Predatory Short-selling and Self-fulfilling Crises*

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Abstract

This paper investigates the mechanism through which short-selling can cause failures of financial firms and state financing. We show that the fundamental problem of short-selling is that it leads to an increase in uncertainty and information asymmetry on the firm’s or the sovereign’s fundamentals. Debtholders, who care about downside risk but not upside risk, are averse to fundamental uncertainty. They may thus run, making default and failure self-fulfilling, while speculators profit from the failure. The paper suggests that naked short-selling can fuel a crisis and should be banned at times of market turbulence. Our work also contributes to the theoretical literature on global games by developing a model that simultaneously endogenizes strategic complementarities, information structure and payoff structure.

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Up until the crash of 2008, the prevailing view – called the efficient market hypothesis – was that the prices of financial instruments accurately reflect all the available information (i.e. the underlying reality). But this is not true...we must understand financial markets through a new paradigm which recognizes that they always provide a biased view of the future, and that the distortion of prices in financial markets may affect the underlying reality that those prices are supposed to reflect. (I call this feedback mechanism “reflexivity.”)

George Soros (2009)

1 Introduction

Short-selling, in particular “naked” short-selling of stocks and “naked” credit default swaps (CDSs), is called to be toughly regulated and even urged to be completely banned. These speculative activities have been blamed for playing an important and direct role in destabilizing the financial system recently.

Short-selling of stocks has been under accusation of constituting a direct cause of the collapses of Bear Stearns and Lehman Brothers. In the late 2008, several countries including US and UK imposed the emergency order to ban the short-selling of financial stocks. In executing the short sale-ban order (Securities Exchange Act NO. 34-58592 / September 18, 2008), the SEC concluded:

*Short selling in the securities of a wider range of financial institutions may be causing sudden and excessive fluctuations of the prices of such securities in such a manner so as to threaten fair and orderly markets.*

Similarly, naked CDSs have been accused of worsening the Greece sovereign debt crisis that has shaken the eurozone. On March 10th, 2010, four EU country leaders called in a joint letter for an

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1 Naked short selling of stocks is the practice of short-selling stocks without first borrowing the stocks. Thus, naked shorting is not limited by the actual number of shares outstanding. An infinite number of shares can be “created” for sell. Naked CDSs mean buying swaps without holding a direct investment in the underlying bonds. Buying CDSs is equivalent to shorting the underlying bonds. Naked CDSs make unlimited shorting of bonds possible.

EU inquiry to “prevent undue speculation” with regard to CDSs.³ On May 19th, 2010, Germany temporarily banned naked short selling and naked credit-default swaps of euro-area government bonds. In the statement, Germany’s financial market regulator, BaFin, said:⁴

*The ban was needed because of **exceptional volatility** in euro-area bonds. Massive short-selling was leading to excessive price movements which could endanger the stability of the entire financial system.*

This paper investigates the specific mechanism through which naked short-selling can cause failures of financial firms and state financing. We develop a theoretical model of *self-fulfilling* bank failures arising from short-selling of bank stocks. We show that the mechanism of speculative attacks on sovereign debt is essentially the same.

Received wisdom usually attributes the danger of short-selling of a bank’s stocks to that short-selling can artificially *bring down* the bank’s stock price.⁵ The low stock price then sparks concern regarding the bank’s fundamental value and may even erode counterparties’ confidence. After observing a very low stock price, the counterparties may be unwilling to roll over their credit to the bank. This in turn causes the failure of the bank. Figure 1 shows the basic logic under the argument.

[Figure 1]

A critical question regarding the above argument, however, is *why counterparties, as debtholders, should care about the stock price.* First, regulatory capital requirements of banks are not calculated based on stock prices.⁶ Second, in a rational-expectations framework, counterparties of a bank

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³Four leaders are French President, Luxembourg’s Prime Minister, German Chancellor and Greek Prime Minister. See “France, Germany back EU speculative trade ban”, EUbusiness, March 11, 2010.
⁴See “Germany to Ban Naked Short-Selling at Midnight”, Bloomberg, May 18, 2010.
⁵See, for example, Soros (2009).
⁶Regulatory capital is instead calculated based on the mark-to-market value of the assets and the book value of banks’ equity capital. The mark-to-market value of the assets is different from the stock price.
would rationally anticipate ‘predatory trading’ of speculators and understand that the low stock
price is due to speculators’ manipulation. So there is no reason as to why counterparties should
care about the artificial stock price.\footnote{In fact, in an official document by Standard & Poor titled “How stock prices can affect an issuer’s credit rating”, the rating agency states clearly: “The price of a company’s stock is one of the many factors that Standard & Poor’s Ratings Services may consider in rating a company’s debt. However, a company’s stock price, by itself, is never the primary factor in how we analyze the creditworthiness of a company. Instead, we emphasize fundamental analysis in our approach” (Standard & Poor’s RatingsDirect, September 26, 2008).}

This paper offers a different explanation for why short-selling of stocks can ‘kill’ a bank. In
our model, we show that the fundamental problem of short selling is that it creates uncertainty
and increases the information asymmetry between the bank and its creditors. Low stock prices
caused by short-selling are the symptom of the problem, but not the true underlying cause. We
show that short-selling not only changes the mean of the stock price (i.e., the first moment), but
also increases its volatility (i.e., the second moment). In the presence of imperfect information
about fundamentals, a higher volatility of the stock price leads to greater uncertainty around the
fundamental value. Creditors, who use the stock price to learn (in a Bayesian fashion) about the
fundamental value, will grow increasingly unsure about the fundamental value in facing a low stock
price. The low price may be due to short-selling or due to a true bad fundamental. In a rational-
expectations equilibrium, the creditors can still correctly assess the fundamental value, on average,
by observing the stock price. However, the precision of the estimation decreases, i.e., the variance
in assessing the fundamental value increases. That is, for a given stock price, both the downside
and the upside risk of the fundamental value increase. Considering that creditors’ (debt) payoff is
concave in fundamental value, creditors mainly care about the downside risk rather than the upside
risk. As a result, uncertainty leads to a reduction in debt value, which may cause creditors to run
and eventually lead to the bank’s failure.

We believe that the mechanism presented above is in line with what happened in reality. In fact,
in both the SEC’s official document and the BaFin’s statement, there appear key words such as
‘fluctuation’ and ‘volatility’. We show that the economic translation for such terms in our model is
‘information asymmetry’. That is, short-selling can lead to a reduction in capital market efficiency
and can exacerbate information asymmetry, which causes creditor runs.
Our model also answers two other important questions regarding short-selling attacks on banks. First, how can dispersed speculators form a collective action to significantly move the stock prices? Second, why do speculators choose to attack banks rather than standard corporations?

We show that the short-selling of financial stocks essentially involves two coordination problems or two runs. The first run is the coordination that is automatically formed among speculators (in the spirit of Morris and Shin (1998)). The more aggressive the speculators are in short selling, the less informative the stock price is. A less informative stock price leads to a higher likelihood of a creditor run, as the debt value is inversely related to the fundamental uncertainty. A higher likelihood of a credit run translates into a higher probability of success for the short-selling attack. In short, speculators have incentives to coordinate with each other in attacking the bank, resulting in aggressive short selling of every speculator. Clearly, the coordination mechanism or the strategic complementarity among speculators in our model is endogenous, which is different with that in the extant literature. The second run is the classic bank run (in the spirit of Diamond and Dybvig (1983)). This run is due to a coordination failure among creditors, which makes them behave conservatively and run on the bank even when the bank’s fundamentals are sound.

Significantly, we show that the two runs interact with, and reinforce each other, with the result of dramatically increasing the probability of a collapse of the bank. Figure 2 shows the two-way feedback loop between the two runs. We have already discussed one way of the feedback: the coordination actions of speculators impact the creditors’ decisions (through the stock price
informativeness channel). The other way of the feedback reflects how the creditors' actions affect speculators' run. We show that the creditors' actions impact the run of speculators through two channels. If the creditors decline to roll over at a higher fundamental value and thus the bank fails at a higher fundamental value, not only does this make the speculators' short-selling attack more likely to succeed, but also increases the speculators' profit in the case of a successful attack. In short, the two runs in our model interact through the endogenous information structure and the endogenous payoff structure. The speculators' run generates the public information for the creditors' run. The creditors' run in turn determines the payoff of the speculators' run.

The model offers two cross-sectional predictions. First, banks with lower fundamentals are more likely to be subject to short-selling attacks, and are more likely to fail. This in turn implies that a bank's fundamentals are the most important factor that determines the probability of both a short-selling attack and bank failure. Second, for given fundamental value, firms with higher maturity mismatches in their balance sheets are more likely to incur short-selling attacks. This implies that predatory short-selling should be more frequently observed on financial firms rather than standard corporations. Our research suggests that naked short-selling should be banned at times of market turbulence, as naked short-selling can fuel a crisis. At times of crisis, rational speculators would coordinate together (through market mechanisms) to attack and kill a weak but viable bank.

Finally, we show that the mechanism of short-selling attacks on sovereign debt through naked CDSs is essentially the same with that of short-selling of bank stocks. CDSs act like insurance. Buying protection through a CDS is equivalent to shorting the underlying bond. A higher spread of a CDS implies a higher default probability for the bond, at least as perceived by market participants. Speculators can coordinate to push up the spread of a CDS and increase its volatility. The uncertainty on the true CDS value, or the true default probability, then leads investors to demand a higher interest rate when a sovereign country like Greece is trying to issue its new debt. Higher borrowing costs impose greater pressure on the country's public finances, or directly make its new debt issuance impossible. The fundamentals of the bonds deteriorate. The default can become self-fulfilling while speculators gain from the default.

Related Literature. Although short-selling stabilizes prices most of the time, our paper studies “destabilizing effects” of short-selling at times of market turbulence.\(^8\) A related work to ours

\(^8\)Friedman (1953), Hart and Kreps (1986), and DeLong et al. (1990) contribute to the old debate of ‘destabilizing
is Brunnermeier and Pedersen (2005), who also study predatory trading. The authors show that predator traders may deliberately dump the asset that a distressed trader holds, depressing the asset's price and triggering the trader's constraints to bind (e.g., margin calls). Their model well captures situations of predatory trading in asset markets where traders like hedge funds typically face some external constraints, such as mark-to-market based margin calls from their brokers. However, for our research on short-selling of bank stocks, the stock price does not represent a relevant constraint for banks, as there are almost no explicit constraints for banks that are contingent to stock prices. In this paper, we instead offer the insight that it is the change in stock price informativeness, rather than the stock price itself, that explains why short-selling can kill a bank. To the best of our knowledge, this insight is new.

Morris and Shin (2004) study market runs. The authors show that sometimes market participants behave extremely conservatively and rush to sell, forming liquidity black holes. In some sense, our paper presents Brunnermeier-Pedersen meets Morris-Shin. In our model, two groups of investors move sequentially. The first group of investors coordinate together to conduct predatory selling, with the objective of triggering the conservative run of the second group. Crucially, we show that there exists a two-way feedback between the actions of these two groups, and that the feedback generates a hurricane-like self-reinforcing spiral.

