathbb{R}$ which is given by:

$$u_i(a, \theta) = -g_{ii}(a_i - \theta)^2 - \sum_{j \neq i} g_{ij}(a_i - a_j)^2. \tag{1}$$

[10] See Jackson, Rogers, and Zenou (2016) and Jackson and Zenou (2015) for comprehensive surveys of this literature.

[11] Golub and Morris (2017) and Bergemann, Heumann, and Morris (2017) also study the properties of equilibria in similar models.

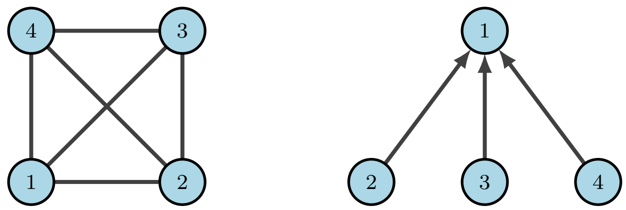


Figure 1: A uniform interaction (left) and a leader-follower interaction (right) (n=4)

The first term in (1) is a distance between i's own action and the payoff state, while the second is the sum of distances between i's action and each of the other agents' actions. The first term is called an *estimation loss*, and the second term is called a *coordination loss*. The coefficient g_{ii} measures i's incentive to minimize the estimation loss, and g_{ij} measures i's coordination motive with agent j. I assume $g_{ii} > 0$ and $g_{ij} \geq 0$ for each $j \neq i$. The sum of each agent's estimation and coordination motives is normalized to be one.[12]

Agents can have different intensities of coordination motives with different agents (e.g., $g_{ij} > g_{ih}$), and coordination motives need not be completely reciprocal (e.g., $g_{ij} > g_{ji}$). An *interaction structure* $G \in \mathbb{R}^{n \times n}$ summarizes agents' coordination motives as

$$G_{ij} = \begin{cases} 0 & \text{if } j = i, \\ g_{ij} & \text{if } j \neq i. \end{cases}$$

(N, G) is regarded as a network with intensities. In particular, a set of agents with whom i has a positive coordinate motive is called i's *neighbors*. The following two interaction structures are used to illustrate my results (see Figure 1).

Example 1. (Uniform interaction)
A *uniform interaction* is a complete network in which a single parameter $r \in [0, 1)$ determines all coordination motives as $g_{ij} = \frac{r}{n-1}$ for each $j \neq i$. Almost all existing papers on the social value of information focus on this interaction structure (e.g., Morris and Shin, 2002; Angeletos and Pavan, 2007; Ui and Yoshizawa, 2015).

Example 2. (Leader-follower interaction)
A *leader-follower interaction* is a special case of bipartite networks where there are a unique agent called a leader (l) and remaining agents called followers (f). The leader does not have any coordination motive and the followers want to coordinate only with the leader. Thus, G is given by $g_{lf} = 0$, $g_{fl} > 0$, and $g_{ff'} = 0$, where f and f' are different followers.

[12] This normalization has no effect on the equilibrium characterization and does not affect the welfare analysis qualitatively.

2.2 Information structure

Each agent i observes a signal vector $\mathbf{s}_i = (x_i, y)$, where x_i is a private signal observed only by agent i and y is a public signal observed by all agents. Formally, $x_i = \theta + \varepsilon_i$ and $y = \theta + \varepsilon_0$, where θ, ε_i, and ε_0 are independently and normally distributed with:

$$\mathbb{E}[\theta] = 0, \mathbb{E}[\varepsilon_i] = \mathbb{E}[\varepsilon_0] = 0, \mathrm{var}[\theta] = \tau_\theta^{-1}, \mathrm{var}[\varepsilon_i] = \tau_x^{-1}, \mathrm{var}[\varepsilon_0] = \tau_y^{-1},$$

and ε_i and ε_j are independent for each $j \neq i$. I call τ_x, τ_y, and τ_θ the precisions of x_i, y, and θ, respectively. The relative precisions of the private and public signal are written as

$$\gamma_x = \frac{\tau_x}{\tau_\theta + \tau_x + \tau_y} \text{ and } \gamma_y = \frac{\tau_y}{\tau_\theta + \tau_x + \tau_y}, \text{ respectively.}$$

3 Equilibrium use of information

3.1 Total coordination motives

Agent i's equilibrium action depends not only on i's own coordination motive but also on his neighbors' coordination motives. For instance, when $g_{ij}g_{jk}$ is positive, i wants to coordinate with j and j wants to coordinate with k. Thus, i has an incentive to predict k's behavior, which I call i's indirect coordination motive with k.

Let $C_i^1 = \sum_{j \neq i} g_{ij}$ be i's first-order coordination motive, $C_i^2 = \sum_{j \neq i} \sum_{k \neq j} g_{ij} g_{jk}$ be i's second-order coordination motive, and let $C_i^k = \sum_{i_1, \dots, i_{k+1}} g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_k i_{k+1}}$ be i's k-order coordination motive, where $i_1 = i$ and $i_l \neq i_{l+1}$. In particular, C_i^1 is called i's direct coordination motive and $(C_i^k)_{k \geq 2}$ is called i's indirect coordination motive. Agent i's *total coordination motive* is a discounted sum of his direct and indirect coordination motive.

Definition 1. Agent i's *total coordination motive* is defined as

$$c_i(\delta, G) = \sum_{k=1}^{\infty} \delta^k C_i^k, \tag{2}$$

where $\delta \in [0, 1]$ is a discount factor.

Agents have a large total coordination motive when their neighbors also have a large total coordination motive.[13] Note that $c_i(\delta, G)$ is well-defined for each $\delta \in [0, 1]$ by the assumptions on G.[14]

[13] Each agent's total coordination motive is an affine transformation of his Katz-Bonacich centrality defined on G. My use of terminology intends to avoid confusion when the Katz-Bonacich centrality is used in directed networks. For instance, in the leader-follower interaction, followers are more central than the leader, which may sound confusing.

[14] The formal argument is as follows. By Debreu and Herstein (1953), G has the largest real positive

3.2 Equilibrium characterization

Agent i's strategy σ_i is a function that maps i's signal vector $\mathbf{s}_i \in \mathbb{R}^2$ to an action $\sigma(\mathbf{s}_i) \in \mathbb{R}$. I focus on pure strategies and on strategies that satisfy $E[\sigma_i^2] < \infty$ to make agents' expected payoffs well-defined. A strategy profile $\sigma^* = (\sigma_i^*)_{i \in N}$ is a (pure strategy) Bayesian Nash equilibrium if $\sigma_i^*(\mathbf{s}_i) \in \arg\max_{a_i} E[u_i((a_i, \sigma_{-i}^*), \theta)|\mathbf{s}_i]$ for all $\mathbf{s}_i \in \mathbb{R}^2$ and $i \in N$, where $\sigma_{-i}^* = (\sigma_j^*)_{j \neq i}$.

Proposition 1. *There exists a unique Bayesian Nash equilibrium:*

$$\sigma_i^*(\mathbf{s}_i) = b_i^y y + b_i^x x_i, \tag{3}$$

where $b_i^y = \gamma_y + \gamma_y c_i(\gamma_x, G)$ *and* $b_i^x = \gamma_x - (1 - \gamma_x) c_i(\gamma_x, G)$.

Proposition 1 shows that the unique equilibrium is linear in signals, and each agent's total coordination motive is a sufficient statistic of G for his sensitivities to public and private information. When agents have no coordination motive, the equilibrium sensitivities are given by the relative precision of signals. The direct and indirect coordination motives increase the relative sensitivity to public information. Intuitively, if i has an indirect coordination motive with k through j, j relies more on the public signal to match his action with k's action, and hence the public signal becomes more useful for i to match his action with j's action.[15] A similar logic applies to higher-order coordination motives.

The equilibrium discount factor is given by the relative precision of private information. Thus, higher-order coordination motives have less impact on the equilibrium use of information when public information becomes more precise (or equivalently private information becomes less precise). When public information is imprecise, agents do not respond much to it unless they have a strong coordination motive. Then, agents can use it as an effective coordination device only when their neighbors also have a strong coordination motive. Thus, the second-order coordination motive plays an important role in agents' use of information. In contrast, when public information is precise, the neighbors respond to it even when they have a weak coordination motive. Hence, agents' use of information is not so sensitive to their second-order coordination motive.

Finally, the following two propositions describe how agents adjust their use of information as public information becomes more precise (see Figure 2).

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eigenvalue $\kappa(G)$ that satisfies $\kappa(G) \leq \max_i \sum_{j \neq i} g_{ij} < 1$. Since $I - \delta G$ is invertible if and only if $\delta\kappa(G) < 1$, $\mathbf{c}(\delta, G) = (I - \delta G)^{-1}\mathbf{1} - \mathbf{1}$ is well-defined, where $\mathbf{c}(\delta, G)$ is a vector of total coordination motives and $\mathbf{1}$ is an n-dimensional vector of ones.

[15] Notice that i and k can be identical in the above argument, which is consistent with the definition of the total coordination motive.

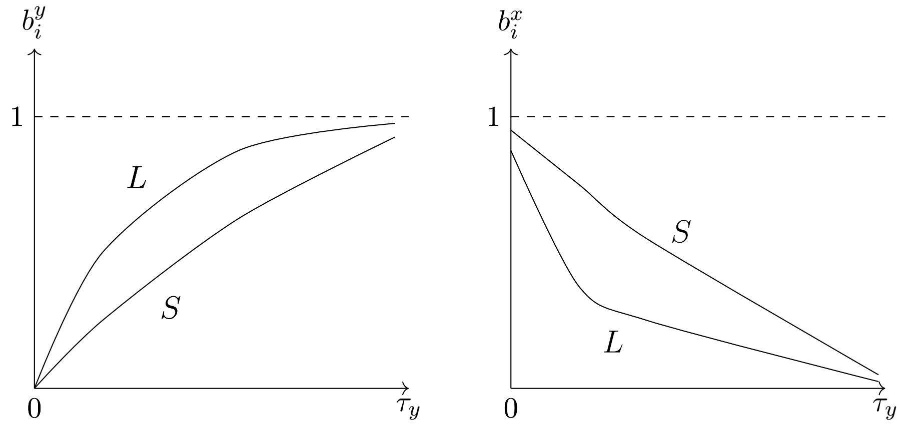

Figure 2: Equilibrium sensitivities to public and private information where agent L has a larger total coordination motive than agent S

Proposition 2. *More precise public information increases (resp. decreases) the equilibrium sensitivity to public information (resp. private information), i.e.,*

$$\frac{\partial b_i^y}{\partial \tau_y} > 0 \text{ and } \frac{\partial b_i^x}{\partial \tau_y} < 0.$$

Proposition 3. *When public information is sufficiently imprecise (resp. private information is sufficiently precise), agents with a larger total coordination motive increase (resp. decrease) their sensitivity to public information (resp. private information) more than those with a smaller total coordination motive do when public information becomes more precise, i.e,* $\frac{\partial b_i^y}{\partial \tau_y} > \frac{\partial b_j^y}{\partial \tau_y}$ *and* $\frac{\partial b_i^x}{\partial \tau_y} < \frac{\partial b_j^x}{\partial \tau_y}$ *if i has a larger total coordination motive than j.*

4 Social value of public information

The goal of this section is to associate the topology of interaction structures to the social value of public information. To avoid repetition, the analysis of the social value of private information is relegated to Appendix A.

4.1 Main results

The first main result presents a strong distributional effect of public information. I split agents into two mutually exclusive and almost exhaustive groups. Define a subset of agents:

$$S_y = \left\{ i \in N : (c_i(1,G)+1)(c_i(1,G)+2) < \sum_{j \neq i} g_{ij}(c_j(1,G)+1)^2 + 1 \right\}.$$

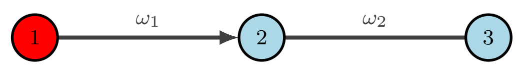

Figure 3: $S_y = \{1\}$ and $L_y = \{2, 3\}$ when ω_2 is sufficiently larger than ω_1

In words, S_y is the set of agents who have a smaller total coordination motive than their neighbors.[16] The left-hand side of the inequality is increasing in agent i's total coordination motive and the right-hand side is increasing in his neighbors' total coordination motives. Also, the discount factor in the total coordination motives is set to be one. Similarly, define a subset of agents L_y, which is generically a complement of S_y:

$$L_y = \left\{ i \in N : (c_i(1, G) + 1)(c_i(1, G) + 2) > \sum_{j \neq i} g_{ij}(c_j(1, G) + 1)^2 + 1 \right\}.$$

Observe that S_y is empty under uniform interactions, but L_y is always nonempty as it contains agents with the largest total coordination motive.

Proposition 4. *There exists a lower bound on the relative precision of private information* $\underline{\gamma_x} \in (0, 1)$ *such that for any* $\gamma_x \geq \underline{\gamma_x}$*, more precise public information decreases the equilibrium payoff of agents in* S_y *and increases that of agents in* L_y*.*

Thus, when private information is relatively precise, agents with a smaller total coordination motive than their neighbors can be harmed by more precise public information. Importantly, the heterogeneity in total coordination motives is necessary for the negative value of public information: when agents have the same total coordination motive, more precise public information necessarily benefits all agents. Example 3 provides a simple interaction structure in which S_y is nonempty.

Example 3. (Nonempty S_y)
Suppose there are three agents such that $g_{12} = \omega_1 > 0$, $g_{23} = g_{32} = \omega_2 > 0$, and $g_{ij} = 0$ otherwise (see Figure 3). It follows that $c_2(1, G) - c_1(1, G) = \frac{\omega_2(1-\omega_1)}{1-\omega_2} \to \infty$ as $\omega_2 \to 1$. Hence, $S_y = \{1\}$ and $L_y = \{2, 3\}$ when ω_2 is sufficiently large.

The above example suggests that more precise public information can even decrease the utilitarian welfare. To state the result, let $\rho_i = \sum_{j \neq i} g_{ji}$ denote *i's in-degree in G*. Agent i's in-degree is the amount of direct coordination motives agent i receives from the other agents.

Proposition 5. *There exists a lower bound on the relative precision of private information* $\underline{\gamma_x} \in (0, 1)$ *such that for any* $\gamma_x \geq \underline{\gamma_x}$*, more precise public information decreases the*

[16] Note that $i \in S_y$ must have at least one neighbor.

utilitarian welfare if

$$\sum_{i \in N} \left((1-\rho_i) \left(c_i(1,G) + 1 \right)^2 + c_i(1,G) \right) < 0. \tag{4}$$

The inequality (4) is satisfied when agents with a high in-degree (at least $\rho_i > 1$) have a sufficiently larger total coordination motive than the other agents. Remember the interaction structure in Example 3. In this interaction structure, agent 2's in-degree is higher than one when $\omega_1 + \omega_2 > 1$, and he has a relatively large total coordination motive than agent 1 when ω_2 is large. Thus, by duplicating agent 1 in the interaction structure, the inequality (4) is satisfied when ω_2 is sufficiently large.

The intuition for the negative value of public information is as follows. Suppose public information is relatively imprecise. By Proposition 3, agents with a larger total coordination motive increase their sensitivity to public information more than those with a smaller total coordination motive do when public information becomes more precise. In particular, this implies that the difference in the sensitivities to public information among agents with different total coordination motives increases.

Consider an agent with a smaller total coordination motive than his neighbors. The neighbors care about aligning their actions with the other agents and increase their sensitivity to public information without fully taking into account the externality on the agent's payoff. The agent is not willing to follow his neighbors' use of information since he cares about aligning his action to the unknown state more than his neighbors do. As a result, the agent incurs more coordination loss when the difference in total coordination motives is sufficiently large.

Proposition 4 shows that the increased coordination loss can dominate a potential decrease in the estimation loss. Moreover, Proposition 5 shows that even at the aggregate level, the increased coordination loss can be first order when agents with a relatively large total coordination motive impose sufficiently large externalities on their neighbors, which are measured by their in-degree. Appendix B provides a more detailed argument with a decomposition of the effect of public information into the effects on the estimation and coordination loss.

4.2 Comparison with the efficient use of information

This section considers the socially optimal use of information to understand externalities in information use. The efficient use of information is defined as a strategy profile that maximizes the sum of agents' expected payoffs while keeping their information dispersed.

Definition 2. A strategy profile $\tilde{\sigma}$ is *efficient* if $\tilde{\sigma}(\mathbf{s}) \in \arg\max_{\sigma(\mathbf{s}) \in \mathbb{R}^n} \sum_{i \in N} \mathbb{E}[u_i(\sigma, \theta) | \mathbf{s}_i]$ for each signal realization, where $\mathbf{s} = (\mathbf{s}_i)_{i \in N}$ is a signal vector profile.

Define an interaction structure $\tilde{G}$ such that for each $i \neq j$,

$$\tilde{g}_{ij} = \frac{g_{ij} + g_{ji}}{1 + \rho_i}.$$ [17]

The efficient strategy is unique and given by the unique equilibrium of a fictitious game where $\tilde{G}$ is the interaction structure.

Proposition 6. *There is a unique efficient strategy profile:*

$$\tilde{\sigma}_i(\mathbf{s}_i) = \tilde{b}_i^y y + \tilde{b}_i^x x_i,$$

where $\tilde{b}_i^y = \gamma_y + \gamma_y c_i(\gamma_x, \tilde{G})$ *and* $\tilde{b}_i^x = \gamma_x - (1 - \gamma_x) c_i(\gamma_x, \tilde{G})$.

$\tilde{G}$ is called the *virtual interaction structure of* G because the social planner wants agents to perceive as if $\tilde{G}$ is the interaction structure they face to implement the efficient strategy. Intuitively, $\tilde{G}$ is more "reciprocal" than G in the following sense. First, when two agents i and j have the same in-degree in G, their coordination motive must be completely reciprocal in $\tilde{G}$, i.e., $\tilde{g}_{ij} = \tilde{g}_{ji}$. Also, i's coordination motive in $\tilde{G}$ incorporates his neighbors' coordination motives with him in G. For instance, consider an interaction structure where i's coordination motive with j is the same as his coordination motive with agent k. Also, j's coordination motive with i is larger than k's coordination motive with i, i.e., $g_{ij} = g_{ik}$ and $g_{ji} > g_{ki}$. Then, i has a larger coordination motive with j than with k in $\tilde{G}$, i.e., $\tilde{g}_{ij} > \tilde{g}_{ik}$.

Remember the interaction structure in Example 3. The virtual interaction structure in this example is given by $\tilde{g}_{12} = \omega_1, \tilde{g}_{21} = \frac{\omega_1}{1+\omega_1+\omega_2}, \tilde{g}_{23} = \frac{2\omega_2}{1+\omega_1+\omega_2}, \tilde{g}_{32} = \frac{2\omega_2}{1+\omega_2}$, and $\tilde{g}_{ij} = 0$ otherwise (See Figure 4). Thus, agent 2 has a positive coordination motive with agent 1 in $\tilde{G}$. Moreover, when ω_2 is sufficiently larger than ω_1, agents 2 and 3 have a larger total coordination motive in $\tilde{G}$ than in G. Hence, while agent 1's equilibrium sensitivity to public information is always smaller than his efficient sensitivity to public information, agents 2 and 3's equilibrium sensitivity to public information is larger than their efficient sensitivity to public information. These inefficiencies in the use of information result in the detrimental effect of public information.

In contrast, when G is a symmetric matrix, it is easy to see that $\tilde{g}_{ij} \geq g_{ij}$ for each $i \neq j$. In this case, public information necessarily increases the utilitarian welfare.

Proposition 7. *If G is a symmetric matrix, agents' equilibrium sensitivity to public information is inefficiently low, and more precise public information necessarily increases the utilitarian welfare.*[18]

[17] Total coordination motives on $\tilde{G}$ is well defined since it satisfies $\sum_{j \neq i} \tilde{g}_{ij} = \frac{C_i^1 + \rho_i}{1+\rho_i} < 1$ for each $i \in N$.

[18] One may conjecture that if $\tilde{G} \geq G$, more precise public information necessarily increases the utilitarian welfare. This is not easy to show for the following reason. It is true that agents' equilibrium sensitivity to public information is inefficiently low when $\tilde{G} \geq G$. However, there still can be an agent whose in-degree is larger than one, and hence I need to find an upper bound on the difference in total coordination motives.

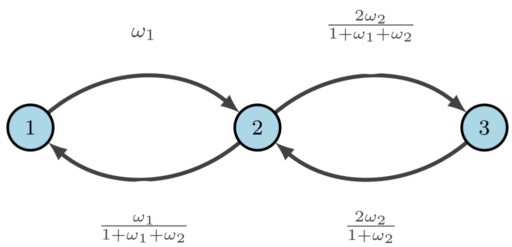

Figure 4: The virtual interaction structure of Example 3

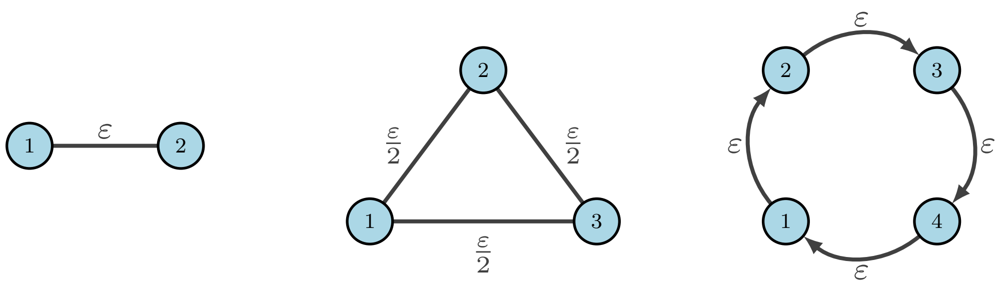

Figure 5: ε-cohesive groups

4.3 When are total coordination motives heterogeneous?

When agents have a weak direct coordination motive, their total coordination motive is small. In contrast, agents have a large total coordination motive when they have a strong cyclic coordination motive. To see this, I introduce a notion of cohesiveness based on Morris (2000).

Definition 3. A subset of agents O_ε is ε-*cohesive* if $\sum_{j \in O_\varepsilon / \{i\}} g_{ij} \geq \varepsilon$ for each $i \in O_\varepsilon$.

Thus, a group of agents is ε-cohesive if each group member has at least ε coordination motive with other members in the group (see Figure 5). I can find a lower-bound for the total coordination motives of agents in O_ε.

Lemma 1. *$c_i(\gamma_x, G) \geq \frac{\gamma_x \varepsilon}{1-\gamma_x \varepsilon}$ for each $i \in O_\varepsilon$.*

Lemma 1 implies that when $\gamma_x \varepsilon$ is close to one, agents in an ε-cohesive group have an extremely large total coordination motive. Hence, the difference in total coordination motives among agents in and out of a highly cohesive group can be very large.

Proposition 8. *Consider an agent $i \notin O_\varepsilon$ such that $g_{ij} > 0$ for some $j \in O_\varepsilon$ and $g_{ij} = 0$ otherwise. Then, more precise public information decreases agent i's equilibrium payoff when $\gamma_x \varepsilon$ is sufficiently large.*[19]

[19]More generally, consider an agent $i \notin O_\varepsilon$ such that $\sum_{j \in O_\varepsilon} g_{ij} > 0$ and $g_{ij} > 0$ for $j \notin O_\varepsilon$ only when j

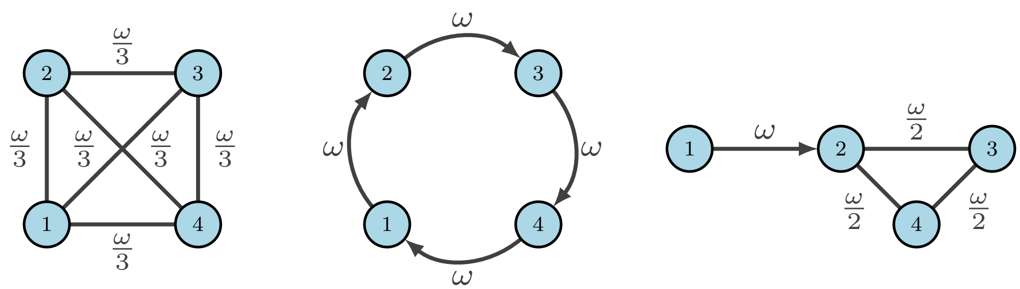

Figure 6: C^1-symmetric interaction structures with $C_i^1 = \omega$ ($n = 4$)

4.4 Strategic equivalence

In some cases, I can translate an interaction structure to a simpler one without changing its properties on the value of information. I say that two interaction structures are *strategically equivalent* if they induce the same unique equilibrium for any information structure. An implication of Proposition 1 is the following.

Lemma 2. *The following four statements are equivalent:*

(i) G *and* G' *are strategically equivalent;*

(ii) $c_i(\delta, G) = c_i(\delta, G')$ *for any* $\delta \in [0,1]$ *and* $i \in N$*;*

(iii) $c_i(\delta, G) = c_i(\delta, G')$ *and* $c_i^{\mathbf{c}}(\delta, G) = c_i^{\mathbf{c}}(\delta, G')$ *for any* $\delta \in [0,1]$ *and* $i \in N$*;*

(iv) Each agent has the same direct and indirect coordination motives in G *and* G'*.*

"Part (*ii*) implies Part (*iv*)" follows by induction since $c_i(\delta, G) \approx \delta C_i^1$ when $\delta \approx 0$. The rest of the proof is not difficult. Lemma 2 implies that the effect of information on each of the estimation and coordination losses is the same in two strategically equivalent interaction structures. This lemma is especially useful for the following class of interaction structures. I say that an interaction structure G is C^1*-symmetric* if agents have a common direct coordination motive in G, i.e., $C_i^1 = C_j^1$ for each $i, j \in N$ (see Figure 6).

Lemma 3. *Any* C^1*-symmetric interaction structure is strategically equivalent to a uniform interaction structure with* $r = C_i^1$*, where* $g_{ij} = \frac{r}{n-1}$ *in the uniform interaction structure.*

In Figure 6, the two interaction structures on the right-hand side are strategically equivalent to the uniform interaction structure on the left-hand side. Thanks to Lemma 3, the existing results on the value of information under uniform interaction structures can

is isolated from O_ε. Thus, i's neighbors include some agents in O_ε but does not include any agent outside of O_ε who has an indirect coordination motive with agents in O_ε. Then, more precise public information decreases i's equilibrium payoff when $\gamma_x \varepsilon$ is sufficiently large.

be extended to C^1-symmetric interaction structures as long as the welfare at the unique equilibrium depends on G only through direct and indirect coordination motives. For instance, the utilitarian welfare under C^1-symmetric interaction structures can be written as $n\left(1-C_i^1\right)\mathbb{E}[(\sigma_i^*)^2]$. Hence, the following result is obtained.