Our model builds on Morris and Shin (1998) and Diamond and Dybvig (1983). By building a connection between the runs in these two papers, we endogenize the strategic complementarities, the information structure and the payoff structure of the global games in our model. Our work therefore contributes to the theoretical literature on global games by simultaneously endogenizing these three elements.
Our work relates to Angeletos and Werning (2006) in that both papers endogenize public information of global games.\textsuperscript{14} Crucially, there are two runs in our model, instead of one run as in Angeletos and Werning (2006). The first run generates the public information for the second run. Angeletos and Werning also study the feedback between asset price, coordination actions and dividends. The feedback in their paper, however, in part depends on the assumption that the payoff (i.e., the dividend) of the first-move group of investors is a function of the coordination actions of the later-move group of investors, while the feedback in our model is completely endogenous. Goldstein, Ozdenoren and Yuan (2009) study the feedback between traders and investors. Their model focuses on the impact of stock prices on firms’ real investment decisions while in our paper there are no real investments. Also, the mechanism of feedback in our paper - the change of stock price informativeness - is different with theirs.

The rest of the paper is organized as follows. Section 2 presents a model of short-selling of bank stocks. Section 3 discusses predictions and policy implications of the model. Section 4 analyzes speculative attacks on sovereign debt through naked CDSs. Section 5 concludes.

2 Model

2.1 The model setup

The model has three dates: $T_0$, $T_1$ and $T_2$. All agents are risk-neutral. There is no discount factor between $T_1$ and $T_2$. We discuss the three types of agents in our model in order: bank, speculators and creditors.

The bank

Consider a bank (investment or commercial bank) which holds one unit of asset. The asset, denoted as $A$, realizes a random cash flow $\tilde{\theta} = \theta + \epsilon$ at time $T_2$, where $\epsilon$ is normally distributed as $\epsilon \sim N(0, \sigma^2)$. The term $\theta$ represents the fundamentals of the bank, and has an ‘improper prior’ over the real line. The value of $\theta$ is realized at $T_0$, while the uncertainty of $\epsilon$ is resolved at $T_2$. The bank finances its asset with short-term debt ($D^S$) and equity ($E$). The short term debt is the borrowing

\textsuperscript{14}The information structure is endogenous also in Dasgupta (2007) and Angeletos, Hellwig and Pavan (2007).
from a continuum of lenders with unit mass. The debt is short-term in the sense that the creditors have the right to decide at $T_1$ whether to roll over their lending or not. If a creditor declines to roll over, her claim is the face value of debt at $T_1$, denoted by $F$. If a creditor rolls over, her claim is the face value of debt plus interest, amounting to a total value of $K$ at $T_2$, where $K > F$. For the equity, $E$, we assume that the bank has a single unit of divisible share outstanding. Figure 3 illustrates the bank’s balance sheet position.

<table>
<thead>
<tr>
<th></th>
<th>Asset (illiquid)</th>
<th>Debt (short-term)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$L$</td>
<td>$F$</td>
</tr>
<tr>
<td>$E$</td>
<td>$\tilde{\theta}$</td>
<td>$K$</td>
</tr>
</tbody>
</table>

Note: $L < F$ and $F < K$

Figure 3

We assume that the asset of the bank is illiquid. The liquidation value of asset $A$ at $T_1$ is $L$, where $L < F$. Therefore, if more than $\frac{L}{F}$ proportion of creditors decline to roll over at $T_1$, the liquidation value is not sufficient to cover the creditors’ claims, and consequently the bank fails. Alternatively, we may think of $L$ as the collateral value of the bank’s asset. This means that the bank can raise at most $L$ amount of cash at $T_1$ by using its asset as collateral. If the demand of cash exceeds $L$ at $T_1$, the bank fails. This is the classic bank run problem. We use the term $L$ (for a given $F$) to measure the maturity mismatch of balance sheet of the bank: the lower the $L$, the more severe the maturity mismatch.

**Speculators**

There is a continuum of speculators with unit mass. At $T_0$, these speculators receive private information regarding $\theta$. The information is imperfect. Specifically, speculator $i$ observes a noisy signal $\theta^i = \theta + \omega^i$, where $\omega^i$ is normally distributed $\omega^i \sim N(0, \delta^2)$ and the noises are independent across speculators. Based on the private information received, a speculator decides whether to short sell the bank’s stock at $T_0$ or not: a speculator can only short sell one unit of stock or decide not to take any position.

Note that the speculators’ choice of shorting is endogenous. Basically, speculators need to gain from their positions. Because only ‘short’ positions can benefit from the fall in the stock price
(i.e., the bank failure), speculators choose to attack by shorting. This is opposite to the attacks on sovereign debt through CDS. When a sovereign defaults, the price of CDS increases rather than decreases. So, only the attacks by long positions in CDS can gain from the default.

If a bank run occurs at $T_1$ and consequently the bank fails, this means that the short-selling attack is successful. Thus, speculators make a gain. At this stage, we assume that the (gross) gain of a successful short-selling attack is a fixed number, $t$, where $t > c$. We will return to this assumption later on. In contrast, if the bank does not fail at $T_1$, the short-selling attack is unsuccessful and the speculators’ (gross) gain is 0. The cost to conduct short selling (e.g., margin or opportunity cost), whether the short-selling is successful or not, is $c$, where $c > 0$.

Speculator $i$’s decision rule at $T_0$ is a map:

$$
\theta^i \rightarrow \text{(Short sell, Not)},
$$

where $\theta^i$ is speculator $i$’s information and (Short sell, Not) is her decision set.

**Stock market**

The stock price at $T_1$ is given by $p = v - ds$, where $v$ is the fundamental value of the stock (i.e., the expected value of the equity at $T_2$), whereas $d$ measures the market liquidity (depth) of the stock, and $s$ represents the aggregate short-selling by speculators at $T_0$.

The above scheme of stock price is based on Grossman and Miller (1988) and is employed by a large literature (e.g., Bernardo and Welch (2004), Morris and Shin (2004), Brunnermeier and Pedersen (2005), Plantin, Sapra, and Shin (2008), etc). Morris and Shin (2004) discuss in detail the theoretical and empirical background of this downward-sloping residual demand curve.

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15 This means that the net gain of a successful short-selling attack is $t - c$, while the net gain of an unsuccessful short-selling is $-c$. Also, the zero gross gain of an unsuccessful short-selling can be an endogenous result in our model. As will be seen later, the expected execution price of a selling order of the stock is $t = v - \frac{1}{2}ds$, while the stock price reverses to its fundamental value $v$ if the bank doesn’t fail. So the expected gross gain is $(v - \frac{1}{2}ds) - v$, whose expectation is 0.

Specifically, we follow the framework in Brunnermeier and Pedersen (2005). In addition to the strategic traders (i.e., speculators), the market is populated by long-term investors such as individual investors and other unsophisticated investors. Those investors are passive and price-takers. They do not have sufficient information, skills, or time to predict short-term events. Their demand is only based on the long-term fundamental information. They are ‘buy and hold’ investors. The demand from an individual investor among them is \( y^i(p) = \frac{v^i - p}{d} \), where \( v^i \) is the individual’s private information regarding the fundamentals and \( p \) is the price.\(^{17}\) Hence, the aggregate demand from those investors is \( Y(p) = \int y^i(p) = \frac{v - p}{d} \), which is a function of the true fundamentals, \( v \), as noises of private information cancel out in aggregating. In equilibrium, net supply equals demand, \( s = Y \), and thus the price is \( p = v - ds \).

Using the above setup, the stock price in our model reflects the long-term fundamental value of the bank. That is, since the long-term investors - the market markers - do not have the ability to predict the short-term events, the stock price is not forward-looking to incorporate the probability of short-term bank failure arising from a bank run at \( T_1 \). Hence, \( v \) is equal to the expected value of equity at \( T_2 \), i.e., \( v = E[\max(0, \theta - K) | \theta] \). In order to have a clean analysis of the model, we abstract from complications and use a simpler version of equity value \( v = E[\theta - K | \theta] \). So \( v = \theta - K \). The mechanism of the model does not depend on this simplification.\(^{18}\) Therefore, the stock price at \( T_1 \) is given by \( p = \theta - K - ds \). We denote \( P \equiv p + K \), where \( P \) is interpreted as the asset value of the bank inferred from the stock price. We have \( P = \theta - ds \).

Further, we assume that the market liquidity \( d \) is random. Pastor and Stambaugh (2002) document time-varying random market liquidity. Randomness in market liquidity can be due to that the size of the market-making sector is random. That is, there are times when a large number of long-term investors are able to provide liquidity to the market and \( d \) is low, whereas there are times when the opposite is true and \( d \) is high. The randomness of \( d \) can also be due to time-varying risk aversion of market makers. We will specify the probability distribution of \( d \) later.

**Creditors**

At \( T_1 \), all creditors observe the stock price \( p \) and hence know \( P \). Further, they receive private  
\(^{17}\)Those investors are passive and price-taking. They do not learn information from the stock price.  
\(^{18}\)We elaborate on this point in the appendix.
information regarding $\theta$. Creditor $j$’s private signal is $\theta^j = \theta + \zeta^j$, where $\zeta^j$ is normally distributed as $\zeta^j \sim N(0, \gamma^2)$. Based on both stock price information and the private signal, each creditor needs to decide whether to roll over her lending. That is, creditor $j$’s decision at $T_1$ is a map: $(\theta^j, P) \mapsto$ (Roll over, Not).

To fully describe the payoff structure of creditors in a bank run, we need to write down the payoff of an individual creditor to roll over (vs. not roll over) as a continuous function of the total number of creditors who roll over. For the payoff structure of a bank run with continuous states, we refer the reader to Diamond and Dybvig (1983), Goldstein and Pauzner (2005) and Liu and Mello (2008). For our purposes, however, it is sufficient to use a discrete-state setup which conveniently simplifies the analysis. More precisely, following the setup in Morris and Shin (2009), we assume the payoff matrix of the bank-run game as follows:

$$
\begin{array}{c|cc}
 & \text{Roll over} & \text{Withdraw} \\
\hline
\text{More than } \frac{L}{P} \text{ proportion withdrawing} & 0 & F \\
\text{Less than } \frac{L}{P} \text{ proportion withdrawing} & K & F \\
\end{array}
$$

In the above payoff matrix, if less than $\frac{L}{P}$ proportion of creditors withdraw, the bank does not fail. New creditors can eventually be found to replace the old creditors who withdraw, and the bank restores to the initial status after the failed run. In this case, the repayments to creditors are not affected: the payoff is the notional value $K$ in case of rolling over, and $F$ in case of withdrawing. In contrast, if more than $\frac{L}{P}$ proportion of creditors withdraw, the bank fails. The creditors who roll over are left with nothing, whereas the creditors who withdraw have a guaranteed payoff of $F$.\footnote{If more than $\frac{L}{P}$ proportion of creditors withdraw and thus the bank fails, the payoff for a creditor who withdraws should be slightly less than the face value $F$ as the total liquidation value is $L$, which is less than $F$. However, the face value $F$ is a close \textit{approximation} of the realized payoff (see figure 4).}

Figure 4 compares the simplified payoff structure (in Morris and Shin (2009) as well as this paper) with the full payoff structure (in Liu and Mello (2008)). Clearly, the former is a close \textit{approximation} of the latter. The simplified setup suffices to catch the key feature of the bank-run game — the strategic-complementary payoffs. That is, if more than $\frac{L}{P}$ proportion of creditors withdraw, it is better for an individual creditor to withdraw as well. If less than $\frac{L}{P}$ proportion of creditors withdraw, it is better for an individual creditor also to hold.
Figure 5 describes the timeline. Figure 6 summarizes the main setups in this subsection.