Proposition 9. *Under C^1-symmetric interaction structures, the utilitarian welfare necessarily increases with public information.*

5 Beauty contest and investor networks

My equilibrium characterization can be used to study the value of information under different objective functions. In particular, this section incorporates investor networks into the beauty contest model by Morris ans Shin (2002). They consider a model of financial markets in the spirit of Keynes's beauty contest: investors try to forecast and outbid others' forecasts in addition to bidding for the fundamental value of the asset. Specifically, I consider the following extension of their payoff function:

$$v_i(a,\theta) = -g_{ii}(a_i-\theta)^2 - \left(\sum_{j\neq i} g_{ij}(a_i-a_j)^2 - \frac{1}{n-2}\sum_{j,k\neq i;j\neq k} g_{jk}(a_j-a_k)^2\right). \quad (5)$$

In contrast to Morris and Shin (2002), there can be a salient investor who attracts coordination motives from the other fringe investors. Other than this difference, I follow their formulation.[20] (5) induces the same equilibrium behavior with (1) but it has an additional externality term. As a result, the sum of the equilibrium payoffs can be written as:

$$E(\tau_y) = -\sum_{i\in N} g_{ii}\mathbb{E}[(\sigma_i^*-\theta)^2].$$

Thus, the so-called beauty contest term (the terms in the large bracket of (5)) generates an incentive to outbid others, but the game of outbidding has a zero-sum nature and it is socially wasteful in this environment.[21]

5.1 A generalized anti-transparency result

Morris ans Shin (2002) show that more precise public information can decrease $E(\tau_y)$ under uniform interactions. The following proposition generalizes their result.

[20]Strictly speaking, Morris and Shin (2002) consider a continuum of agents and (5) is an extension of the finite analogue of their payoff function.

[21]The same results follow as long as the social gain from minimizing the coordination losses is sufficiently small.

Proposition 10. *There exists a lower-bound on the relative precision of private information* $\underline{\gamma_x} \in (0,1)$ *such that more precise public information decreases* $E(\tau_y)$ *for any* $\gamma_x \geq \underline{\gamma_x}$ *if*

$$\sum_{i \in N} g_{ii}((c_i(1,G))^2 - 1) > 0. \tag{6}$$

If otherwise, more precise public information always increases $E(\tau_y)$*.*

The inequality (6) is satisfied if a weighted sum of the squared total coordination motives is sufficiently large. As Morris and Shin argue, agents with at least some coordination motive overreact to public information relative to the efficient action that maximizes $E(\tau_y)$. When an agent has a large total coordination motive (and γ_x is large), more precise public information accelerates this overreaction and makes the agent's action more distant from the state. An additional insight from Proposition 4 is that the degree of the overreaction can be measured by the agent's total coordination motive.

An implication of Proposition 10 is that there must be at least one agent whose total coordination motive is larger than one to have the detrimental effect.

Proposition 11. *A necessary condition for the detrimental effect of public information is that there is at least one agent who has an indirect coordination motive.*

The above result sharply distinguishes two "similar" financial markets: one in which all investors engage in outbidding and the other in which almost all investors engage in outbidding. The following example clarifies this point.

Example 4. (Necessity of indirect coordination motives)
Consider a leader-follower interaction where the leader is interpreted as a salient investor whose investment practice is based on the fundamental value of the asset, and the followers are interpreted as fringe investors who try to outbid the salient investor. As this interaction structure has no indirect coordination motive, Proposition 11 shows that public information is always beneficial regardless of the number of fringe investors and how strongly the fringe investors try to outbid the salient investor. Intuitively, the salient investor's use of information coincides with the socially desirable one. Thus, while the fringe investors may engage mostly in outbidding, they try to outbid the fundamental-based bid, and they are still incentivized to forecast the fundamental value. As a result, their overreaction to public information is kept minimal.

Proposition 10 also implies that the existence of a few agents with an extremely large total coordination motive qualitatively changes the value of public information. To see this, suppose there is an ε-cohesive group O_ε in G, and let $\omega = \min_{i \neq O_\varepsilon} \sum_{j \in O_\varepsilon} g_{ij}$ denote the minimum of the direct coordination motives with O_ε from the agents outside of O_ε.

Proposition 12. *A sufficient condition for the detrimental effect of public information is that G has an ε-cohesive group O_ε with $\varepsilon > \frac{1}{1+\omega}$.*

Again, this proposition sharply distinguishes the two "similar" financial markets.

Example 5. (Sufficiency of indirect coordination motives)
Consider a variant of a leader-follower interaction in which there are two salient investors who try to outbid each other with degree ε. In contrast to Example 4, this financial market features indirect coordination motives and two salient investors form an ε-cohesive group. By Proposition 12, for any number of fringe investors and their direct coordination motives, I can find a sufficiently large $\varepsilon < 1$ under which public information can be detrimental to welfare. The intuition is similar to that of Example 4. In this case, the use of information of the two salient investors is far from the socially desirable one. Even though the fringe investors may hardly engage in outbidding, they try to outbid the forecasts that are distant from the fundamental-based forecast. As a result, they overreact to public information much more than they do in the previous example.

5.2 Optimal dissemination of public information

A natural solution for the detrimental effect of public information may be limiting the number of agents who observe the disseminated signal.[22] This section considers optimal dissemination problems in which a social planner chooses both the observability of the disseminated signal and its precision in order to maximize the utilitarian welfare.

For this purpose, I consider the following extension. There is a *semipublic* signal z observed only by a subset of agents $P \subseteq N$ instead of a completely public signal y. If $i \in P$, agent i observes both x_i and z, and if $i \notin P$, he observes only x_i. I assume $z = \theta + \varepsilon_z$, where $\varepsilon_z \sim N(0, \tau_z^{-1})$ and ε_z is independent of θ and ε_i for each $i \in N$. Clearly, the original information structure is a special case of this information structure with $P = N$. Let $p = |P|$ denote the cardinality of P and let G_P denote the $p \times p$ sub-matrix of G that is induced by agents in P.[23]

The new information structure makes the equilibrium representation more complicated since agents' higher-order expectations can differ depending on their identities (e.g.,

[22] Cornand and Heinemann (2008) consider a similar dissemination problem but with a complete network of a continuum of agents.

[23] For instance, if $P = \{1, 3, 5\}$, then

$$G_P = \begin{pmatrix} 0 & g_{13} & g_{15} \\ g_{31} & 0 & g_{35} \\ g_{51} & g_{53} & 0 \end{pmatrix}.$$

$\mathbb{E}_i\mathbb{E}_j\mathbb{E}_k[\theta] \neq \mathbb{E}_i\mathbb{E}_k\mathbb{E}_j[\theta]$ if $j \in P$ and $k \notin P$). Then, I cannot clearly separate the information structure from the interaction structure as in Proposition 1. To avoid this complication, I focus on the case where agents have no prior information about θ, i..e, they have the improper uniform prior with $\tau_\theta = 0$.[24]

With this assumption, the higher-order expectations of agents who observe only the private signal become very simple and given by x_i. Then, I can truncate the iterated expectations of agents who observe only private signals as follows: $\mathbb{E}_{i_1}\mathbb{E}_{i_2}\cdots\mathbb{E}_{i_l}\cdots\mathbb{E}_{i_k}[\theta] = \mathbb{E}_{i_1}\mathbb{E}_{i_2}\cdots\mathbb{E}_{i_l}[\theta]$ for any sequence of agents $i_1, i_2, \ldots, i_k$ with $i_{l+1} \notin P$ and $l+1 \leq k$. After all the truncations are completed, the remaining higher-order expectations among agents in P no longer depend on the identity of agents since they are ex ante symmetric in information.

Let $\gamma = \frac{\tau_x}{\tau_x+\tau_z}$ denote the relative precision of x_i to z. Given $P \subseteq N$, agent i's strategy σ_i^P is a function of z and x_i when $i \in P$ and of x_i when $i \notin P$. The following proposition characterizes a unique linear equilibrium.

Proposition 13. *There exists a unique linear equilibrium σ^P such that $\sigma_i^P(x_i) = x_i$ for each $i \notin P$, and for each $i \in P$,*

$$\sigma_i^P(z, x_i) = b_{i,P}^z z + b_{i,P}^x x_i, \tag{7}$$

where $b_{i,P}^z = 1 - \gamma + (1-\gamma)c_i(\gamma, G_P)$ and $b_{i,P}^x = 1 - b_{i,P}^z$.

As expected from the previous discussion, the equilibrium use of information of agents in P is almost identical with the one in Proposition 1 except that the total coordination motive is defined on G_P not on G. Thus, all the intuition and the comparative statics carry over by replacing G with G_P. However, it is worth noting that agent i's total coordination motive in G and in G_P can be very different. For instance, whenever the leader is not in P in the leader-follower interaction, each follower's total coordination motive in G_P is zero.

As before, I consider the utilitarian welfare at the unique equilibrium:

$$E(\tau_z, P) = -\sum_{i\in N} g_{ii}\mathbb{E}[(\sigma_i^P - \theta)^2]. \tag{8}$$

Suppose there is an upper bound on the precision of z, i.e., $\tau_z \leq \bar{\tau}_z$. I say that an information dissemination (τ_z^*, P^*) is *optimal* if it maximizes $E(\tau_z, P)$ over $0 \leq \tau_z \leq \bar{\tau}_z$ and $P \subseteq N$. The following proposition characterizes the optimal information dissemination.

[24] Both Morris and Shin (2002) and Cornand and Heinemann (2008) assume this assumption. By continuity, qualitatively same results hold when τ_θ is sufficiently small.

Proposition 14. *An information dissemination (τ_z^*, P^*) is optimal if and only if*

$$\tau_z^* = \bar{\tau}_z \text{ and } P^* \in \arg\min_{P \subseteq N} \sum_{i \in P} g_{ii}((c_i(\bar{\gamma}, G_P))^2 - 1), \tag{9}$$

where $\bar{\gamma} = \frac{\tau_x}{\tau_x + \bar{\tau}_z}$.

For the optimal precision of the semipublic signal, the social planner chooses the maximum precision given the optimal choice of P. To understand what P^* looks like, suppose p agents receive the semipublic signal. Then, the social planner chooses p agents whose induced sub-network minimizes a weighted sum of the squared total coordination motives.[25] This is intuitive since as in the original setup, agents in P overuse the semipublic signal relative to the efficient level and the degree of this overuse is measured by the total coordination motives in G_P.

Also, the social planner needs to know whether giving the semipublic signal to an additional agent is better or not. (9) says that informing a new agent i is better when i does not have a large total coordination motive in $G_{P\cup\{i\}}$ (i.e., $c_i(\bar{\gamma}, G_{P\cup\{i\}}) < 1$), and i does not impose large externalities on agents in P (i.e., $c_j(\bar{\gamma}, G_{P\cup\{i\}}) \approx c_j(\bar{\gamma}, G_P)$ for each $j \in P$). Thus, the social planner faces a trade-off between maximizing the number of agents who observe z and "minimizing" the weighted sum of the squared total coordination motives in the induced sub-network. The following two corollaries are immediate from the above observations.

Corollary 1. *$i \in P^*$ if i has no indirect coordination motive and nobody wants to coordinate with i.*

Corollary 2. *If $\max_{i \in N} c_i(\bar{\gamma}, G) > 1$, $P^* \neq N$.*

Finally, the following example illustrates an interesting implication of Proposition 14.

Example 6. (Non-monotonic dissemination in hierarchies)
The optimal information dissemination in hierarchical networks can be non-monotonic in the sense that P^* can exclude agents at some layer while agents at one layer above and below are included. Consider a bottom-up hierarchical interaction in Figure 7. In this example, P^* excludes the two agents at the middle layer when the maximum relative precision of private information $\bar{\gamma}$ is in the middle range.

Corollary 3. *There exist cutoff values γ_1 and γ_2 with $0 < \gamma_1 < \gamma_2 < 1$ such that $P^* = N$ if $\bar{\gamma} < \gamma_1$, $P^* = N/\{T\}$ if $\gamma_1 < \bar{\gamma} < \gamma_2$, and $P^* = N/\{M_1, M_2\}$ if $\gamma_2 < \bar{\gamma}$.*

[25] In particular, when $g_{ii} = g_{jj}$, agents in P can be relabeled as $P = \{1, \ldots, p\}$ and $c_i(\bar{\gamma}, G_P) \geq c_{i+1}(\bar{\gamma}, G_P)$ for each $i = 1, \ldots, p-1$. Hence, it is optimal to choose P that induces the smallest total coordination motive vector.

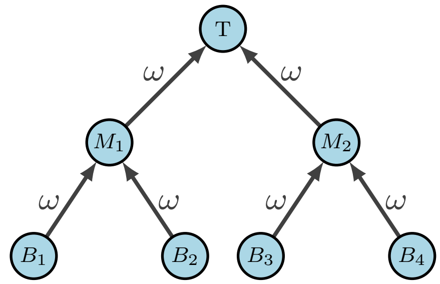

Figure 7: Bottom-up hierarchical interaction (n=7)

6 Discussion

6.1 Strategic substitutability

The main analysis of this paper assumes that all agents have weakly positive coordination motives. I can relax this assumption and accommodate anti-coordination motives by assuming $\sum_{j\neq i}|g_{ij}| < 1$ for each $i \in N$. The same equilibrium characterization is obtained, but an indirect coordination motive between i and j can increase or decrease i's use of public information depending on the number of anti-coordination motives within the indirect coordination motive.

For instance, suppose i has an anti-coordination motive with j and j has a coordination motive with k. Then, j relies *more* on y to match his action with k's action, and hence i has an incentive to rely *less* on y to mismatch his action with j's. Overall, this indirect coordination motive decreases i's sensitivity to public information. Instead, suppose j has an anti-coordination motive with k. Then, j relies *less* on y to mismatch his action with k's action, and hence y becomes less valuable as a coordination device between i and j. In turn, i, who wants to mismatch his action with j's, can rely *more* on y without matching his action with j's, and hence i puts more weight on y. In this way, an indirect coordination motive that includes an odd (resp. even) number of anti-coordination motives decreases (resp. increases) the sensitivity to public information.