Figure 5

1. The value of θ is realized.
2. Speculators receive private signals regarding θ.
3. Based on her private information, each speculator decides whether to short sell.
4. The stock price is formed.
5. Creditors receive private signals regarding θ.
6. Based on both the stock price and their private signals, creditors decide whether to roll over.
7. If more than L/F proportion of creditors don’t roll over, the bank fails and goes to bankruptcy.
8. If the bank doesn’t fail earlier, the cash flow of the asset is realized and the creditors are repaid based on the initial contract.
2.2 A simple example for the mechanism of the model

In order to facilitate the analysis in the next subsection, here we use a simple example to illustrate the key mechanism of the model.

Specifically, we assume that $d$ follows the simple two-state Bernoulli distribution: $d$ can be either 0 or 1 with 50%-probability each. That is, the size of market-maker sector is sometimes small and speculators can influence the stock price, i.e., $d = 1$, while sometimes there are a large number of long-term traders providing liquidity to the market, who completely digest the selling of speculators, i.e., $d = 0$. Intuitively, this is equivalent to that the market depth is deterministic (i.e., not random), but speculators face noise-trading risk. In one state, noise traders are few, $d$ is a positive number (normalized to be 1) reflecting the true market depth, and hence speculators can influence the stock price. In the other state, noise traders are abundant. They come out and absorb the selling of speculators, and $d = 0$. Further, as will be shown later, in equilibrium, $s$ is deterministic and is public information. We assume that $s = 2$. We further assume the following parameter values: $\theta = 7$, $K = 6$, and $\sigma = 0$.

We compute the debt value in a rational-expectations equilibrium. Note that $P = \theta - ds$. If there is no short-selling, $P = \theta = 7$. The debt value, $D = \min(\theta, K)$, is 6. Suppose there is short-selling. Given the fundamental value $\theta = 7$, the price $P$ can be 7 or 5, depending whether the market is liquid enough. In the case of $P = 7$, the creditors rationally expect (with Bayesian inference) that $\theta$ can be either 9 or 7 with equal probabilities, that is, $E(\theta|P = 7) = 8$.\footnote{Note that $\theta$ has an ‘improper prior’ over the real line.} In the
case of $P = 5$, the creditors rationally expect that $\theta$ can be either 7 or 5 with equal probabilities, that is, $E(\theta|P = 5) = 6$. Therefore, the estimation of $\theta$ is unbiased, i.e., $E[(\theta|P)|\theta] = \theta$. For the debt value, however, we can easily obtain $E(D|P = 7) = 6$ and $E(D|P = 5) = 5.5$. Hence, $E[(D|P)|\theta] < K$. That is, short-selling creates uncertainty and increases information asymmetry, causing a reduction in the expected debt value.

Figure 7 demonstrates the example. Clearly, in the figure, under short-selling, the posterior distribution of $\theta$ is a mean-preserving spread of the true fundamentals $\theta$. Because $D$ is concave with respect to $\theta$, short-selling reduces the expected debt value.

| $\theta$ | $P$ | $\theta|P$ | $D|P$ | $E(D|P)$ | $E(\theta|P)$ |
|----------|-----|-----------|------|---------|-------------|
| No short-selling: | 7 | 7 | 7 | 6 | |
| Short-selling: | 7 | 5 | 7 | 6 | 5.5 | 6 |

![Figure 7](image)

From the above example, short-selling does decrease the (expected) stock price. However, the lower stock price itself is not the underlying cause for the reduction in debt value and the bank failure. In fact, creditors can rationally expect that a low stock price may be due to short-selling. Thus, when estimating the fundamental value $\theta$, they would take short-selling pressure into consideration and offset the possible price derivation. That is, creditors can correctly estimate the fundamental $\theta$, on average. Therefore, the key reason for the reduction in the (expected) debt value is the increase in uncertainty, rather than the fall in the stock price.

### 2.3 The equilibrium of the model

In the formal analysis of the model, we assume that $d$ is with the normal distribution $d \sim N(0, h^2)$. Noting that $d$ is typically positive, however, the assumption of normal distribution is to obtain
closed-form solutions of the model (for global game reasons) and is unrelated to the mechanism of the model.\footnote{We can assume $d \sim N(a, h^2)$, where $a > 0$. In this case, the price $P$ at $T_1$ follows the distribution $P \sim N(\theta - as, h^2 s^2)$. So short selling not only depresses the mean of stock prices but also increases the variance. However, as illustrated in the previous example, the danger of short-selling does not lie with the change in the mean rather than in the variance. In fact, in equilibrium, creditors can rationally anticipate the (expected) magnitude of short-selling and hence offset the term $-as$. Using the language of statistics, from $P$ there is an \textit{unbiased} estimation of $\theta$. That is, short-selling does not induce \textit{bias} in the estimation of the fundamental value, but affects its \textit{efficiency}.}

With the above setup, the price $P$ at $T_1$ has the distribution $P \sim N(\theta, h^2 s^2)$.\footnote{With this distribution, it seems that ‘long’ attacks can also increase uncertainty and cause the failure of the bank. However, speculators need to gain from their positions and therefore choose to attack by shorting.}

We consider the threshold (monotone) equilibrium of the model. That is, both the speculators and the creditors use threshold strategies. Specifically, the speculators’ strategy is $\theta_i \mapsto \begin{cases} 
\text{Not short} & \theta_i \geq \theta^* \\
\text{Short} & \theta_i < \theta^* 
\end{cases}$, where $\theta^*$ is the threshold. The creditors use the strategy $(\theta^i, P) \mapsto \begin{cases} 
\text{Roll over} & \theta^i \geq \theta^{**}(P) \\
\text{Not Roll over} & \theta^i < \theta^{**}(P) 
\end{cases}$, where $\theta^{**}(P)$ is the threshold, which itself is a function of $P$.

The procedure to work out the equilibrium is by backward induction, from $T_1$ to $T_0$. We analyze the equilibria in the following four cases.

\textbf{2.3.1 (Case 1) The constant payoff to short-selling with no creditor runs}

In this benchmark case, we assume that the creditors of the bank can coordinate their actions and avoid a creditor run. We may think that there is only one single large lender of the bank, so there exists no coordination problem among creditors. Further, we assume that $0 < h < +\infty$ and $\gamma = +\infty$, so that the creditors’ decision does not depend on their private information. With these simplifications, we can shut down the effect of the coordination problem among creditors and focus on the coordination problem among speculators.

Conditional on the speculators using the threshold $\theta^*$ and the fundamental value being $\theta$, the proportion of speculators short selling at $T_0$ is $s(\theta; \theta^*) = \Pr(\theta^i < \theta^*|\theta) = \Phi(\frac{\theta^*-\theta}{\delta})$, where $\Phi(\cdot)$ is the
cumulative distribution function (cdf) of the standard normal distribution. So \( s(\theta; \theta^*) \) is increasing in \( \theta^* \) and decreasing in \( \theta \). In particular, if there is no short-selling at all (e.g., short-selling is banned): \( \theta^* = -\infty \), then we have \( s(\theta; \theta^* = -\infty) \equiv 0 \).

We find the equilibrium in three steps.

**Step 1** Given the speculators’ strategy at \( T_0 \), we work out the creditors’ optimal strategy at \( T_1 \). If the creditors decline rolling over at \( T_1 \), their payoff is \( F_{23} \). If the creditors roll over, their payoff is the debt value \( D(\theta) = E[\min(\bar{\theta}, K)|\theta] \). Clearly, \( D(\theta) \) is globally concave with respect to \( \theta \). A creditor infers the debt value based on the stock price information and her private signal. At \( T_1 \), the price \( P \) is distributed as \( P \sim N(\theta, (h \cdot s(\theta; \theta^*))^2) \) while the private signal has the distribution \( \theta^j \sim N(\theta, \gamma^2) \). Thus, the posterior expectation value of the debt is \( E[D(\theta)|P, \theta^j; \theta^*] \).

A creditor rolls over her lending if and only if

\[
E[D(\theta)|P, \theta^j; \theta^*] \geq F. \quad (1)
\]

Let us consider the case \( \theta^* = -\infty \). In this case, there is no short-selling and the stock market is efficient, i.e., \( P = \theta \). That is, the creditors know \( \theta \) perfectly through the stock price. We have theorem 1.

**Theorem 1** If no short selling exists and there is no bank run, the bank fails whenever \( \theta < \theta^{FB} \), where \( \theta^{FB} \) satisfies \( D(\theta^{FB}) = F \).

If \( \theta^* > -\infty \), there exists short-selling. The price information \( P \) becomes noisy and is not perfectly informative. Because we assume that \( 0 < h < +\infty \) and \( \gamma = +\infty \) in this benchmark, creditors’

\[23\] Of course, if all creditors decline to roll over, their payoff is \( L \) rather than \( F \). But here this setup is only for benchmark purposes. When we consider the creditors’ run below, an individual creditor who declines to roll over obtains \( F \) as in Diamond and Dybvig (1983).

\[24\] We can write \( D(\theta) = E[\max(0, \min(\bar{\theta}, K))|\theta] \). In this case, \( D(\theta) \) is concave with respect to \( \theta \) when \( \theta \) is sufficiently large. This alternative expression does not affect the results of our model. In fact, in our model, the parameter condition requires that \( \frac{\gamma}{K} \) is close to 1. This condition guarantees that all relevant \( \theta s \) are sufficiently large. So the probability that \( D(\theta) \) is non-concave is very small. We can neglect the region of \( \theta \) in which \( D(\theta) \) is non-concave.
private information is too noisy relative to the stock price information. Creditors only look at the public information. We rewrite the creditors’ decision rule as $(\theta^1, P) \rightarrow \begin{cases} \text{Roll over} & P \geq P^{**} \\ \text{Not} & P < P^{**} \end{cases}$, where $P^{**}$ is the threshold. We have

$$E[D(\theta)|P = P^{**}; \theta^*] = F \quad \text{or} \quad \int_{-\infty}^{+\infty} D(\theta) \cdot f(\theta|P = P^{**}; \theta^*) d\theta = F,$$  

where $f(\theta|P; \theta^*)$ is the posterior density of $\theta$.

Considering that $\theta$ has an ‘improper prior’, it is easy to obtain the posterior density $f(\theta|P; \theta^*) = \frac{f(P|\theta; \theta^*)}{\int_{-\infty}^{+\infty} f(P|\theta; \theta^*) d\theta}$, where $f(P|\theta; \theta^*)$ is the density function of $P \sim N(\theta, (h \cdot \Phi(\frac{\theta - \theta^*}{\delta}))^2)$.

Figure 8 plots a family of the posterior density function $f(\theta|P; \theta^*)$.

25This decision rule can be equivalently written as $(\theta^1, P) \rightarrow \begin{cases} \text{Roll over} & \theta^1 \geq \theta^{**(P)} \\ \text{Not} & \theta^1 < \theta^{**(P)} \end{cases}$, where $\theta^{**(P)} = \begin{cases} +\infty & \text{if } P < P^{**} \\ -\infty & \text{if } P \geq P^{**} \end{cases}$.
We need to show that the bank failure threshold $P^{**}$ increases in $\theta^*$. Intuitively, if $\theta^*$ increases, the distribution $P|\theta$ becomes more volatile, so does the posterior distribution of $\theta|P$. It then follows that the creditors demand a higher ‘premium’ because their debt payoff is concave.

**Lemma 1** $P^{**}$ is continuous with respect to $\theta^*$ and takes values on the bounded interval $(\theta^{FB}, \overline{P})$. Furthermore, $P^{**}$ increases in $\theta^*$ when $\theta^*$ lies within an intermediate region $(\theta^L, \theta^H)$.

Proof: See the Appendix.

Lemma 1 highlights two results. First, that the influence of short-selling on creditors’ decisions is limited. That is, $P^{**}$ is bounded. Second, that $P^{**}$ increases in $\theta^*$. That is, the more aggressive the short selling, the more likely that the creditors decline to roll over. Figure 8 illustrates the idea of the second result. When $\theta^*$ increases, the posterior distribution $\theta|P$ gets fatter tails, meaning that both downside risk and upside risk of $\theta$ increase. But the creditors mainly care about the downside risk because $D(\theta)$ is concave. So they set a higher threshold $P^{**}$. A technical complication is that if $\theta^*$ is extremely low or high, the distribution of $f(\theta|P; \theta^*)$ does not become fatter in both tails when $\theta^*$ increases. In these cases, $P^{**}$ is not increasing in $\theta^*$. However, we prove in Step 3 that speculators never choose an extremely low or high $\theta^*$. All relevant $\theta^*$s that creditors possibly choose lie within $(\theta^L, \theta^H)$.

Figure 9 illustrates that $P^{**}$ increases in $\theta^*$ for a set of parameter values, and $\theta^{FB} = 11.644$. 

![Figure 9](image-url)
Step 2  Given the strategy used by the creditors, we calculate the probability of bank failure at $T_1$. Let $\Pr(\theta; \theta^*)$ be the probability of bank failure for a realized fundamental $\theta$. Because $P \sim N(\theta, (h \cdot s(\theta; \theta^*))^2)$, we obtain $\Pr(\theta; \theta^*) = \Pr(P < P^{**}(\theta^*) | \theta; \theta^*) = \Phi\left(\frac{P^{**}(\theta^*) - \theta}{h \cdot s(\theta; \theta^*)}\right)$. We have Lemma 2.

Lemma 2  The probability of bank failure, $\Pr(\theta; \theta^*)$, decreases in $\theta$. For a given $\theta$, $\Pr(\theta; \theta^*)$ increases in $\theta^* (\in (\theta^L, \theta^H))$ for a high enough $\theta$.

Proof: see the Appendix.

Mathematically, an increase in $\theta^*$ has two effects: changing both the threshold $P^{**}(\theta^*)$ and the standard deviation $h \cdot s(\theta; \theta^*)$. The corresponding economic intuition is as follows. The increase in $\theta^*$ causes compound information asymmetry: both the posterior distribution $\theta | P$ and the distribution $P | \theta$ itself become more diffuse. The diffusion in the posterior distribution $\theta | P$ leads to the high uncertainty of creditors in conjecturing the true fundamental $\theta$. The uncertainty causes creditors to set a higher threshold $P^{**}(\theta^*)$. On the other hand, the distribution $P | \theta$ itself becomes more diffuse.
The spread in $P$ makes it harder to reveal the true fundamental $\theta$. Thus, ‘pooling’ increases. The bank with a high $\theta$ may end up realizing a low $P$ while the bank with a low $\theta$ may realize a high $P$.

Geometrically, in Figure 10, the increase in $\theta^*$ not only causes the cdf curve $\Pr(\theta; \theta^*)$ to shift to the right (the first effect), but also ‘squeezes’ the curve to make it flatter (the second effect). Clearly, when $\theta$ is high enough such as $\Pr(\theta; \theta^*) \leq \frac{1}{2}$, both effects are positive and, hence, $\Pr(\theta; \theta^*)$ is certainly increasing in $\theta^*$. Also, the higher the $\frac{F}{K}$, the wider the region of $\theta$ in which $\Pr(\theta; \theta^*)$ is increasing in $\theta^*$.26

Figure 11 shows the result of Lemma 2 for a set of parameter values.

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**Step 3** We go back to $T_0$ and work out the equilibrium strategy of speculators. First, we find two dominance regions of signal where speculators have a dominant strategy independent of their beliefs about other speculators’ actions. From Lemma 1, we know that the creditors’ threshold is bounded, $P^{**} \in (\theta^{FE}, \overline{P})$. That is, short-selling can influence the decisions of the creditors to a certain level, which is not unlimited. All speculators realize this. Therefore, when a speculator receives a very strong signal $\theta^i$, she is not going to short sell whatever her beliefs about other speculators.26

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26Intuitively, if $\frac{F}{K}$ is high, the curve of $D(\theta)$ around the threshold $\theta = P^{**}$ is very flat. The upside risk is little while the downside risk is large. Therefore, $\frac{dP^{**}(\theta^*)}{d\theta}$ is large. A slight increase in $\theta^*$ leads to a big increase in $P^{**}$. So the first effect is dominating. Consequently, $\Pr(\theta; \theta^*)$ is increasing in $\theta^*$ for a wide region of $\theta$. See the Appendix for details.
speculators’ actions. When her signal $\theta^i$ is very weak, she is going to short sell whatever her beliefs about other speculators’ actions. Formally,

**Lemma 3** There exists a lower boundary $\underline{\theta}$ and an upper boundary $\overline{\theta}$ such that a speculator definitely short sells if her signal $\theta^i < \underline{\theta}$ and definitely doesn’t short sell if $\theta^i > \overline{\theta}$, whatever her belief about other speculators’ actions and whatever her belief about creditors’ beliefs on speculators’ actions. That is, a speculator’s threshold $\theta^*$ can never be outside the interval $[\underline{\theta}, \overline{\theta}]$.

Proof: See the Appendix.

When a speculator’s signal falls within $[\underline{\theta}, \overline{\theta}]$, her action depends on her belief about other speculators’ actions. Conditional on all other speculators using the threshold strategy with the threshold as $\theta^*$ and creditors using the strategy in Step 1 as the response to speculators, we compute the optimal strategy of speculator $i$. She short sells if and only if

$$\int_{-\infty}^{+\infty} \left[ \Pr(\theta; \theta^*) \cdot t - c \right] \cdot dG(\theta|\theta^i) \geq 0,$$

where $G(\theta|\theta^i)$ is the conditional cdf of $\theta$.

The threshold of speculator $i$, denoted as $\theta^{i*}$, satisfies:

$$\int_{-\infty}^{+\infty} \left[ \Pr(\theta; \theta^*) \cdot t - c \right] \cdot dG(\theta|\theta^{i*}) = 0 \quad \text{or}$$

$$\int_{-\infty}^{+\infty} \Pr(\theta; \theta^*) \cdot dG(\theta|\theta^{i*}) = \frac{c}{t}. \quad (3)$$

We prove that, given $\theta^*$, there exists a unique solution of $\theta^{i*}$ in (3). Further, if $\frac{c}{t}$ is small enough (e.g., $\frac{c}{t} \leq \frac{1}{2}$) or $\frac{c}{K}$ is high enough, $\theta^{i*}$ is increasing in $\theta^*$.

**Lemma 4** $\theta^{i*}$ is increasing in $\theta^*$ ($\in (\theta^L, \theta^H)$). That is, there exist strategic complementarities among speculators in short-selling.

Proof: See the Appendix.
Lemma 4 is an important result of the model. It formally shows that there exist strategic complementarities among short-sellers: if other speculators are aggressive in short selling, it is optimal for an individual speculator to be aggressive as well. Importantly, the strategic complementarities among short-sellers in our paper are *endogenous*. It is through the impact on creditors’ rollover decisions that the speculators create strategic complementarities among themselves. Figure 12 illustrates the strategic complementarity results for a set of parameter values.

![Figure 12](image)

After showing the strategic complementarity and the existence of two dominance regions of actions, we are able to prove that there exists a unique equilibrium of the benchmark model. By symmetric equilibrium, we have

\[ \theta^{i*} = \theta^*. \]  

(4)

We need to prove that equation system (3)-(4) has a unique solution.\(^{27}\)

**Theorem 2** There exist a unique equilibrium of the benchmark model. In the equilibrium, speculators short sell at a threshold higher than \( \overline{\theta} \), i.e., \( \theta^* > \overline{\theta} \).

\(^{27}\)Our proof for the unique equilibrium is equivalent to the proof with iterated deletion of strictly dominated strategies. According to Milgrom and Roberts (1990) and Morris and Shin (2003), if a symmetric game with strategic complementarities has a unique symmetric Nash equilibrium, then the strategy played in that unique Nash equilibrium is also the unique strategy surviving iterated deletion of strictly dominated strategies.
Proof: See the Appendix.

Figure 12 presents the equilibrium, where the intersection between the curve $\theta^*(\theta^*)$ and the 45 degree line gives the unique equilibrium $\theta^*$. The equilibrium $\theta^*$ is higher than $\bar{\theta}$. Intuitively, if all other speculators commit not to short sell, the optimal threshold for an individual speculator to short sell is $\frac{c}{t}$. However, speculators cannot commit not to short sell. As every speculators think that others may short sell and because of the strategic complementarities among them, in equilibrium, we have $\theta^* > \frac{c}{t}$.

Figure 13 summarizes the basic logic to find the benchmark equilibrium.

We have two properties regarding the equilibrium $\theta^*$.

**Corollary 1** A sufficient condition to guarantee $\theta^* > \theta^{FB}$ is $\frac{c}{t} < k$, where $k$ is slightly smaller than $\frac{1}{2}$.