Most of the results in Section 4 remain true, but there are several important differences. First, public information can decrease welfare even under undirected interaction structures. This is because public information makes agents' actions more correlated which is bad for the agents who have strong anti-coordination motives. Second, heterogeneity in total coordination motives is positively associated with the social value of public information since it makes agents' actions less correlated. Finally, agents who have similar anti-coordination motives may have very different total coordination motives.[26] Thus,

[26] A simple example is when there are three agents such that $g_{12} = g_{21} = \varepsilon$ and $g_{ij} = 0$ otherwise. Then, agent 1 has a larger total coordination motive than agent 2 for small $\varepsilon > -1$.

with anti-coordination motives, the characteristics of interaction structures that admit heterogeneous total coordination motives are much more subtle.

6.2 Asymmetric information structure

Throughout the paper, the information structures are assumed to be ex ante symmetric for the tractability. An important implication of this assumption is that heterogeneity in the equilibrium information use is generated solely by the payoff heterogeneity. The asymmetry in information structures can increase or decrease heterogeneity in agents' use of information. For instance, more precise public information can be detrimental to welfare even when some of agents have relatively imprecise private information.

To see this, remember the interaction structure in Example 3. Consider a new information structure such that agents 2 and 3 have a more precise private signal than agent 1 does. Let τ_1 denote the precision of agent 1's private signal and let τ_2 denote that of the other two agents. Under this new information structure, I can show that more precise public information decreases agent 1's equilibrium payoff even when $\tau_1 \leq \tau_y$ if τ_2 is sufficiently larger than τ_y and τ_1, and ω_2 is sufficiently larger than ω_1. Thus, as long as agent 2 has a sufficiently precise private information, he responds to public information more than agent 1. As a result, agent 1 incurs more coordination loss with agent 2.

7 Conclusion

This paper studied how underlying interaction structures relate to the social value of information in a coordination game. I first characterized the equilibrium information use using total coordination motives, which is reminiscent of the Katz-Bonacich centrality in the network literature.

Based on this characterization, the main results of this paper identified a novel channel through which public information can be detrimental to welfare. The first main result showed a distributional effect of public information. When private information is relatively precise, more precise public information benefits agents with a smaller total coordination motive than their neighbors and harms the others. Moreover, I showed that this negative effect of public information can be dominant at the aggregate level.

I also provided a generalization of the anti-transparency result of Morris and Shin (2002). I showed that public information can be detrimental to welfare in Morris and Shin's beauty contest model when agents have a sufficiently large total coordination motive. An interesting implications is the necessity and sufficiency of indirect coordination motives.

That is, the coordination motives of a few agents can qualitatively change the desirability of transparent communication.

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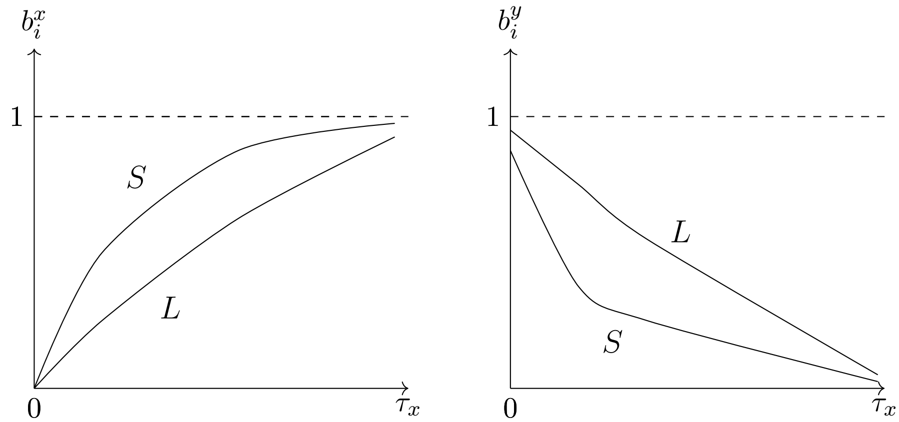

Figure 8: Equilibrium sensitivities to private and public information where agent L has a larger total coordination motive than agent S

Appendix A Social value of private information

A.1 Equilibrium sensitivity to private information

The equilibrium sensitivity to private information is described by the following two propositions (see Figure 8).

Proposition 15. *More precise private information increases (resp. decreases) the equilibrium sensitivity to private information (resp. public information), i.e.,*

$$\frac{\partial b_i^x}{\partial \tau_x} > 0 \textit{ and } \frac{\partial b_i^y}{\partial \tau_x} < 0.$$

Proposition 16. *When private information is sufficiently imprecise (resp. public information is sufficiently precise), agents with a smaller total coordination motive increase (resp. decrease) their sensitivity to private information (resp. public information) more than those with a larger total coordination motive do when private information becomes more precise, i.e,* $\frac{\partial b_i^x}{\partial \tau_x} > \frac{\partial b_j^x}{\partial \tau_x}$ *and* $\frac{\partial b_i^y}{\partial \tau_x} < \frac{\partial b_j^y}{\partial \tau_x}$ *if i has a smaller total coordination motive than j.*

A.2 Welfare

As in the analysis of public information, I split agents into two mutually exclusive and almost exhaustive groups. Define $L_x = \{i \in N : (1 - C_i^1)^2 < \sum_{j \neq i} g_{ij}(1 - C_j^1)^2\}$ and $S_x = \{i \in N : (1 - C_i^1)^2 > \sum_{j \neq i} g_{ij}(1 - C_j^1)^2\}$. In words, L_x (resp. S_x) is the set of agents who have a relatively large (resp. small) direct coordination motive than their neighbors. Notice that L_x is an empty set under C^1 symmetric interaction structures, and S_x is

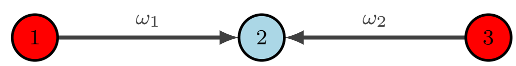

Figure 9: More precise private information can decrease welfare when ω_1 and ω_2 are sufficiently large

always non-empty since the condition must be satisfied by an agent with the smallest direct coordination motive.

Proposition 17. *There exists an upper bound on the relative precision of private information* $\overline{\gamma_x} \in (0, 1)$ *such that for any* $\gamma_x \leq \overline{\gamma_x}$, *more precise private information decreases the equilibrium payoff of agents in* L_x *and increases that of agents in* S_x.

Proposition 17 says that if an agent has a relatively large (resp. small) direct coordination motive, he suffers (resp. benefits) from more precise private information when it is relatively imprecise. Note that when private information is extremely imprecise, only lower-order coordination motives have an impact on agents' use of information. Thus, heterogeneity in total coordination motives is captured by heterogeneity in direct coordination motives. Intuitively, agents suffer from more precise private information since their neighbors have a smaller total coordination motive than them, and increase their sensitivity to private information more than they do when private information is imprecise.

More precise private information can also decrease the utilitarian welfare.

Proposition 18. *There exists an upper bound on the relative precision of private information* $\overline{\gamma_x} \in (0, 1)$ *such that more precise private information decreases the utilitarian welfare for any* $\gamma_x \leq \overline{\gamma_x}$ *if*

$$\sum_{i \in N} (1 - \rho_i) \left(1 - (C_i^1)^2\right) < 0. \tag{10}$$

The inequality (10) is satisfied when agents with a high in-degree (at least $\rho_i > 1$) have a relatively small direct coordination motive than the other agents. The following is a simple example in which (10) can be satisfied.

Example 7. (Negative social value of private information)
Suppose there are three agents such that $g_{12} = \omega_1$, $g_{32} = \omega_2$, and $g_{ij} = 0$ otherwise (see Figure 9). In this interaction structure, agent 2 has an in-degree higher than one when $\omega_1 + \omega_2 > 1$. As expected, (10) is satisfied when $\omega_1 + \omega_2$ is sufficiently large since then agents 2 overresponds to private information from agents 1 and 3's perspective.

As in the case of public information, completely reciprocal coordination motives restrict the degree of externalities and more precise private information necessarily increases the utilitarian welfare under this restriction.

Proposition 19. *If G is a symmetric matrix, more precise private information always increases the utilitarian welfare.*

Appendix B Decomposing the effect of public information

This appendix decomposes the effect of public information on the equilibrium payoffs to the one on the estimation loss and the one on the coordination loss.

To describe marginal changes in the equilibrium sensitivity to information, I introduce a *weighted total coordination motive*.

Definition 4. Agent i's $\mathbf{q}$-*weighted total coordination motive* is the i-th coordinate of

$$\mathbf{c}^{\mathbf{q}}(\delta, G) = \sum_{k=1}^{\infty} \delta^k G^k \mathbf{q}, \tag{11}$$

where $\mathbf{c}^{\mathbf{q}}(\delta, G) = (c_1^{\mathbf{q}}(\delta, G), \ldots, c_n^{\mathbf{q}}(\delta, G))'$ and $\mathbf{q} = (q_1, \ldots, q_n)' \in \mathbb{R}_+^n$ is a vector of weights.

In the following, $c_i(\gamma_x, G)$ is sometimes called i's *unweighted* total coordination motive. Agent i's total coordination motive is a special case of this $\mathbf{q}$-weighted total coordination motive with $\mathbf{q} = \mathbf{1}$. In words, i's $\mathbf{q}$-weighted total coordination motive is the sum of direct and indirect coordination motives in which each direct and indirect coordination motives with agent j is multiplied by q_j.

I first examine the effect of public information on the estimation loss. Let $D_{i\theta} = \mathbb{E}[(\sigma_i^* - \theta)^2]$.[27] I can rewrite $D_{i\theta}$ as:

$$D_{i\theta} = \mathrm{var}[b_i^x \varepsilon_i] + \mathrm{var}[b_i^y \varepsilon_0 + (b_i^x + b_i^y)\theta] - 2\mathrm{cov}[\theta, \sigma_i^*].\text{[28]} \tag{12}$$

The first term of (12) is the variance of the idiosyncratic noise, called the idiosyncratic variance of i's action. Clearly, this term is decreasing in τ_y since i puts less weight on the private signal. The second term is the variance of the common noise, called the common variance of i's action. This term is increasing in τ_y. To see this, consider a hypothetical agent j who has the same total coordination motives with i. Then, the common variance of i's action coincides with $\mathrm{cov}[\sigma_i^*, \sigma_j^*]$, which is increasing in τ_y since more precise information causes more correlated actions. The last term is the negative of the covariance of i's action and θ, which is decreasing in τ_y since i's action becomes less noisy and more correlated with θ. Thus, the effect of public information on the estimation loss is ambiguous.

[27] $D_{i\theta}$ is a mnemonic for a distance between σ_i^* and θ.

[28] Remember that $\sigma_i^* = b_i^x \varepsilon_i + b_i^y \varepsilon_0 + (b_i^x + b_i^y)\theta$. This is a particularly useful decomposition under uniform interaction structures as shown by Ui and Yoshizawa (2014). In their symmetric environments, the third term of (12) can also be expressed as a function of the idiosyncratic and common variances by using the first order condition.

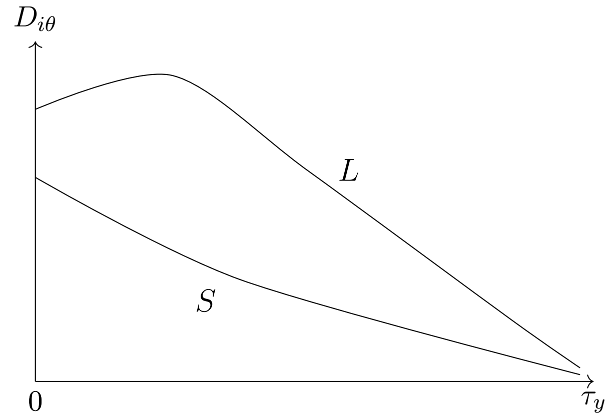

Figure 10: Effect of public information on the estimation loss where agent L has a larger unweighted and $\mathbf{c}-$*weighted* total coordination motive than agent S and $c_L(1,G) > 1 > c_S(1,G)$

Lemma 4. *The marginal effect of public information on i's estimation loss is:*

$$\frac{\partial D_{i\theta}}{\partial \tau_y} = -\gamma_0^2 - \tau_x^{-1}\gamma_0 c_i(\gamma_x, G)\left((2-3\gamma_x)c_i(\gamma_x, G) + 2(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)\right).$$

Lemma 4 shows that the effect of public information is smaller for agents with a larger unweighted and **c**-weighted total coordination motive when $\gamma_x < \frac{2}{3}$ and is larger for them when γ_x is sufficiently large (see Figure 10). This is intuitive since agents with a larger unweighted and **c**-weighted total coordination motive respond to public information more when public information is imprecise. Then, the increase of the common variance should be larger for them than for those who have a small total coordination motive.

Next, let $D_{ij} = \mathbb{E}[(\sigma_i^* - \sigma_j^*)^2]$ denote the coordination loss between i and j at the unique equilibrium.[29] I decompose D_{ij} as:

$$D_{ij} = \text{var}[b_i^x\varepsilon_i + b_j^x\varepsilon_j] + \text{var}[(b_i^y - b_j^y)\varepsilon_0 + (b_i^x + b_i^y - b_j^x - b_j^y)\theta]. \tag{13}$$

The first term of (13) is the idiosyncratic variance of the sum of two agents' actions. As before, this idiosyncratic variance is unambiguously decreased by public information. The second term is the common variance of the difference of two agents' actions. This term can increase since agents have different sensitivities to public information. Thus, the total effect of public information on D_{ij} is ambiguous.