Proof: See the Appendix.

In Figure 12, the equilibrium threshold is $\theta^* = 15.974$, which is higher than $\theta^{FB} = 11.644$.

Also, we show that the equilibrium threshold $\theta^*$ is an increasing function of $t$. That is, the higher the gain of short-selling, the more aggressive the short-sellers are. Essentially, we need to prove that the solution of $\theta^*$ in equation (5) is increasing in $t$.

$$\int_{-\infty}^{+\infty} [\Pr(\theta; \theta^*) \cdot t - c] \cdot dG(\theta|\theta^*) = 0. \quad (5)$$

**Corollary 2** The equilibrium threshold $\theta^*$ of speculators is an increasing function of $t$.

Proof: See the Appendix.
2.3.2 (Case 2) The endogenous payoff to short-selling with no creditor runs

So far we have assumed that the gain from short selling is a constant value $t$. In practice, however, the gain depends on the level at which speculators start to short sell. The expectation of the executed selling price when a speculator short sells the stock is $\frac{1}{2}[v + (v - ds)] = v - \frac{1}{2}ds$. If the bank fails, speculators spend zero money to buy the stock back. So the gain of short-selling is $v - \frac{1}{2}ds$. Conditional on the fundamental $\theta$, the expected gain is $t = E(v - \frac{1}{2}ds|\theta) = \theta - K$. Therefore, under the new scheme of the gain of short-selling, equation (5) is replaced by:

$$\int_{-\infty}^{+\infty} [\Pr(\theta; \theta^*) \cdot (\theta - K) - c] \cdot dG(\theta|\theta^*) = 0. \quad (6)$$

By comparing (6) with (5), we can see a spiral effect arising under the endogenous payoff scheme: the higher the gain $t = \theta - K$ of short-selling, the higher the threshold $\theta^*$ by Corollary 2; a higher threshold $\theta^*$ drives expected gain further higher because $E(\theta - K) = \theta^* - K$; and so on.

Intuitively, as Morris and Shin (2008) argue, two factors determine the probability of a run: First, the threshold for coordination (not) to run, and, second, the (opportunity) cost of miscoordination. In our model, the bank failure threshold $P^{**}$ affects the aggressiveness of speculators through both these two channels. A high $P^{**}$ not only makes the short-selling attack easy to succeed (i.e., the first channel) but also increases the gain of short-selling in case of a successful attack (i.e., the second channel). These two forces jointly push up the equilibrium threshold level $\theta^*$. Figure 14 illustrates the equilibrium idea.

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28 As in Morris and Shin (2004), the selling orders are placed in the queue for execution by matching with the limit buy orders posted by the market-making sector. The place of a selling order in the queue for execution is uniformly distributed in the interval $[0, s]$, where $s$ is the aggregate sell.
2.3.3 (Case 3) The constant payoff to short-selling with creditor runs

In this subsubsection, we consider the coordination problem among creditors. Creditors cannot coordinate their actions to behave like one single large lender. That is, there is a risk of creditor run, which leads the bank to fail at a higher fundamental value. This, in turn, induces the speculators to start to short sell at a higher threshold.

Recall that the creditors receive both public information (the stock price) and private information: \( P \sim N(\theta, (h \cdot s(\theta; \theta^*))^2) \) and \( \theta^j \sim N(\theta, \gamma^2) \). In order to model the creditor run, we assume that \( 0 < h < +\infty \) and \( 0 < \gamma < +\infty \). That is, both private and public information play a role in creditors’ rollover decisions. We have already defined the threshold \( \theta^{**}(P) \), which is the threshold at which creditors decline to roll over. Here we define another threshold \( \hat{\theta}(P) \), which is the threshold of fundamental value \( \theta \) below which the bank fails (for a given \( P \)). Solving the creditor run problem is to figure out these two thresholds. Fixing \( P \), we solve \( \theta^{**} \) and \( \hat{\theta} \) simultaneously.

Given the threshold \( \theta^{**} \), the proportion of creditors declining to roll over conditional on the realized fundamentals \( \theta \) is \( \text{pr}(\theta^j < \theta^{**} | \theta) = \Phi(\frac{\theta^{**} - \theta}{\gamma}) \). Recall that the bank fails if more than \( \frac{L}{P} \) proportion of creditors decline to roll over. Then, we can work out \( \hat{\theta} \) by solving the following equation:

\[
\Phi(\frac{\theta^{**} - \hat{\theta}}{\gamma}) = \frac{L}{P}.
\]

(7)

Since \( \hat{\theta} \) decreases in \( L \) in (7), equation (7) implies that the more severe the maturity mismatch is (i.e., the lower the \( L \)), the higher the threshold \( \hat{\theta} \) for a given \( \theta^{**} \).

Next, we consider the position for an individual creditor. In equilibrium, the creditor at margin who just receives the signal \( \theta^{**} \) should be indifferent between rolling over and not. That is,

\[
\int_{-\infty}^{\hat{\theta}} 0 \cdot d\theta + \int_{\hat{\theta}}^{+\infty} D(\theta) \cdot f(\theta | P, \theta^j = \theta^{**}; \theta^*) d\theta = F,
\]

(8)

where \( f(\theta | P, \theta^j; \theta^*) \) is the posterior pdf.

By solving the system of equations (7)-(8), we can obtain \( \theta^{**} \) and \( \hat{\theta} \), both of which are functions of \( P \). The two thresholds \( \theta^{**}(P) \) and \( \hat{\theta}(P) \) fully characterize the creditor run.
We need to examine existence and uniqueness of the threshold equilibrium. By (7), we have
\[ \tilde{\theta} = \theta^{**} - \gamma \cdot \Phi^{-1}(\frac{L}{F}) \]. Thus, we can combine (7) and (8) as
\[ \int_{\theta^{**} - \gamma - \Phi^{-1}(\frac{L}{F})}^{+\infty} D(\theta) \cdot f(\theta|P, \theta^j = \theta^{**}; \theta^*) d\theta = F. \tag{9} \]

If the private signal is infinitely more precise than the public signal, it is easy to prove that (9) admits a unique solution of \( \theta^{**}(P) \).\(^{29}\) However, in our setting that the public information is not infinitely small, we will show that (9) admits either zero or two solutions.\(^{30}\) In fact, if \( P \) is very low, (9) has no solution. For higher values of \( P \), (9) admits two solutions. Because of the non-uniqueness of the threshold equilibria, we need to conduct equilibrium refinements.

We adopt the following equilibrium selection criterion: If there exist multiple threshold equilibria, the creditors coordinate on the Pareto-dominant one. That is, the creditors use the lowest threshold. If \( P \) is low and consequently there exist no threshold equilibria, the creditors use the strategy ‘Not roll over’ whatever their private signals are. After the refinements, the creditors’ equilibrium strategy can be written as \((\theta^j, P) \rightarrow \begin{cases} 
    \text{Not roll over} & P < \hat{P} \\
    \text{Not roll over} & P \geq \hat{P} \text{ } \& \text{ } \theta^j < \theta^{**}(P) \\
    \text{Roll over} & P \geq \hat{P} \text{ } \& \text{ } \theta^j \geq \theta^{**}(P)
\end{cases} \),

where \( \hat{P} \) is the lowest \( P \) for equation (9) to have solutions, and \( \theta^{**}(P) \) is the lowest solution if (9) admits solutions.

In what follows we prove that there exists a unique equilibrium among the creditors after the refinements.

**Lemma 5** The creditor-run game has a unique equilibrium (after the refinements). The threshold boundary of the equilibrium expands outward as \( L \) decreases. If \( \frac{\gamma}{h} \) is sufficiently high, the boundary expands outward as \( \theta^* \) increases.

Proof: see the Appendix.

\(^{29}\)In fact, in the limit \( \gamma \to 0 \) for given \( h \), (9) can be translated to \( \frac{L}{F} \cdot D(\theta^{**}) = F \). The uniqueness of solution of \( \theta^{**} \) is straightforward. Also, in the limit \( h \to +\infty \) for a given \( \gamma \), it is easy to prove that there exists a unique solution (see the appendix).

\(^{30}\)In the appendix, we will prove that \( \int_{\theta^{**} - \gamma - \Phi^{-1}(\frac{L}{F})}^{+\infty} D(\theta) \cdot f(\theta|P, \theta^j = \theta^{**}; \theta^*) d\theta \) is a \( \cap \)-shape function in \( \theta^{**} \).
Figure 15 shows how the threshold boundary of the equilibrium changes if $\theta^*$ increases or if $L$ decreases. Intuitively, the credit risk of a financial institution can be decomposed into solvency risk and liquidity risk. In our model, an increase in $\theta^*$ magnifies uncertainty on the fundamental value, which leads to higher solvency risk. A lower $L$ means a more severe coordination problem among creditors, which causes higher liquidity risk. Both sources of risk lead to creditors running at a higher threshold $\theta^{**}(P)$.

It is worth noting that an increase in $\theta^*$ has two effects: not only it increases the aggregate uncertainty (i.e., the aggregate of public information and private information), but also changes the relative weights of public and private information (i.e., the weight of public information decreases). Although these two effects may have opposite impacts on the debt value, we prove that if $\frac{1}{k}$ is sufficiently high, the former dominates the latter.

After we have $\theta^{**}(P)$, it is easy to obtain $\hat{\theta}(P)$. Then, we can compute the bank failure probability, $\Pr(\theta; \theta^*, L)$. Similar to Lemma 2, we have the following property regarding $\Pr(\theta; \theta^*, L)$.

**Lemma 6** The bank failure probability $\Pr(\theta; \theta^*, L)$ decreases in $\theta$. For a given $\theta$, $\Pr(\theta; \theta^*, L)$ increases in $\theta^*$ when $\theta$ is high enough, and decreases in $L$.

Proof: see the Appendix.

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31 See Morris and Shin (2009).
We go back to $T_0$ to obtain the equilibrium threshold $\theta^*$ of speculators. With the result of Lemma 6, obtaining the equilibrium is analogous to that of Case 1. We have

$$\int_{-\infty}^{+\infty} [\Pr(\theta; \theta^*, L) \cdot t - c] \cdot dG(\theta|\theta^*) = 0. \quad (10)$$

Similar to theorem 2, we can prove that equation (10) has a unique solution, which means that there is a unique threshold equilibrium among speculators. Equation (9) and (10) together characterize the equilibrium of the model. Equation (9) describes the creditor run while equation (10) shows the speculator run. Two runs interact with each other.

**Theorem 3** There exists a unique threshold equilibrium of the model, which is characterized by the system of equations (9)-(10).

The equilibrium idea in this subsubsection can be summarized by figure 16. In equilibrium, $\theta^*$ is endogenous while $L$ is exogenous.

![Figure 16](image)

It is easy to obtain the following corollary to theorem 3.

**Corollary 3** In equilibrium, $\theta^*$ is decreasing in $L$.

Proof: see the Appendix.

Intuitively, a decrease in $L$ leads to a more severe coordination problem among creditors. The bank fails at a higher fundamental value. Consequently, it is easy to bring down the bank. This in turn induces speculators to start to short sell at a higher threshold.
2.3.4 (Case 4) The endogenous payoff to short-selling with creditor runs

If the gain of short-selling is endogenous, equation (10) is replaced by

$$\int_{-\infty}^{+\infty} [\Pr(\theta^*; \theta, L) \cdot (\theta - K) - c] \cdot dG(\theta^*) = 0. \quad (11)$$

Combining equations (9) and (11), we have the full equilibrium of the model. Figure 17 summarize the complete feedback loops of the model.

The key insight of this case of equilibrium is that a slight decrease in $L$ can drastically increase the level of $\theta^*$. The reason is that there exists an additional feedback loop on $\theta^*$ because of the endogenous payoff structure of short-selling. In Figure 17, a decrease in $L$, pushing up the level of $\theta^{**}$, will lead to a chain of feedback between $\theta^*$ and $\theta^{**}$. Further, $\theta^*$ reinforces itself because of the endogenous payoff structure. Therefore, $\theta^*$ rises along the compound upward spirals.

3 Predictions and implications of the model

3.1 Predictions of the model

We have two cross-sectional predictions of the model.

**Prediction 1:** Banks with lower fundamentals are more likely to incur short-selling attacks and fail with higher probability.

In equilibrium, $\theta^*$ is determined. Thus, we have the short-selling attack probability and the bank failure probability in equilibrium, which are $s(\theta; \theta^*)$ and $\Pr(\theta; \theta^*, L)$, respectively. We have
proved that $s(\theta; \theta^\star)$ is decreasing in $\theta$. Also, $\Pr(\theta; \theta^\star, L)$ is decreasing in $\theta$ by Lemma 6. Prediction 1 follows. Prediction 1 implies that the bank failure is fundamental-driven in our model. Speculation only exacerbates the problem but is not the origin of the problem.

**Prediction 2:** The more severe the maturity mismatch is, the more likely the short-selling attack is and the higher probability that the firm fails.

By Corollary 3 and Lemma 6, we have Prediction 2. Prediction 2 explains why banks rather than standard corporations are more likely to incur short-selling attacks. In our model, the maturity mismatch of balance sheets is a key driving factor for the probability of short-selling attacks and the likelihood of firm failure. Banks are more likely and more vulnerable to short-selling attacks because banks typically have much higher maturity mismatches than standard corporations.

### 3.2 Policy implications

We discuss policy implications of the model from the ex-post and ex-ante perspectives.

Ex post, short-selling destroys value and is socially inefficient. Our model shows that short-selling can cause a failure to a viable bank. Banks with weak fundamentals and severe maturity mismatches are most vulnerable to short-selling attacks. Those banks are first ones to fall. Also, naked short-selling can potentially create systematic risk. In our model, if the fall of one bank or some banks imposes negative externality to other banks by reducing the fundamentals of other banks, then even healthy banks might become weaker. This would trigger a new round of short-selling attack. In this way, short-selling can lead to a chain of collapse of financial institutions.

From the ex-ante perspective, however, the ban on short-selling has side effects. Suppose in our model that the fundamental value $\theta$ and the liquidity status $L$ are partially determined by the bank manager’s effort ex ante (before $T_0$). The higher the effort, the higher the probability that the bank realizes a high fundamental value $\theta$. Hence, if there exist short-selling threats, the bank manager has an incentive to work harder ex ante. In fact, by Lemma 6 and Figure 10 in our model, short-selling increases the bank failure probability *disproportionally* for different fundamentals. For a high fundamental value, the bank is little affected by short-selling. In contrast,
if the fundamental value is in the intermediate region, short-selling dramatically increases the failure probability. Consequently, with short-selling threats, the bank manager has an incentive to work hard ex ante, trying to realize a high fundamental value thus to minimize the risk of being exposed to high chance of failure.

4 Discussion: Speculative attacks on sovereign debt by CDSs

So far we analyze short-selling of stocks. Similarly, short-selling can happen on bonds. There are two ways to short a bond: either shorting the bond directly (i.e., borrow and sell) or buying CDS protection of the bond.

Recent years have seen a tremendous growth in CDS trading. A CDS contract is essentially an insurance contract to cover a potential default of the underlying bond. The price (spread) of a CDS is decreasing in the value of the underlying bond. CDS contracts are commonly used by banks and hedge funds to hedge and reduce their risk. However, in recent years, “naked” CDSs are popular, which are estimated to make up 80% of the total value of the CDS market.\textsuperscript{32} Naked CDS entitles buying and selling protection against default of an entity without having any sort of credit exposure towards it. That is, naked CDS contracts don’t require an insurable interest to be present. It is popularly argued that naked CDS is equivalent to “allowing you to buy insurance on your neighbor’s house. This way you have an incentive to burn down their house.”

Buying a CDS is equivalent to shorting the underlying bond. However, shorting a bond directly faces difficult practical problems, for example, shorting is constrained by the total amount of the issued bond outstanding. In contrast, shorting by naked CDS has no such restrictions.\textsuperscript{33} Unlimited shorting of bonds is facilitated by the CDS market. In theory, speculators can ‘short’ a bond unlimitedly by taking a large long position in its CDS.

We argue that the mechanism of speculative attacks on sovereign debt through naked CDS is essentially the same with that of short-selling of bank stocks. Speculators can coordinate and push


\textsuperscript{33}In fact, the gross amount of CDS far exceeds all “real” corporate bonds and loans outstanding. Typically, the market value of the CDS contracts is 3-4 times of the value of the underlying bond. For financial names, this ratio is even higher.
up the CDS price (spread) of sovereign debt. Although investors are aware that the jump in the CDS spread may be due to speculation, information asymmetry exacerbates, and uncertainty on the bond’s true fundamental value increases. When a sovereign country is trying to issue its new debt (for the purpose to stimulate the real economy or to repay the mature old debt), debtholders demand a higher rate of return in response of the increased uncertainty. The new debt issuance can be prohibitively expensive or become impossible at all. The high financing costs lead to the country’s economic fundamentals deteriorating. The default may become a closer reality. If the default probability increases, the spread of the CDS immediately shots up. As speculators take long positions in CDS when they attack, they profit from their positions when the price goes up.

5 Concluding remarks

The paper provides a theoretical model to understand how naked short-selling could threaten the fairly- and orderly-functioning of financial markets. Unlimited shorting is made possible by naked trading. We argue that naked short-selling can create uncertainty and increase information asymmetry. The exacerbation of information asymmetry leads to creditors’ concern about downside risk of underlying fundamentals and thus they decline debt rollover. The withdrawals by creditors cause failures of financial firms and state financing. Our framework also helps understand the role of rumors in financial markets. By definition, rumors contain no material informational value. Rumors would not change investors’ assessment on the ‘mean’ of fundamentals, but instead create uncertainty and increase the ‘variance’. Investors who have a concave payoff, like debtholders, are averse to uncertainty. Uncertainty reduces the value of their claims. Thus, investors run, making default and failure self-fulfilling, while rumor-creators profit from the failure.

34 For example, after Greece sovereign debt crisis struck, on February 3rd 2010 the Portuguese Treasury and Government Debt Agency only sold out €300 million of 12-month treasury bills, well below the anticipated sale of €500 million in bills. See “Portugal Bond Auction Disappoints”, the Wall Street Journal, February 3, 2010.
6 Appendix

(NOT FOR PUBLICATION) Proof in section 2.1:

We consider \( v = E_T [\max(0, \tilde{\theta} - K)] \). We denote \( v \equiv E(\theta) \). Thus, the equity value \( E(\theta) \) is convex in \( \theta \). Then, the distribution of \( p \) becomes \( p \sim N(\theta, (h \cdot s(\theta; \theta^*))^2) \). We prove that as long as \( E(\theta) \) is weakly convex in \( \theta \), the conditional expectation \( E(D(\theta)|p; \theta^*) \) is decreasing in \( \theta^* \).

Intuitively, the inference route is \( p \rightarrow \theta \rightarrow D(\theta) \). As long as \( E(\theta) \) is weakly convex in \( \theta \), the inverse function \( \theta = E^{-1}(p) \) is concave. Therefore, if \( \theta^* \) increases (i.e. the volatility increases), then the conditional variance \( \text{var}(\theta|p) \) goes up and the conditional mean \( E(\theta|p) \) goes down. Thus, the conditional expectation \( E(D(\theta)|p) \) goes down.

(NOT FOR PUBLICATION) Proof of Lemma 1:

As \( P \sim N(\theta, (h \cdot \Phi(\frac{\theta - \theta^*}{h}))^2) \), we can write down explicitly the posterior density function \( f(\theta|P; \theta^*) \). Considering \( \theta \) has an ‘improper prior’, it is easy to obtain

\[
f(\theta|P; \theta^*) = \frac{f(P|\theta, \theta^*)}{\int_{-\infty}^{+\infty} f(P|\theta, \theta^*)d\theta} = \frac{1}{\sqrt{2\pi(h \cdot \Phi(\frac{\theta - \theta^*}{h}))}} \exp\left(-\frac{(P - \theta)^2}{2(h \cdot \Phi(\frac{\theta - \theta^*}{h}))^2}\right).
\]

As all the functions involved in equation (2) are continuous, the continuity of \( P^{**} \) is straightforward. To prove boundedness, we consider two extreme cases: \( \theta^* = -\infty \) and \( +\infty \). When \( \theta^* = -\infty \), we saw in Theorem 1 that \( P^{**} = \theta^{FB} \). When \( \theta^* = +\infty \), we have \( \theta \sim N(P, h^2) \) and thus \( P^{**} \) is finite. As \( P^{**} \) is continuous in \( \theta^* \), therefore, \( P^{**} \) is bounded.

We prove that \( P^{**} \) is not a monotonic function of \( \theta^* \). In fact, when \( \theta^* \) is very low or very large relative to \( P \), the posterior distribution \( f(\theta|P; \theta^*) \) is no longer two-side fatter in tails as \( \theta^* \) increases. Figure A-1 shows what happens if \( \theta^* \) is very high relative to \( P \). In these cases, the posterior distribution \( f(\theta|P; \theta^*) \) is fatter only in the right tail as \( \theta^* \) increases. So, \( P^{**} \) is not increasing in \( \theta^* \). The full picture of \( P^{**} \) is that \( P^{**} \) increases first and then decreases and approaches to a finite number. Figure A-2 shows how \( P^{**}(\theta^*) \) evolves when \( \theta^* \) is in a wider range. We define a region of \( \theta^* \) where \( P^{**} \) is increasing before it decreases as \((\theta^L, \theta^H)\).