Let $\Sigma_{ij}(\gamma_x, G) = c_i(\gamma_x, G) + c_j(\gamma_x, G)$ denote the *sum* of unweighted total coordination motives, and let $\Delta_{ij}(\gamma_x, G) = c_i(\gamma_x, G) - c_j(\gamma_x, G)$ denote the *difference* of unweighted total coordination motives. Similarly, define $\Sigma_{ij}^{\mathbf{c}}(\gamma_x, G) = c_i^{\mathbf{c}}(\gamma_x, G) + c_j^{\mathbf{c}}(\gamma_x, G)$

[29] D_{ij} is a mnemonic for a distance between σ_i^* and σ_j^*.

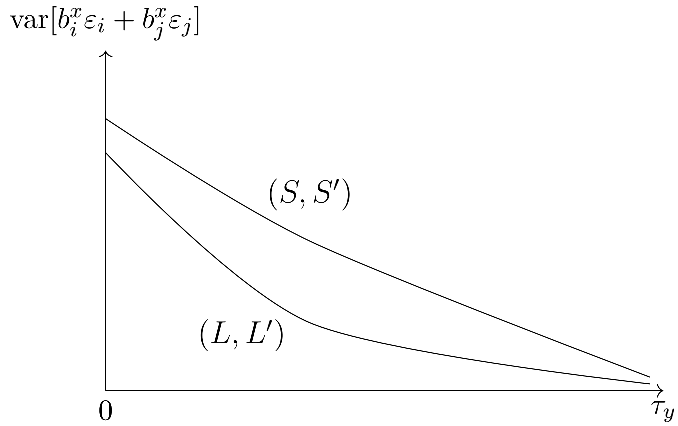

Figure 11: Effect of public information on the idiosyncratic variance where $\Sigma_{LL'} > \Sigma_{SS'}$

and $\Delta^{\mathbf{c}}_{ij}(\gamma_x, G) = c^{\mathbf{c}}_i(\gamma_x, G) - c^{\mathbf{c}}_j(\gamma_x, G)$ for the **c**-weighted total coordination motives. Without loss of generality, I assume $\Delta_{ij}(\gamma_x, G) \geq 0$.

Lemma 5. *The marginal effect of public information on the coordination loss between i and j is given by:*

$$\begin{aligned}\frac{\partial D_{ij}}{\partial \tau_y} &= 2\tau_x^{-1}\gamma_0((1-\gamma_x)\Sigma_{ij}(\gamma_x, G) - 2\gamma_x)((1-2\gamma_x)\Sigma_{ij}(\gamma_x, G) + (1-\gamma_x)\Sigma^{\mathbf{c}}_{ij}(\gamma_x, G) - 2\gamma_x) \\ &\quad + \gamma_0^2 \Delta_{ij}(\gamma_x, G)\left((4\gamma_x - 3)\Delta_{ij}(\gamma_x, G) + (1-\gamma_x)\Delta^{\mathbf{c}}_{ij}(\gamma_x, G)\right). \qquad (14)\end{aligned}$$

The first term of (14) corresponds with the derivative of the idiosyncratic variance. This term is increasing in both $\Sigma_{ij}(\gamma_x, G)$ and $\Sigma^{\mathbf{c}}_{ij}(\gamma_x, G)$ if $\gamma_x < 1/2$ and is decreasing in $\Sigma_{ij}(\gamma_x, G)$ if γ_x is sufficiently large (see Figure 11). The second term is the derivative of the common variance. It is increasing in both $\Delta_{ij}(\gamma_x, G)$ and $\Delta^{\mathbf{c}}_{ij}(\gamma_x, G)$ if $\gamma_x > 3/4$ and is decreasing in $\Delta_{ij}(\gamma_x, G)$ if γ_x is sufficiently small (see Figure 12).[30]

Also, (14) implies that when γ_x is close to 1, the gain from the decreased idiosyncratic variance is measured by $\Sigma_{ij}(\gamma_x, G)$, and the loss from the increased common variance is measured by $(\Delta_{ij}(\gamma_x, G))^2$. By continuity, public information can increase the coordination loss when $(\Delta_{ij}(1, G))^2$ is sufficiently larger than $\Sigma_{ij}(1, G)$, i.e., when i's total coordination motive is sufficiently larger than j's when the discount factor is given by 1.

Proposition 20. *There exists an lower bound on the relative precision of private information $\underline{\gamma_x} \in (0, 1)$ such that more precise public information increases the coordination loss between i and j for any $\gamma_x \geq \underline{\gamma_x}$ if $(\Delta_{ij}(1, G))^2 > 4\left(\Sigma_{ij}(1, G) + 2\right)$.*

Proposition 20 may sound counterintuitive since agents rely more on the public signal and this change seems to reduce the coordination loss. In fact, this intuition is valid

[30]When γ_x is close to 0, $\Delta^{\mathbf{c}}_{ij}(\gamma_x, G)$ is much smaller than $\Delta_{ij}(\gamma_x, G)$ and becomes negligible.

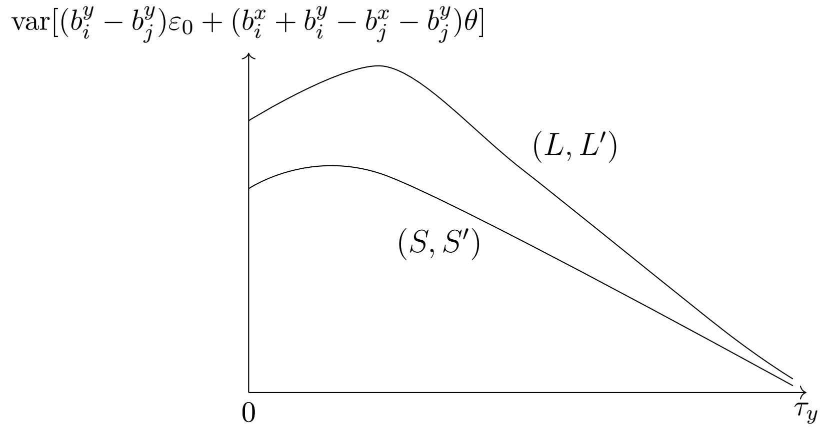

Figure 12: Effect of public information on the common variance where $\Delta_{LL'} > \Delta_{SS'} > 0$

when agents have the same sensitivity to public information. To see this, the derivative of $\mathrm{var}[(b_i^y - b_j^y)\varepsilon_0]$ is written as:

$$\frac{\partial \mathrm{var}[(b_i^y - b_j^y)\varepsilon_0]}{\partial \tau_y} = -(b_i^y - b_j^y)^2 \tau_y^{-2} + 2(b_i^y - b_j^y)\frac{\partial (b_i^y - b_j^y)}{\partial \tau_y}\tau_y^{-1}. \tag{15}$$

Since the first term of (15) is negative, the entire derivative is positive whenever the two agents have the same sensitivity to public information. Thus, more precise public information can increase the coordination loss only when agents have different sensitivities to public information.

Appendix C Proofs

C.1 Preliminary observations

Lemma 6.

$$\frac{\partial c_i(\gamma_x, G)}{\partial \gamma_x} = \gamma_x^{-1}(c_i(\gamma_x, G) + c_i^{\mathbf{c}}(\gamma_x, G)).$$

Proof. Let $M \equiv (I - \gamma_x G)^{-1}$. Then differentiating $(I - \gamma_x G)M = I$ with respect to γ_x yields:

$$(I - \gamma_x G)\frac{\partial M}{\partial \gamma_x} = GM.$$

Hence, we have

$$\begin{aligned}
\frac{\partial M}{\partial \gamma_x}\mathbf{1} &= (I - \gamma_x G)^{-1} G (I - \gamma_x G)^{-1}\mathbf{1} \\
&= \gamma_x^{-1}((I - \gamma_x G)^{-1} - I)(\mathbf{1} + \mathbf{c}(\gamma_x, G)) \\
&= \gamma_x^{-1}(\mathbf{c}(\gamma_x, G) + \mathbf{c}^{\mathbf{c}}(\gamma_x, G) + \mathbf{1} + \mathbf{c}(\gamma_x, G) - \mathbf{c}(\gamma_x, G) - \mathbf{1}) \\
&= \gamma_x^{-1}(\mathbf{c}(\gamma_x, G) + \mathbf{c}^{\mathbf{c}}(\gamma_x, G)).
\end{aligned}$$

□

Lemma 7.

$$\frac{\partial b_i^x}{\partial \tau_x} = \tau_x^{-1}(1-\gamma_x)(\gamma_x - (1-2\gamma_x)c_i(\gamma_x, G) - (1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)).$$

Proof. Since $b_i^x = 1-(1-\gamma_x)(1+c_i(\gamma_x, G))$, we have

$$\begin{aligned}
\frac{\partial b_i^x}{\partial \tau_x} &= \frac{\partial \gamma_x}{\partial \tau_x}(1+c_i(\gamma_x, G)) - (1-\gamma_x)\frac{\partial \gamma_x}{\partial \tau_x}\frac{\partial c_i(\gamma_x, G)}{\partial \gamma_x} \\
&= \frac{\partial \gamma_x}{\partial \tau_x}\left(1+c_i(\gamma_x, G) - (1-\gamma_x)\frac{\partial c_i(\gamma_x, G)}{\partial \gamma_x}\right) \\
&= \tau_x^{-1}(1-\gamma_x)(\gamma_x + \gamma_x c_i(\gamma_x, G) - (1-\gamma_x)(c_i(\gamma_x, G) + c_i^{\mathbf{c}}(\gamma_x, G))) \\
&= \tau_x^{-1}(1-\gamma_x)(\gamma_x - (1-2\gamma_x)c_i(\gamma_x, G) - (1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)).
\end{aligned}$$

The third equality follows from Lemma 6. □

Lemma 8.

$$\frac{\partial b_i^y}{\partial \tau_x} = -\frac{\gamma_y}{1-\gamma_x}\frac{\partial b_i^x}{\partial \tau_x}.$$

Proof. Since $b_i^y = \gamma_y(1+c_i(\gamma_x, G))$, we have

$$\begin{aligned}
\frac{\partial b_i^y}{\partial \tau_x} &= \frac{\partial \gamma_y}{\partial \tau_x}(1+c_i(\gamma_x, G)) + \gamma_y\frac{\partial \gamma_x}{\partial \tau_x}\frac{\partial c_i(\gamma_x, G)}{\partial \gamma_x} \\
&= -\gamma_0\gamma_y\left(1+c_i(\gamma_x, G) - (1-\gamma_x)\frac{\partial c_i(\gamma_x, G)}{\partial \gamma_x}\right) \\
&= -\frac{\tau_y}{\tau_y+\tau_\theta}\frac{\partial \gamma_x}{\partial \tau_x}\left(1+c_i(\gamma_x, G) - (1-\gamma_x)\frac{\partial c_i(\gamma_x, G)}{\partial \gamma_x}\right) \\
&= -\frac{\gamma_y}{1-\gamma_x}\frac{\partial b_i^x}{\partial \tau_x}.
\end{aligned}$$

The last equality follows from Lemma 7. □

Lemma 9.

$$\frac{\partial b_i^x}{\partial \tau_y} = -\gamma_0(\gamma_x - (1-2\gamma_x)c_i(\gamma_x, G) - (1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)).$$

Proof. Since $b_i^x = 1-(1-\gamma_x)(1+c_i(\gamma_x, G))$, we have

$$\begin{aligned}
\frac{\partial b_i^x}{\partial \tau_y} &= \frac{\partial \gamma_x}{\partial \tau_y}(1+c_i(\gamma_x, G)) - (1-\gamma_x)\frac{\partial \gamma_x}{\partial \tau_y}\frac{\partial c_i(\gamma_x, G)}{\partial \gamma_x} \\
&= \frac{\partial \gamma_x}{\partial \tau_y}\left(1+c_i(\gamma_x, G) - (1-\gamma_x)\frac{\partial c_i(\gamma_x, G)}{\partial \gamma_x}\right) \\
&= -\gamma_0(\gamma_x + \gamma_x c_i(\gamma_x, G) - (1-\gamma_x)(c_i(\gamma_x, G) + c_i^{\mathbf{c}}(\gamma_x, G))) \\
&= -\gamma_0(\gamma_x - (1-2\gamma_x)c_i(\gamma_x, G) - (1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)).
\end{aligned}$$

□

Lemma 10.

$$\frac{\partial b_i^y}{\partial \tau_y} = \gamma_0(1-\gamma_y+(1-2\gamma_y)c_i(\gamma_x,G)-\gamma_y c_i^{\mathbf{c}}(\gamma_x,G)).$$

Proof. Since $b_i^y = \gamma_y(1+c_i(\gamma_x,G))$, we have

$$\begin{aligned}
\frac{\partial b_i^y}{\partial \tau_y} &= \frac{\partial \gamma_y}{\partial \tau_y}(1+c_i(\gamma_x,G)) + \gamma_y \frac{\partial \gamma_x}{\partial \tau_y}\frac{\partial c_i(\gamma_x,G)}{\partial \gamma_x} \\
&= \gamma_0^2(\tau_x+\tau_\theta)(1+c_i(\gamma_x,G)) - \gamma_0^2\tau_y(c_i^{\mathbf{c}}(\gamma_x,G)+c_i(\gamma_x,G)) \\
&= \gamma_0((1-\gamma_y)(1+c_i(\gamma_x,G)) - \gamma_y(c_i(\gamma_x,G)+c_i^{\mathbf{c}}(\gamma_x,G))) \\
&= \gamma_0(1-\gamma_y+(1-2\gamma_y)c_i(\gamma_x,G)-\gamma_y c_i^{\mathbf{c}}(\gamma_x,G)).
\end{aligned}$$

□

Lemma 11.