However, the above technique complication doesn’t affect the results of the model for two reasons. Firstly, the length of the region \((\theta^L, \theta^H)\) is increasing in the parameter \( \delta \). We can obtain
an arbitrary length of \((\theta^L, \theta^H)\) by adjusting the parameter \(\delta\). In comparison with Figure A-2, Figure A-3 shows that the interval \((\theta^L, \theta^H)\) expands as \(\delta\) increases. We prove in theorem 2 that we can choose some parameter value of \(\delta\) to make sure that all \(\theta^*\)s that speculators possibly choose are within \((\theta^L, \theta^H)\). That is, \(P^{**}\) is increasing in \(\theta^*\) for all relevant \(\theta^*\)s. Also, although the interval \((\theta^L, \theta^H)\) expands as \(\delta\) increases, the upper limit \(\bar{P}\) changes little whatever \(\delta\) is. So the boundary \((\theta^{FB}, \bar{P})\) is almost independent with the choice of \(\delta\). Secondly, \(P^{**}\) decreases very slowly when \(\theta^*\) is large. This property determines that the unique threshold equilibrium of the model is intact even if we don’t limit the parameter choice of \(\delta\).
Proof of Lemma 2:

We check how \( \Pr(\theta^*; \theta^*) \) evolves as \( \theta^* \) changes. We have

\[
\frac{\partial \Phi(\frac{P^{**}(\theta^*)-\theta}{\sigma})}{\partial \theta^*} = \phi\left(\frac{P^{**}(\theta^*)-\theta}{\sigma}\right)\left[\frac{dP^{**}(\theta^*)}{\partial \theta^*} - \frac{P^{**}(\theta^*)-\theta}{\sigma} \cdot \phi\left(\frac{\theta^*-\theta}{\delta}\right)\right].
\]

Note that \( \frac{dP^{**}(\theta^*)}{\partial \theta^*} > 0 \) and \( P^{**}(\theta^*) - \theta \) can be positive or negative.

If \( \theta > P^{**}(\theta^*) \), then it is certain that \( \frac{\partial \Phi(\frac{P^{**}(\theta^*)-\theta}{\sigma})}{\partial \theta^*} > 0 \). If \( \theta < P^{**}(\theta^*) \), the effects are mix. However, when \( \theta \) is close to \( P^{**}(\theta^*) \), the term \( P^{**}(\theta^*) - \theta \) is close to 0. So the total effect is still positive. Also, if \( \frac{F}{K} \) is high, the relevant payoffs of creditors become very flat. The upside risk becomes less while the downside risk doesn't change too much. The creditors demand a higher 'premium'. A tiny increase in \( \theta^* \) leads to a big increase \( P^{**}(\theta^*) \). The first effect dominates the second effect. In comparison with Figure 11, Figure A-4 shows a higher \( F \) leads to \( \Pr(\theta; \theta^*) \) being increasing in \( \theta^* \) for a wider region of \( \theta \).
Finally, we prove that $\Pr(\theta; \theta^*)$ is decreasing in $\theta$ for a given $\theta^*$. When $\theta > P^{**}(\theta^*)$, it is easy to prove that $\Phi\left(\frac{P^{**}(\theta^*) - \theta}{\delta}\right)$ is decreasing in $\theta$. When $\theta < P^{**}(\theta^*)$, we have

$$\frac{\partial \Phi\left(\frac{P^{**}(\theta^*) - \theta}{\delta}\right)}{\partial \theta} = \phi\left(\frac{P^{**}(\theta^*) - \theta}{\delta}\right) \cdot \frac{1}{\delta} \cdot \frac{-\Phi\left(\frac{P^{**}(\theta^*) - \theta}{\delta}\right) - \Phi\left(\frac{P^{**}(\theta^*) - \theta}{\delta}\right) \left(\frac{-1}{2}\right)(P^{**}(\theta^*) - \theta)}{\Phi\left(\frac{P^{**}(\theta^*) - \theta}{\delta}\right)} = \phi\left(\frac{P^{**}(\theta^*) - \theta}{\delta}\right) \cdot \frac{1}{\delta} \cdot \frac{-\Phi\left(\frac{\theta^* - \theta}{\delta}\right) + \phi\left(\frac{\theta^* - \theta}{\delta}\right) P^{**}(\theta^*) - \theta}{\Phi\left(\frac{\theta^* - \theta}{\delta}\right)}.$$

Let $z = \frac{\theta^* - \theta}{\delta}$. We have $-\Phi\left(\frac{\theta^* - \theta}{\delta}\right) + \phi\left(\frac{\theta^* - \theta}{\delta}\right) P^{**}(\theta^*) - \theta = -\Phi(z) + \phi(z)(z + \frac{P^{**}(\theta^*) - \theta^*}{\delta}).$

Because $\theta^* \in (\theta^L, \theta^H)$, $P^{**}(\theta^*) - \theta^*$ is finite for a given interval $(\theta^L, \theta^H)$. We can choose a sufficient large $\delta$ to make sure that $-\Phi(z) + \phi(z)(z + \frac{P^{**}(\theta^*) - \theta^*}{\delta}) < 0$. So $\Pr(\theta; \theta^*)$ is decreasing in $\theta$. Figure 11 shows this property is true for very general parameter choices.

**Proof of Lemma 3:**

Conditional on that creditors use the threshold $P^{**}$ and all other speculators use the threshold $\theta^*$, the net payoff to short selling for an individual speculator who receives signal $\theta^i$ is $V(P^{**}, \theta^*, \theta^i) = \int_{-\infty}^{+\infty} \Pr(P < P^{**}\mid \theta; \theta^*) \cdot t - c \cdot dG(\theta|\theta^i)$, where $G(\theta|\theta^i)$ is the conditional cdf. By $P \sim N(\theta, (\phi(\theta))^2)$, we have $V(P^{**}, \theta^*, \theta^i) = \int_{-\infty}^{+\infty} \Phi\left(\frac{P^{**} - \theta}{\delta}\right) \cdot t - c \cdot dG(\theta|\theta^i)$. It is easy to prove that

$$\frac{\partial V(P^{**}, \theta^*, \theta^i)}{\partial P^{**}} > 0.$$

We need to check the monotonicity of $\Phi\left(\frac{P^{**} - \theta}{\delta}\right)$ with respect to $\theta$. When $\theta > P^{**}$, it is easy to prove that $\Phi\left(\frac{P^{**} - \theta}{\delta}\right)$ is decreasing in $\theta$. When $\theta < P^{**}$, it involves some technique.
complications. In this case, if $\theta^*$ is extremely low, then $\Phi(-\frac{P^{**}-\theta}{h\Phi(\frac{\theta}{\delta})})$ is not decreasing around $\theta = \theta^*$. In fact,

$$
\frac{d\Phi(-\frac{P^{**}-\theta}{h\Phi(\frac{\theta}{\delta})})}{d\theta} = \phi\left(-\frac{P^{**}-\theta}{\Phi(\frac{\theta}{\delta})}\right) 
+ \frac{1}{h\Phi(\frac{\theta}{\delta})} \cdot \frac{-\Phi(\frac{\theta}{\delta})-\Phi(\frac{\theta^*}{\delta})(-\frac{1}{\delta})(P^{**}-\theta)}{\Phi^2(\frac{\theta}{\delta})} 
+ \phi\left(-\frac{\theta^*}{\delta}\right) + \phi\left(-\frac{\theta^*}{\delta}\right)(P^{**}-\theta)\right].
$$

Let $z = \frac{\theta^*}{\delta}$. we have

$$
-\Phi\left(-\frac{\theta^*}{\delta}\right) + \phi\left(-\frac{\theta^*}{\delta}\right) \frac{P^{**}-\theta}{\delta} = -\Phi(z) + \phi(z)(z + \frac{P^{**}-\theta^*}{\delta}).
$$

So $\Phi(-\frac{P^{**}-\theta}{h\Phi(\frac{\theta}{\delta})})$ is non-decreasing in $\theta$ when $\theta^*$ is extremely low relative to $P^{**}$. However, the non-decreasing region is small and it is around $\theta = \theta^*$ (i.e. $z = 0$) considering that $\phi(z)$ is highest when $z = 0$. If $\theta^i$ is relatively high, the conditional probability $\frac{dG(\theta^i; \theta^*)}{d\theta}$ is small when $\theta$ is extremely low. Therefore, we can neglect the non-decreasing region.

Consider that creditors use the lowest threshold $P^{**} = \theta^{FB}$. We can prove that for each $\theta^*$ there exists a $\theta$ such that as long as the signal $\theta^i$ is less than $\theta$, the speculator’s payoff is certainly positive $V(P^{**}, \theta^*, \theta^i) > 0$. In fact, we can consider two extreme cases of $\theta^*$ first: $\theta^* = -\infty$ and $+\infty$. It is easy to prove that there exist such $\theta$s. For all other $\theta^*$s, of course there exist such $\theta$s. The minimum $\theta$ among all these is the one we desire.

Similarly, by considering that creditors use the highest threshold $P^{**}$, we can prove the existence of an upper boundary $\theta$.

Finally, as the boundary $(\theta^{FB}, P)$ is independent with the choice of $\delta$, the $\theta$ and $\theta$ change little as $\delta$ changes.

**Proof of Lemma 4:**

$\Pr(\theta; \theta^*)$ is decreasing in $\theta$ and is between 0 and 1. Because $\xi = 1$, there exists a unique solution of $\theta^*$ in (3).

From the analysis in Lemma 2, we have the following result: There exists a value $Y$ such that for any $y \in [0, Y]$, $\theta$ is increasing with $\theta^*$ in solving the implicit function $\Pr(\theta; \theta^*) = y$. That is, in figure 10, for the bottom part of curves such that $\Pr(\theta; \theta^*) \leq Y$, the curves shift to right as $\theta^*$ increases.

It is easy to prove that $Y$ is greater than $\frac{1}{2}$ and increases in $\frac{E}{F}$. 38
Therefore, if $\xi$ is small enough (e.g. $\xi \leq \frac{1}{2}$) or $\frac{F}{K}$ is high enough, $\theta^i*$ is increasing with $\theta^*$ in solving (3).

**Proof of Theorem 2:**

Figure A-5 illustrates the idea of the proof of Theorem 2: the unique intersection between the curve $\theta^i*(\theta^*)$ and the 45 degree line represents the equilibrium. By Lemma 4, $\theta^i*(\theta^*)$ is increasing in $\theta^*$ for $\theta^* \in (\theta^L, \theta^H)$. Also, by Lemma 3, $\theta^i*$ is bounded, $\theta^i* \in [\underline{\theta}, \overline{\theta}]$. Further, we can prove that $[\underline{\theta}, \overline{\theta}] \subset (\theta^L, \theta^H)$ if $\delta$ is sufficiently high. So we have $\theta^i*(\theta^* = \underline{\theta}) > \theta^i*(\theta^* = \theta^L+) \geq \underline{\theta}$, which means that the point $(\underline{\theta}, \theta^i*(\underline{\theta}))$ is above the 45 degree line. Similarly, we have $\theta^i*(\theta^* = \overline{\theta}) < \overline{\theta}$, meaning that the point $(\overline{\theta}, \theta^i*(\overline{\theta}))$ is below the 45 degree line. Therefore, there exist intersections between the curve $\theta^i*(\theta)$ and the 45 degree line. As for the uniqueness of intersection, we have to rely on simulation as the functions are implicit and we do not have a closed-form of solution.