$$\frac{\partial (b_i^x+b_i^y)}{\partial \tau_x} = \frac{\gamma_\theta}{1-\gamma_x}\frac{\partial b_i^x}{\partial \tau_x}.$$

Proof. Immediate from Lemma 8. □

Lemma 12.

$$\frac{\partial (b_i^x+b_i^y)}{\partial \tau_y} = \gamma_0\gamma_\theta(1+2c_i(\gamma_x,G)+c_i^{\mathbf{c}}(\gamma_x,G)).$$

Proof. Since $b_i^x + b_i^y = 1-\gamma_\theta(1+c_i(\gamma_x,G))$, we have

$$\begin{aligned}
\frac{\partial (b_i^x+b_i^y)}{\partial \tau_y} &= -\frac{\partial \gamma_\theta}{\partial \tau_y}(1+c_i(\gamma_x,G)) - \gamma_\theta \frac{\partial \gamma_x}{\partial \tau_y}\frac{\partial c_i(\gamma_x,G)}{\partial \gamma_x} \\
&= \gamma_0\left(\gamma_\theta(1+c_i(\gamma_x,G)) + \gamma_\theta(c_i(\gamma_x,G)+c_i^{\mathbf{c}}(\gamma_x,G))\right) \\
&= \gamma_0\gamma_\theta(1+2c_i(\gamma_x,G)+c_i^{\mathbf{c}}(\gamma_x,G)).
\end{aligned}$$

□

Lemma 13.

$$\frac{\partial \mathbb{E}[\sigma_i^2]}{\partial \tau_x} = \tau_x^{-2} b_i^x(\gamma_x-(1-3\gamma_x)c_i(\gamma_x,G)-2(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x,G)).$$

Proof. First, remember that $\mathbb{E}[\sigma_i^2] = (b_i^x)^2\tau_x^{-1} + (b_i^y)^2\tau_y^{-1} + (b_i^x+b_i^y)^2\tau_\theta^{-1}$. By Lemmas 7, 8, and 11, we have

$$\begin{aligned}
\frac{\partial \mathbb{E}[\sigma_i^2]}{\partial \tau_x} &= 2b_i^x\tau_x^{-1}\frac{\partial b_i^x}{\partial \tau_x} - (b_i^x)^2\tau_x^{-2} + 2b_i^y\tau_y^{-1}\frac{\partial b_i^y}{\partial \tau_x} + 2(b_i^x+b_i^y)\tau_\theta^{-1}\frac{\partial (b_i^x+b_i^y)}{\partial \tau_x} \\
&= 2\frac{\tau_x^{-1}}{1-\gamma_x}\frac{\partial b_i^x}{\partial \tau_x}b_i^x - (b_i^x)^2\tau_x^{-2} \\
&= \tau_x^{-2}b_i^x(2(\gamma_x-(1-2\gamma_x)c_i(\gamma_x,G)-(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x,G)) - (1-(1-\gamma_x)(1+c_i(\gamma_x,G))) \\
&= \tau_x^{-2}b_i^x(\gamma_x-(1-3\gamma_x)c_i(\gamma_x,G)-2(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x,G)).
\end{aligned}$$

□

Lemma 14.
$$\frac{\partial \mathbb{E}[\sigma_i^2]}{\partial \tau_y} = \tau_x^{-1}\gamma_0(\gamma_x + 4\gamma_x c_i(\gamma_x, G) + (3\gamma_x - 2)c_i(\gamma_x, G)^2 + 2\gamma_x c_i^{\mathbf{c}}(\gamma_x, G) - 2(1-\gamma_x)c_i(\gamma_x, G)c_i^{\mathbf{c}}(\gamma_x, G)).$$

Proof. By Lemmas 9, 10, and 12, we have

$$\begin{aligned}
\frac{\partial \mathbb{E}[\sigma_i^2]}{\partial \tau_x} &= 2b_i^x \tau_x^{-1}\frac{\partial b_i^x}{\partial \tau_y} + 2b_i^y \tau_y^{-1}\frac{\partial b_i^y}{\partial \tau_y} - (b_i^y)^2\tau_y^{-2} + 2(b_i^x + b_i^y)\tau_\theta^{-1}\frac{\partial (b_i^x + b_i^y)}{\partial \tau_y} \\
&= 2\gamma_0\left((1 + (1-\gamma_x^{-1})c_i(\gamma_x, G))\frac{\partial b_i^x}{\partial \tau_y} + (1 + c_i(\gamma_x, G))\frac{\partial b_i^y}{\partial \tau_y} + (\gamma_\theta^{-1} - 1 - c_i(\gamma_x, G))\frac{\partial (b_i^x + b_i^y)}{\partial \tau_y}\right) - (b_i^y)^2 \\
&= 2\gamma_0\left(-\gamma_x^{-1}c_i(\gamma_x, G)\frac{\partial b_i^x}{\partial \tau_y} + \gamma_\theta^{-1}\frac{\partial (b_i^x + b_i^y)}{\partial \tau_y}\right) - (b_i^y)^2\tau_y^{-2} \\
&= \frac{\gamma_0}{\tau_x}(\gamma_x + 4\gamma_x c_i(\gamma_x, G) + (3\gamma_x - 2)c_i(\gamma_x, G)^2 + 2\gamma_x c_i^{\mathbf{c}}(\gamma_x, G) - 2(1-\gamma_x)c_i(\gamma_x, G)c_i^{\mathbf{c}}(\gamma_x, G)).
\end{aligned}$$

□

Lemma 15. *Let $E_i(\tau_y) = -\mathbb{E}[(\sigma_i - \theta)^2]$.*
$$\frac{\partial E_i(\tau_y)}{\partial \tau_y} = \tau_x^{-1}\gamma_0(\gamma_x + c_i(\gamma_x, G)((2 - 3\gamma_x)c_i(\gamma_x, G) + 2(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G))).$$

Proof. By Lemmas 9, 10, and 12, we have

$$\begin{aligned}
\frac{\partial E_i(\tau_y)}{\partial \tau_y} &= -2b_i^x \tau_x^{-1}\frac{\partial b_i^x}{\partial \tau_y} - 2b_i^y \tau_y^{-1}\frac{\partial b_i^y}{\partial \tau_y} + (b_i^y)^2\tau_y^{-2} + 2(1 - b_i^x - b_i^y)^2\tau_\theta^{-1}\frac{\partial (b_i^x + b_i^y)}{\partial \tau_y} \\
&= \gamma_0\left(-2\gamma_x^{-1}c_i(\gamma_x, G)\frac{\partial b_i^x}{\partial \tau_y}\right) + \gamma_0^2(1 + c_i(\gamma_x, G))^2 \\
&= -2\tau_x^{-1}\gamma_0\left(c_i(\gamma_x, G)(\gamma_x - (1-2\gamma_x)c_i(\gamma_x, G) - (1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G))\right) + \gamma_0^2(1 + c_i(\gamma_x, G))^2 \\
&= \tau_x^{-1}\gamma_0\left(-2(c_i(\gamma_x, G)(\gamma_x - (1-2\gamma_x)c_i(\gamma_x, G) - (1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)) + \gamma_x(1 + c_i(\gamma_x, G))^2\right) \\
&= \tau_x^{-1}\gamma_0(\gamma_x + c_i(\gamma_x, G)((2 - 3\gamma_x)c_i(\gamma_x, G) + 2(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)))
\end{aligned}$$

□

Lemma 16.
$$\frac{\partial E_i(\tau_y)}{\partial \tau_x} = \gamma_0^2 + \tau_x^{-2}(1-\gamma_x)c_i(\gamma_x, G)((3\gamma_x - 1)c_i(\gamma_x, G) - 2(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)).$$

Proof. By Lemmas 7, 8, and 11, we have

$$\begin{aligned}
\frac{\partial E_i(\tau_y)}{\partial \tau_x} &= 2b_i^x \tau_x^{-1}\frac{\partial b_i^x}{\partial \tau_x} - (b_i^x)^2\tau_x^{-2} + 2b_i^y \tau_y^{-1}\frac{\partial b_i^y}{\partial \tau_x} - 2(1 - b_i^x - b_i^y)\tau_\theta^{-1}\frac{\partial (b_i^x + b_i^y)}{\partial \tau_x} \\
&= 2\frac{\tau_x^{-1}}{1-\gamma_x}\frac{\partial b_i^x}{\partial \tau_x}((1-\gamma_x)b_i^x - \tau_x\tau_y^{-1}\gamma_y b_i^x - \tau_x\tau_\theta^{-1}\gamma_\theta(1 - b_i^x - b_i^y)) - (b_i^x)^2\tau_x^{-2} \\
&= 2\frac{\tau_x^{-1}}{1-\gamma_x}\frac{\partial b_i^x}{\partial \tau_x}(b_i^x - \gamma_x) - (b_i^x)^2\tau_x^{-2} \\
&= -2\tau_x^{-1}\frac{\partial b_i^x}{\partial \tau_x}c_i(\gamma_x, G) - (b_i^x)^2\tau_x^{-2} \\
&= -2\tau_x^{-2}(1-\gamma_x)c_i(\gamma_x, G)(\gamma_x - (1-2\gamma_x)c_i(\gamma_x, G) - (1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)) - (b_i^x)^2\tau_x^{-2} \\
&= -\gamma_0^2 - \tau_x^{-2}(1-\gamma_x)c_i(\gamma_x, G)((3\gamma_x - 1)c_i(\gamma_x, G) - 2(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)).
\end{aligned}$$

$\square$

C.2 Proof of Proposition 1

Proof. I first find a unique linear equilibrium, then I show its uniqueness. For notational simplicity, I write $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot|\mathbf{s}_i]$ and $\mathbb{E}_i\mathbb{E}_j[\cdot] = \mathbb{E}_i[\mathbb{E}_j[\cdot]]$.

For the existence, suppose that agents follow a linear strategy $b_i^x x_i + b_i^y y$. I show that $b_i^x = \gamma_x c_i^{\mathbf{g}}(\gamma_x, G)$ and $b_i^y = \frac{\gamma_y}{1-\gamma_x}\left(c_i^{\mathbf{g}}(1,G) - \gamma_x c_i^{\mathbf{g}}(\gamma_x, G)\right)$, where $c_i^{\mathbf{g}}(\gamma_x, G)$ is the i-th coordinate of $\mathbf{c}^{\mathbf{g}}(\gamma_x, G) = (I - \gamma_x G)^{-1}\mathbf{g}$. Agent i's best response is given by:

$$\begin{aligned} b_i^x x_i + b_i^y y &= g_{ii}\mathbb{E}_i[\theta] + \sum_{j\neq i} g_{ij}\mathbb{E}_i[\sigma_j] \\ &= g_{ii}(\gamma_x x_i + \gamma_y y) + \sum_{j\neq i} g_{ij}((b_j^x + b_j^y)(\gamma_x x_i + \gamma_y y) + b_j^y y) \\ &= \gamma_x(g_{ii} + \sum_{j\neq i} g_{ij} b_j^x) x_i + (\gamma_y g_{ii} + \sum_{j\neq i} g_{ij}(\gamma_y b_j^x + b_j^y) y) \end{aligned} \tag{16}$$

Since the equation (16) holds for any $x_i, y \in \mathbb{R}$, it must follow $b_i^x = \gamma_x(g_{ii} + \sum_{j\neq i} g_{ij} b_j^x)$ and $b_i^y = \gamma_y g_{ii} + \sum_{j\neq i} g_{ij}(\gamma_y b_j^x + b_j^y)$. In a matrix notation, I can rewrite these two equations as:

$$(I - \gamma_x G)\mathbf{b}^x = \gamma_x \mathbf{g}, \text{and } (I - G)\mathbf{b}^y - \gamma_y G\mathbf{b}^x = \gamma_y \mathbf{g},$$

where $\mathbf{b}^x = (b_1^x, \ldots, b_n^x)'$ and $\mathbf{b}^y = (b_1^y \ldots, b_n^y)'$. By the argument in Footnote 14, the matrix $I - \delta G$ is invertible for any $\delta \in [0,1]$. Thus, $\mathbf{b}^x = \gamma_x \mathbf{c}^{\mathbf{g}}(\gamma_x, G)$ as desired. For the coefficient of y, observe that

$$\begin{aligned} (I - G)^{-1} - (I - \gamma_x G)^{-1} &= \sum_{k=0}^{\infty} G^k - \sum_{k=0}^{\infty} \gamma_x^k G^k \\ &= \sum_{k=0}^{\infty} (1 - \gamma_x^k) G^k \\ &= (1 - \gamma_x) G \left(I + (1 + \gamma_x) G + (1 + \gamma_x + \gamma_x^2 G^2 + \cdots)\right) \\ &= (1 - \gamma_x) G \left((I + G + G^2 + \cdots) + \gamma_x (I + G + G^2 + \cdots) G + \cdots\right) \\ &= (1 - \gamma_x) G \sum_{k=0}^{\infty} G^k \left(I + \gamma_x G + \gamma_x^2 G^2 + \cdots\right) \\ &= (1 - \gamma_x)(\sum_{k=0}^{\infty} G^k) G (\sum_{k=0}^{\infty} \gamma_x^k G^k) \\ &= (1 - \gamma_x)(I - G)^{-1} G (I - \gamma_x G)^{-1}. \end{aligned}$$

Thus, I can write

$$\begin{aligned}
\mathbf{b}^y &= \gamma_y \left((I-G)^{-1}\mathbf{g} + (I-G)^{-1}G\mathbf{b}^x\right) \\
&= \gamma_y \left((I-G)^{-1} + \gamma_x (I-G)^{-1}G(I-\gamma_x G)^{-1}\right)\mathbf{g} \\
&= \frac{\gamma_y}{1-\gamma_x}\left(\mathbf{c}^{\mathbf{g}}(1,G) - \gamma_x \mathbf{c}^{\mathbf{g}}(\gamma_x, G)\right)\mathbf{g}.
\end{aligned}$$

Next, I show the uniqueness. The following lemma presents the higher-order expectations and their coefficients.

Lemma 17. *For any sequence of agents $i_0, \cdots, i_n$ with $i_k \neq i_{k+1}$, it follows that*

$$E_{i_0}E_{i_1}\cdots E_{i_n}[\theta] = \gamma_x^{n+1}x_{i_0} + \sum_{k=0}^{n}\gamma_x^k\gamma_y y \tag{17}$$

Proof. When $n=0$, (17) holds by the property of multivariate normal distribution. Suppose (17) holds for $n = l-1$. Then I have

$$\begin{aligned}
E_{i_0}E_{i_1}\cdots E_{i_l}[\theta] &= E_{i_0}\left[\gamma_x^l x_{i_1} + \sum_{k=0}^{l-1}\gamma_x^k\gamma_y y\right] \\
&= \gamma_x^l(\gamma_x x_{i_0} + \gamma_y y) + \sum_{k=0}^{l-1}\gamma_x^k\gamma_y y \\
&= \gamma_x^{l+1}x_{i_0} + \sum_{k=0}^{l}\gamma_x^k\gamma_y y.
\end{aligned}$$

□

Let W_i^k denote the sum of length-k walks emanating from agent i, where each walk ending in agent j is weighted by g_{jj} (i.e., $W_i^k \equiv \sum_{i_1,\ldots,i_k; i_{l-1}\neq i_l} g_{i_1 i_2}g_{i_2,i_3}\cdots g_{i_{k-1}i_k}g_{i_k i_k}$). Iterative substitutions of the first order condition yield:

$$\begin{aligned}
\sigma_i(\mathbf{s}_i) &= g_{ii}\mathbb{E}_i[\theta] + \sum_{j\neq i} g_{ij}g_{jj}\mathbb{E}_i\mathbb{E}_j[\theta] + \sum_{j\neq i}\sum_{k\neq j} g_{ij}g_{jk}\mathbb{E}_i\mathbb{E}_j[\sigma_k] \\
&= \mathbb{E}_i[\theta] + \sum_{j\neq i} g_{ij}g_{jj}\mathbb{E}_i\mathbb{E}_j[\theta] + \sum_{j\neq i}\sum_{k\neq j} g_{ij}g_{jk}g_{kk}\mathbb{E}_i\mathbb{E}_j E_k[\theta] + \sum_{j\neq i}\sum_{k\neq j}\sum_{h\neq k} g_{ij}g_{jk}g_{jh}\mathbb{E}_i\mathbb{E}_j E_k[\sigma_h] \\
&\;\;\vdots \\
&= \sum_{n=0}^{\infty}\sum_{i_1,\ldots,i_n; i_k\neq i_{k+1}} g_{ii_1}g_{i_1 i_2}\cdots g_{i_{n-1}i_n}g_{i_n i_n}E_iE_{i_1}E_{i_2}\cdots E_{i_n}[\theta] \\
&= \sum_{n=0}^{\infty} W_i^n(\gamma_x^{n+1}x_i + \sum_{k=0}^{n}\gamma_x^k\gamma_y y).
\end{aligned}$$

The last equality follows from Lemma 17. Thus, the coefficient of the private signal is given by $\sum_{n=0}^{\infty} W_i^n\gamma_x^{n+1} = \gamma_x c_i^{\mathbf{g}}(\gamma_x, G)$. Also, the coefficient of the public signal is given by

$$\sum_{n=0}^{\infty} \frac{\gamma_y(1-\gamma_x^{n+1})}{1-\gamma_x} W_i^n = \frac{\gamma_y}{1-\gamma_x}\left(\sum_{n=0}^{\infty} W_i^n - \gamma_x \sum_{n=0}^{\infty} \gamma_x^n W_i^n\right)$$
$$= \frac{\gamma_y}{1-\gamma_x}\left(c_i^{\mathbf{g}}(1,G) - \gamma_x c_i^{\mathbf{g}}(\gamma_x, G)\right).$$

The obtained coefficients coincide with those obtained in the existence part of the proof. Thus, the linear equilibrium is actually a unique equilibrium of the game. Finally, it is easy to check that the obtained equilibrium reduces to (3) when $\sum_{j\in N} g_{ij} = 1$ since it follows that $\mathbf{c}^{\mathbf{g}}(1,G) = (I-G)^{-1}(I-G)\mathbf{1} = \mathbf{1}$ and

$$\mathbf{c}^{\mathbf{g}}(\gamma_x, G) = (I-\gamma_x G)^{-1}(I-G)\mathbf{1} = \mathbf{c}(\gamma_x, G) + \mathbf{1} - \gamma_x^{-1}\mathbf{c}(\gamma_x, G).$$ [31]

□

C.3 Proof of Proposition 2

Proof. Since $b_i^y = \frac{\tau_y}{\tau_y+\tau_\theta}(1-b_i^x)$, it suffices to show that $\frac{\partial b_i^x}{\partial \tau_x} > 0$ and $\frac{\partial b_i^x}{\partial \tau_y} < 0$. This is immediate since $b_i^x = \gamma_x c_i^{\mathbf{g}}(\gamma_x, G)$ and $c_i^{\mathbf{g}}(\gamma_x, G)$ is increasing in τ_x and decreasing in τ_y. □

C.4 Proof of Proposition 3

Proof. By Lemma 9, as $\gamma_x \to 1$,

$$\gamma_0^{-1}\frac{\partial b_i^x}{\partial \tau_y} = -\gamma_x + (1-2\gamma_x)c_i(\gamma_x, G) - (1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G) \to -1 - c_i(1,G).$$

Also, by Lemma 10, as $\gamma_y \to 0$,

$$\gamma_0^{-1}\frac{\partial b_i^y}{\partial \tau_y} = 1 - \gamma_y + (1-2\gamma_y)c_i(\gamma_x, G) - \gamma_y c_i^{\mathbf{c}}(\gamma_x, G) \to 1 + c_i(\gamma_x, G).$$

Thus, the result follows. □

C.5 Proof of Propositions 4 and 5

Proof. Let U_i denote agent i's equilibrium payoff. Then I can write $U_i = \mathbb{E}[(\sigma_i^*)^2] - \sum_{j\neq i}\mathbb{E}[(\sigma_j^*)^2]$. Let $\frac{\partial \mathbf{U}}{\partial \tau_y} = \left(\frac{\partial U_i}{\partial \tau_y}\right)_{i\in N}$ and $\frac{\partial \sigma^2}{\partial \tau_y} = \left(\frac{\partial \mathbb{E}[(\sigma_i^*)^2]}{\partial \tau_y}\right)_{i\in N}$. Using these notation, Lemma 14 implies:

$$\tau_x(\tau_x+\tau_y+\tau_\theta)\frac{\partial \mathbf{U}}{\partial \tau_y} = \tau_x(\tau_x+\tau_y+\tau_\theta)(I-G)\frac{\partial \sigma^2(\boldsymbol{\tau})}{\partial \tau_y}$$
$$\to (I-G)^{-1}\mathbf{1} - \mathbf{1} + (I-G)\Lambda(I-G)^{-1}\mathbf{1} \text{ as } \gamma_x \to 1,$$

[31] Note that $(I-\gamma_x G)^{-1}G\mathbf{1} = (G+\gamma_x G^2+\cdots)\mathbf{1} = \left(\gamma_x^{-1}(I+\gamma_x G+\gamma_x^2 G^2) - \gamma_x^{-1}I\right)\mathbf{1} = \gamma_x^{-1}\mathbf{c}(\gamma_x, G)$.

where Λ is a $n \times n$ diagonal matrix whose (i,i)-th entry is $c_i(1,G)+1$. Hence, $\frac{\partial U_i}{\partial \tau_y} < 0$ (resp. > 0) for sufficiently large γ_x if

$$(c_i(1,G)+1)(c_i(1,G)+2) < \sum_{j \neq i} g_{ij}(c_j(1,G)+1)^2 + 1 \text{ (resp. } > 0).$$

Thus, Proposition 4 follows. For the utilitarian welfare, observe that

$$\begin{aligned} \textstyle\sum_{i\in N} & \left((c_i(1,G)+1)(c_i(1,G)+2) - \sum_{j \neq i} g_{ij}(c_j(1,G)+1)^2 - 1 \right) \\ = & \sum_{i \in N} \left((1-\rho_i)(c_i(1,G)+1)^2 + c_i(1,G) \right). \end{aligned}$$

Hence, Proposition 5 follows. □

C.6 Proof of Proposition 6

Proof. The sum of the payoff functions can be written as:

$$\sum_{i \in N} u_i(a,\theta) = -\sum_{i \in N} g_{ii}(a_i - \theta)^2 - \sum_{i \in N}\sum_{j \neq i} g_{ij}(a_i - a_j)^2.$$

Dropping the terms which are independent of agent i's action yields:

$$-g_{ii}(a_i - \theta)^2 - \sum_{j \neq i}(g_{ij} + g_{ji})(a_i - a_j)^2.$$

By dividing the above expression by $1 + \rho_i$, the result follows. □

C.7 Proof of Proposition 7

Proof. Since the utilitarian welfare can be written as $\sum_{i \in N}(1-\rho_i)\mathbb{E}[(\sigma_i^*)^2]$ and $\rho_i \leq 1$ when G is symmetric, it suffices to show that $\frac{\partial \mathbb{E}[(\sigma_i^*)^2]}{\partial \tau_y} > 0$ for each $i \in N$. But this follows from Lemma 14 and from the fact that $\gamma_x - (1-\gamma_x)c_i(\gamma_x, G) \geq 0$. □

C.8 Proof of Lemma 1

Proof. Let $c_\varepsilon = \min_{i \in O_\varepsilon} c_i(\gamma_x, G)$. Then it follows that:

$$\begin{aligned} c_\varepsilon &\geq \gamma_x \varepsilon + \gamma_x \sum_{j \in S_\varepsilon; j \neq i} g_{ij} c_j(\gamma_x, G) \\ &\geq \gamma_x \varepsilon + \gamma_x \varepsilon c_\varepsilon. \end{aligned}$$

Hence, $c_i(\gamma_x, G) \geq c_\varepsilon \geq \frac{\gamma_x \varepsilon}{1 - \gamma_x \varepsilon}$ for each $i \in O_\varepsilon$. □

C.9 Proof of Proposition 8

Proof. By assumption, $c_i(\gamma_x, G) = g_{ij}c_j(\gamma_x, G)$, where $i \notin O_\varepsilon$ and $j \in O_\varepsilon$. Hence, $i \in S_y$ if and only if $1 - g_{ij} < g_{ij}c_j(1,G)$. Since $c_j(1,G) \to \infty$ as $\varepsilon \to 1$, the result follows. □

C.10 Proof of Lemma 2

Proof. The equivalence between (i) and (ii) is immediate from Proposition 1. (iv) implies (ii) by definition. The other direction can be shown by mathematical induction. Suppose $\mathbf{c}(\delta, G) = \mathbf{c}(\delta, G')$ for any $\delta \in [0, 1]$. Then the direct coordination motives in G and G' must agree since if otherwise, there must be an agent who has different total coordination motives in G and G' for sufficiently small $\delta > 0$. The same logic applies for the k-order coordination motives when higher-order coordination motives agree up to $k-1$ order. Since (iv) implies (iii) and (iii) implies (ii), the proof is completed. □

C.11 Proof of Lemma 3

Proof. This is obvious since $c_i(\gamma_x, G) = \frac{\gamma_x C_i^1}{1-\gamma_x C_i^1}$ in a C^1-symmetric interaction structure. □

C.12 Proof of Proposition 9

Proof. This directly follows from Lemma 14. □

C.13 Proof of Proposition 10

Proof. The goal is to show that $\frac{\partial E(\tau_y)}{\partial \tau_y} \geq 0$ for any information structure if and only if $\sum_{i \in N} g_{ii}(1 + c_i(1, G))(1 - c_i(1, G)) \geq 0$. By Lemma 15,

$$\frac{\partial E_i(\tau_y)}{\partial \tau_y} = \tau_x^{-1}\gamma_0\left(\gamma_x + c_i(\gamma_x, G)((2-3\gamma_x)c_i(\gamma_x, G) + 2(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G))\right)$$

For the only if part, observe that $\frac{\partial E_i(\tau)}{\partial \tau_y}$ is continuous in γ_x and the terms in the bracket converge to $1 - (c_i(1, G))^2$ as $\gamma_x \to 1$. Hence, the necessity follows. For the if part, observe that $1 - (c_i(\gamma_x, G))^2$ is weakly decreasing in γ_x. It follows that

$$\begin{aligned}
0 &\leq \gamma_x \sum_{i \in N} g_{ii}(1 - (c_i(1, G))^2) \\
&\leq \gamma_x \sum_{i \in N} g_{ii}(1 - (c_i(\gamma_x, G))^2) \\
&\leq \sum_{i \in N} g_{ii}(\gamma_x + c_i(\gamma_x, G)((2-3\gamma_x)c_i(\gamma_x, G) + 2(1-\gamma_x)c_i^{\mathbf{c}}(\gamma_x, G)))) \\
&= \tau_x\gamma_0^{-1} \sum_{i \in N} g_{ii} \frac{\partial E_i(\tau_y)}{\partial \tau_y}.
\end{aligned}$$

This completes the proof of Proposition 10. □

C.14 Proof of Proposition 11

Proof. This is immediate from Proposition 10. □

C.15 Proof of Proposition 12

Proof. Suppose $\varepsilon > \frac{1}{1+\omega}$. By Lemma 1, $c_i(1, G) \geq \frac{\omega\varepsilon}{1-\varepsilon} > 1$ for $i \notin O_\varepsilon$. Also, $c_j(1, G) \geq \frac{\varepsilon}{1-\varepsilon} > 1$ for $j \in O_\varepsilon$. Hence, the result follows from Proposition 10. $\square$

C.16 Proof of Proposition 13

Proof. I show that the unique linear equilibrium is obtained as a limit of a unique equilibrium when $\tau_\theta \to 0$. Throughout the proof, I fix an information dissemination $P \subseteq N$, and omit P from the notation (e.g., b_i^x for $b_{i,P}^x$). Without loss of generality, assume $P = \{1, \ldots, p\}$. Also, I write $\gamma_x = \frac{\tau_x}{\tau_x+\tau_z+\tau_\theta}$, $\gamma_z = \frac{\tau_z}{\tau_x+\tau_z+\tau_\theta}$, and $\gamma_{x\theta} = \frac{\tau_x}{\tau_x+\tau_\theta}$ with slight abuse of notation. First, the existence of equilibria follows from Proposition 4 of Ui (2016) since $I - G$ is invertible.[32]

Next, I show the uniqueness. As before, we derive higher-order expectations and their coefficients explicitly. Remember that, by iteratively substituting the first order conditions, we obtain:

$$\sigma_{i_1}(\mathbf{s}_{i_1}) \quad = \quad g_{i_1 i_1}\mathbb{E}_{i_1}[\theta] + \sum_{k=2}^{\infty} \sum_{i_2,\ldots,i_k; i_l \neq i_{l-1}} g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_{k-1} i_k} g_{i_k i_k} \mathbb{E}_{i_1}\mathbb{E}_{i_2} \cdots \mathbb{E}_{i_k}[\theta]$$

Since signals follow a multivariate normal distribution, the conditional expectations are linear in signals given by: $\mathbb{E}_{i_1}\mathbb{E}_{i_2} \cdots \mathbb{E}_{i_k}[\theta] = \alpha_{i_1,i_2,i_3,\ldots,i_k} x_{i_1} + \beta_{i_1,i_2,i_3,\ldots,i_k} z$, where $\alpha_{i_1,i_2,i_3,\ldots,i_k}$ and $\beta_{i_1,i_2,i_3,\ldots,i_k}$ are constants and non-negative. I show that the sum of these two constants are less than or equal to $\gamma_x + \gamma_y$.

Lemma 18. *For any integer $k \geq 1$ and sequence of agents $i_1, \cdots, i_k$, it holds that*

$$\alpha_{i_1,i_2,i_3,\ldots,i_k} + \beta_{i_1,i_2,i_3,\ldots,i_k} \leq \gamma_x + \gamma_z \tag{18}$$

Proof. I show this by mathematical induction. When $k = 1$, this is obvious since $\mathbb{E}_{i_1}[\theta] = \gamma x_{i_1}$ if $i \notin P$ and $\mathbb{E}_{i_1}[\theta] = \gamma_x x_{i_1} + \gamma_z z$ if $i \in P$. Suppose the statement holds up to k, and take any sequence of agents $i_1, \cdots, i_{k+1}$. Then we have:

$$\begin{aligned} E_{i_1} E_{i_2} \cdots E_{i_{k+1}}[\theta] \quad &= \quad E_{i_1}\left[\alpha_{i_2,\ldots,i_{k+1}} x_{i_1} + \beta_{i_2,\ldots,i_{k+1}} z\right] \\ &= \begin{cases} \gamma(\alpha_{i_2,\ldots,i_{k+1}} + \beta_{i_2,\ldots,i_{k+1}}) x_{i_1} \text{ if } i_1 \notin P \\ \gamma_x \alpha_{i_2,\ldots,i_{k+1}} x_{i_1} + (\gamma_z \alpha_{i_2,\ldots,i_{k+1}} + \beta_{i_2,\ldots,i_{k+1}}) z \text{ if } i_1 \in P \end{cases} \end{aligned}$$

Hence, we have:

$$\alpha_{i_1,i_2,\ldots,i_{k+1}} + \beta_{i_1,i_2,\ldots,i_{k+1}} \quad \leq \quad \max\{\gamma(\gamma_x + \gamma_z), (\gamma_x + \gamma_z)\alpha_{i_2,\ldots,i_{k+1}} + \beta_{i_2,\ldots,i_{k+1}}\} \leq \gamma_x + \gamma_z.$$

$\square$

[32] Proposition 4 of Ui (2016) also implies that there is a unique equilibrium if $I - G$ is positive definite.

This implies that: for each $n \geq 2$,

$$
\begin{aligned}
\alpha_{i_1}^n &\equiv g_{i_1 i_1}\alpha_{i_1 i_1} + \sum_{k=2}^{n} \sum_{i_2,\ldots,i_k; i_l \neq i_{l-1}} g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_{k-1} i_k} g_{i_k i_k} \alpha_{i_1,i_2,i_3,\ldots,i_k} \\
&\leq (\gamma_x + \gamma_z)\left(g_{i_1 i_1} + \sum_{k=2}^{n} \sum_{i_2,\ldots,i_k; i_l \neq i_{l-1}} g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_{k-1} i_k} g_{i_k i_k} \right) \\
&= (\gamma_x + \gamma_z)(1 - C_{i_1}^{n+1})
\end{aligned}
$$

Since $\alpha_{i_1}^n$ is increasing in n and $1 - C_{i_1}^{n+1}$ converges to 1 as $n \to \infty$, $\alpha_{i_1}^n$ also converges to a real number, say $\alpha_{i_1}^{\infty}$. I can similarly define $\beta_{i_1}^n$ and show that $\beta_{i_1}^n$ converges to a real number $\beta_{i_1}^{\infty}$ as $n \to \infty$. Hence, I have shown that any equilibrium must take the following form:

$$\sigma_{i_1}(\mathbf{s}_{i_1}) = \alpha_{i_1}^{\infty} x_{i_1} + \beta_{i_1}^{\infty} z.$$

It remains to show that $\alpha_{i_1}^{\infty} = 1$ and $\beta_{i_1}^{\infty} = 0$ if $i_1 \notin P$, and $\alpha_{i_1}^{\infty} = b_{i_1}^x + o(\tau_\theta^{-1})$ and $\beta_{i_1}^{\infty} = b_{i_1}^z + o(\tau_\theta^{-1})$ if $i_1 \in P$. First, suppose $i_1 \notin P$.

Lemma 19. *For any integer $k \geq 1$ and a sequence of agents $i_1, i_2, \cdots, i_k$, it holds that*

$$\lim_{\tau_\theta \to 0} (\alpha_{i_1,i_2,\ldots,i_k} + \beta_{i_1,i_2,\ldots,i_k}) = 1. \tag{19}$$

Proof. I show this by mathematical induction. For $k = 1$, this is obvious since $\gamma_{x\theta} \to 1$ and $\gamma_z \to 1 - \gamma_x$ as $\tau_\theta \to 0$. Suppose the statement holds for up to k. Remember that

$$
\begin{aligned}
\alpha_{i_1,i_2,\ldots,i_{k+1}} &= \begin{cases} \alpha_{i_1,i_2,\ldots,i_k} + \beta_{i_1,i_2,\ldots,i_k} \text{ if } i_1 \notin P \\ \gamma_x \alpha_{i_1,i_2,\ldots,i_k} \text{ if } i_1 \in P, \end{cases} \\
\beta_{i_1,i_2,\ldots,i_{k+1}} &= \begin{cases} 0 \text{ if } i_1 \notin P \\ \gamma_z \alpha_{i_1,i_2,\ldots,i_k} + \beta_{i_1,i_2,\ldots,i_k} \text{ if } i_1 \in P. \end{cases}
\end{aligned}
$$

Thus, the statement holds for $k+1$, as desired. □

Hence, for any agent $i_1 \notin P$, we have

$$\lim_{\tau_\theta \to 0} \lim_{n \to \infty} \alpha_{i_1}^n = \lim_{n \to \infty} \lim_{\tau_\theta \to 0} \alpha_{i_1}^n = 1.$$

I can swap the two limits since $\sup_{\tau_\theta \geq 0} \alpha_{i_1}^n \leq 1$ and we can apply the dominated convergence theorem.

Finally, suppose $i_1 \in P$. Define $A_k \equiv \{i_2, \ldots, i_k : i_l \neq i_{l-1} \text{ and } i_2, \ldots, i_k \in P\}$ and $B_k \equiv \{i_2, \ldots, i_k : i_l \neq i_{l-1}, i_2, \ldots, i_{k-1} \in P, \text{ and } i_k \notin P\}$, By using these two subsets of agents, I can write:

$$\begin{aligned}\sigma_{i_1}(\mathbf{s}_{i_1}) = g_{i_1 i_1}\mathbb{E}_{i_1}[\theta] \quad &+ \quad \sum_{k=2}^{\infty}\sum_{A_k} g_{i_1 i_2} g_{i_2 i_3}\cdots g_{i_{k-1} i_k} g_{i_k i_k}\mathbb{E}_{i_1}\mathbb{E}_{i_2}\cdots\mathbb{E}_{i_k}[\theta] \\ &+ \quad \sum_{k=2}^{\infty}\sum_{B_k} g_{i_1 i_2} g_{i_2 i_3}\cdots g_{i_{k-1} i_k}\mathbb{E}_{i_1}\mathbb{E}_{i_2}\cdots\mathbb{E}_{i_{k-1}}[\sigma_{i_k}]\end{aligned}$$

Define $A_{i_1}^k = \sum_{A_k} g_{i_1 i_2} g_{i_2 i_3}\cdots g_{i_{k-1} i_k}$ and $B_{i_1}^k = \sum_{B_k} g_{i_1 i_2} g_{i_2 i_3}\cdots g_{i_{k-1} i_k}$. Then observe that

$$\begin{aligned} g_{i_1 i_1}\gamma \quad &+ \quad \sum_{k=2}^{\infty}\sum_{A_k} g_{i_1 i_2} g_{i_2 i_3}\cdots g_{i_{k-1} i_k} g_{i_k i_k}\gamma^k \\ &= \quad (A_{i_1}^1 - B_{i_1}^1)\gamma + (A_{i_2}^2 - B_{i_1}^2)\gamma^2 + (A_{i_2}^3 - B_{i_1}^3)\gamma^3 + \cdots \\ &= \quad c_i^{\mathbf{g}_P}(\gamma, G_P) - \sum_{k=1}^{\infty}\gamma^k B_{i_1}^k,\end{aligned}$$

where $\mathbf{g}_P = (g_{11}, g_{22}, \cdots, g_{pp})'$ is a p-dimensional vector. Thus, it follows that

$$\begin{aligned}\sum_{k=2}^{\infty} \quad &\sum_{B_k} g_{i_1 i_2} g_{i_2 i_3}\cdots g_{i_{k-1} i_k}\gamma^{k-1} \\ &= \quad B_{i_1}^1\gamma + B_{i_1}^2\gamma^2 + B_{i_1}^3\gamma^3 + \cdots \\ &= \quad \sum_{k=1}^{\infty}\gamma^k B_{i_1}^k\end{aligned}$$

Therefore, I have:

$$\lim_{\tau_\theta\to 0}\lim_{n\to\infty}\alpha_{i_1}^n \quad = \quad \lim_{n\to\infty}\lim_{\tau_\theta\to 0}\alpha_{i_1}^n = c_i^{\mathbf{g}_P}(\gamma, G_P).$$

It is easy to see that $c_i^{\mathbf{g}_P}(\gamma, G_P) = 1 - (1-\gamma)c_i(\gamma, G_P)$ and $\lim_{\tau_\theta\to 0}\lim_{n\to\infty}\beta_{i_1}^n = 1 - \lim_{\tau_\theta\to 0}\lim_{n\to\infty}\alpha_{i_1}^n$. This concludes the proof. $\square$

C.17 Proof of Proposition 14

Proof. For notational convenience, I write $c_i(\gamma, G_P) = 0$ for any agent $i \notin P$. Then for each agent $i \in N$,

$$\sigma_i^P(\mathbf{s}_i) - \theta = (1 - (1-\gamma)c_i(\gamma, G_P))\varepsilon_i + (1-\gamma)c_i(\gamma, G_P)\varepsilon_z.$$

Thus, it holds that

$$\begin{aligned} -\mathbb{E}[(\sigma_i^P - \theta)^2] \quad &= \quad -((\gamma - (1-\gamma)c_i(\gamma, G_P))^2\tau_x^{-1} + (1-\gamma)^2(c_i(\gamma, G_P)+1)^2\tau_z^{-1}) \\ &= \quad -\tau_x^{-1}((1-\gamma)(c_i(\gamma, G_P)+1)(c_i(\gamma, G_P)-1)+1)\end{aligned}$$

This implies that

$$E(\tau_z, P) \quad = \quad \tau_x^{-1}\left((1-\gamma)\sum_{i\in P} g_{ii}(1+c_i(\gamma, G_P))(1-c_i(\gamma, G_P)) - \sum_{i\in N} g_{ii}\right)$$

Hence, given τ_z, P^* is optimal if and only if $P \in \arg\min_{P'\subseteq N}\sum_{i\in P'} g_{ii}((c_i(\gamma, G_{P'}))^2 - 1)$. Since $i \in P^*$ only when $c_i(\gamma, G_P) \leq 1$ and $\frac{\partial \mathbb{E}[(\sigma_i^P - \theta)^2]}{\partial \tau_z} \leq 0$ for any information structure by Lemma 15, setting $\tau_z^* = \bar{\tau}_z$ is optimal given P^*. □