We prove that $[\underline{\theta}, \overline{\theta}] \subset (\theta^L, \theta^H)$. We saw in the proof of lemma 3 that the boundary $[\underline{\theta}, \overline{\theta}]$ changes little as $\delta$ changes. In the proof of Lemma 1, we show the interval $(\theta^L, \theta^H)$ expands as $\delta$ increases. Therefore, if $\delta$ is sufficiently high, there exists an interval $(\theta^L, \theta^H)$ such that $[\underline{\theta}, \overline{\theta}] \subset (\theta^L, \theta^H)$. The simulation results in the text confirm this.

**Proof of Corollary 1:**

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In fact, if \( \frac{c}{t} < k \), we can prove \( \bar{\theta} > \theta^{FB} \). By \( \theta^* > \bar{\theta} \), we obtain \( \theta^* > \theta^{FB} \). Consider that creditors use the lowest threshold \( P^{**} = \theta^{FB} \) and the creditors can use any \( \theta^* \in (-\infty, +\infty) \). In the two extreme cases \( \theta^* = -\infty \) and \( +\infty \), the bank failure probability \( \Phi \left( \frac{\theta^{FB} - \theta}{h \cdot \Phi(\frac{1}{2})} \right) \) is symmetric around the point with coordinate \( (\theta^{FB}, \frac{1}{2}) \). If \( \frac{c}{t} = \frac{1}{2} \), then in solving \( V(\theta^{FB}, \theta^*, \bar{\theta}) = 0 \), where \( V(\theta^{FB}, \theta^*, \bar{\theta}) = \int_{-\infty}^{+\infty} [\Phi \left( \frac{\theta^{FB} - \theta}{h \cdot \Phi(\frac{1}{2})} \right) \cdot t - c] \cdot dG(\theta|i) \), we have \( \bar{\theta} = \theta^{FB} \). When \(-\infty < \theta^* < +\infty\), \( \Phi \left( \frac{\theta^{FB} - \theta}{h \cdot \Phi(\frac{1}{2})} \right) \) is not symmetric around the point \( (\theta^{FB}, \frac{1}{2}) \). The curve \( \Phi \left( \frac{\theta^{FB} - \theta}{h \cdot \Phi(\frac{1}{2})} \right) \) declines steeper at \( \theta > \theta^{FB} \) than at \( \theta < \theta^{FB} \). So \( \int_{-\infty}^{+\infty} \Phi \left( \frac{\theta^{FB} - \theta}{h \cdot \Phi(\frac{1}{2})} \right) \cdot dG(\theta|i = \theta^{FB}) < \frac{1}{2} \). Therefore, in order to restore \( V(\theta^{FB}, \theta^*, \bar{\theta} = \theta^{FB}) = 0 \), we need to have \( \frac{c}{t} \) to be slightly lower than \( \frac{1}{2} \).

**Proof of Corollary 2:**

In solving (3), it is easy to obtain the comparative static result \( \frac{\partial \theta^*}{\partial t} > 0 \). That is, the curve \( \theta^*(\theta^*) \) shifts up when \( t \) increases. So in Figure 12 the intersection between the curve \( \theta^*(\theta^*) \) and the 45 degree line increases in \( t \). That is, the equilibrium threshold \( \theta^* \) is increasing in \( t \).

**Proof of Lemma 5:**

The posterior density can be explicitly written as

\[
f(\theta|P, \theta^*; \theta^*) = \frac{f(P|\theta^*) \cdot f(\theta|i)}{\int_{-\infty}^{+\infty} f(P|\theta^*) \cdot f(\theta|i) d\theta} = \frac{\left( \frac{1}{\sqrt{2\pi h \cdot \Phi(\frac{1}{2})}} \right) \exp \left( \frac{-\left( P - \theta \right)^2}{2h \cdot \Phi(\frac{1}{2})} \right)}{\left( \frac{1}{\sqrt{2\pi h \cdot \Phi(\frac{1}{2})}} \right) \exp \left( \frac{-\left( \theta - \theta^* \right)^2}{2h \cdot \Phi(\frac{1}{2})} \right)}.
\]

We define \( U(\theta^{**}, P; \theta^*) = \int_{\theta^{**} - \gamma \cdot \Phi^{-1}(\frac{1}{2})}^{+\infty} D(\theta) \cdot f(\theta|P, \theta^i = \theta^{**}; \theta^*) d\theta \). In the limit \( h \to +\infty \) for a given \( \gamma \), \( U(\theta^{**}, P; \theta^*) \equiv U(\theta^{**}) = \int_{\theta^{**} - \gamma \cdot \Phi^{-1}(\frac{1}{2})}^{+\infty} D(\theta) \cdot f(\theta|\theta^i = \theta^{**}) d\theta \). Considering \( D(\theta) \) is an increasing function, clearly, \( U(\theta^{**}) \) is monotonically increasing in \( \theta^{**} \). So there exists a unique solution for \( U(\theta^{**}) = F \).

We are interested in the case of \( 0 < h < +\infty \) and \( 0 < \gamma < +\infty \). Clearly, \( \frac{\partial U(\theta^{**}, P; \theta^*)}{\partial P} > 0 \). We prove that for any given \( P \), \( U(\theta^{**}, P; \theta^*) \) is a \( \cap \)-shape function of \( \theta^{**} \), i.e., increases first and then decreases. Intuitively, an increase in \( \theta^{**} \) has two effects on \( U(\theta^{**}, P; \theta^*) \). It not only shifts the density function \( f(\theta|P, \theta^i = \theta^{**}; \theta^*) \) to the right (i.e. the mean of \( \theta \) increases) but also pushes up the lower boundary of integral \( \theta^{**} - \gamma \cdot \Phi^{-1}(\frac{1}{2}) \). The first effect on \( U(\theta^{**}, P; \theta^*) \) is positive while the second effect is negative. Also, the speed of the mean moving is lower than that of the lower
boundary moving, which is because the mean is the weighted average of $P$ and $\theta^{**}$ and $P$ doesn’t change. In fact, only when $h = +\infty$, two speeds are same. Considering that $D(\theta)$ is increasing in $\theta$, we can conclude that $U(\theta^{**}, P; \theta^*)$ increases in $\theta^*$ first and then decreases. Figure A-6 reports the simulation result of $U(\theta^{**}, P; \theta^*)$.

![Figure A-6](image_url)

If $L$ decreases, the lower boundary of integral $\theta^{**} - \gamma \cdot \Phi^{-1}(\frac{L}{P})$ increases. Thus, $U(\theta^{**}, P; \theta^*)$ decreases. In order to restore the equation $U(\theta^{**}, P; \theta^*) = F,$ $P$ has to increase for a given $\theta^{**}$, considering that $U(\theta^{**}, P; \theta^*)$ is increasing in $P$.

Consider that $\frac{\gamma}{h}$ is sufficiently high, that is, $\gamma$ is large relative to $h$. Note that the value of $D(\theta)$ is very small when $\theta$ is low (say, when $\theta < 0$). Thus, the change of the lower boundary $\theta^{**} - \gamma \cdot \Phi^{-1}(\frac{L}{P})$ has a very limited effect on the value of $U(\theta^{**}, P; \theta^*)$ unless $\theta^{**}$ is very big. At the same time, the density function $f(\theta | P, \theta^j = \theta^{**}; \theta^*)$ shifts to the right when $\theta^{**}$ increases. Therefore, in aggregate, for a given $P$, $U(\theta^{**}, P; \theta^*)$ is increasing in $\theta^{**}$ up to $\theta^{**}$ reaching a high value. Or, $U(\theta^{**}, P; \theta^*)$ starts to decrease only when $\theta^{**}$ is very high. (In fact, in the extreme $\frac{\gamma}{h} = +\infty$, $U(\theta^{**}, P; \theta^*)$ is constant in $\theta^{**}$). That is, the peak of $U(\theta^{**}, P; \theta^*)$ occurs at $\theta^{**} > P$. Then, we have the point $(\hat{P}, \theta^{**}(\hat{P}))$ is above the 45 degree line.

An increase in $\theta^*$ not only increases the volatility of $f(\theta | P, \theta^j = \theta^{**}; \theta^*)$ but also changes its mean. The latter is because the weights between the public and private information changes (i.e.
the weight of public information goes down). However, in the extreme case \( \frac{\gamma}{h} = +\infty \), the second effect disappears and only the first effect exists. Therefore, if \( \frac{\gamma}{h} \) is sufficiently high, the first effect dominates the second effect.

**Proof of Lemma 6:**

We need to prove that \( \hat{\theta}(P) \) decreases in \( L \) and increases in \( \theta^* \). We have proved that \( \theta^{**}(P) \) decreases in \( L \) and increases in \( \theta^* \). Considering \( \hat{\theta} = \theta^{**} - \gamma \cdot \Phi^{-1}(\frac{L}{h}) \), \( \hat{\theta}(P) \) is obviously increasing in \( \theta^* \). Also, because \( \gamma \cdot \Phi^{-1}(\frac{L}{h}) \) is increasing in \( L \). Thus, clearly, \( \hat{\theta}(P) \) decreases in \( L \). We compute the bank failure probability, which is given by

\[
\text{Pr}(\theta < \hat{\theta}(P); \theta^*, L) = \begin{cases} \text{Pr}(\theta < \hat{\theta}(P); \theta^*, L) & \text{if } \theta < \hat{\theta}(P) \\ \text{Pr}(P < \hat{P}; \theta^*, L) & \text{if } \theta > \hat{\theta}(P) \end{cases}
\]

We compute the bank failure probability, which is given by

\[
\text{Pr}(\theta < \hat{\theta}(P); \theta^*, L) = \text{Pr}(\theta < \hat{\theta}(P); \theta^*, L) = \begin{cases} \Phi(\frac{\hat{\theta}(P) - \theta}{\Phi^{-1}(\frac{L}{h})}) & \text{if } \theta < \hat{\theta}(P) \\ \Phi(\frac{\hat{P} - \theta}{\Phi^{-1}(\frac{L}{h})}) & \text{if } \theta > \hat{\theta}(P) \end{cases}
\]

(\text{NOT FOR PUBLICATION}) **Proof of Corollary 3:**

The proof is similar with that of Corollary 2. Let \( \theta^{**}(\theta^*) \) be the solution to the equilibrium equation

\[
\int_{-\infty}^{+\infty} [\text{Pr}(\theta; \theta^*, L) \cdot t - c] \cdot dG(\theta|\theta^{**}) = 0.
\]

Because \( \text{Pr}(\theta; \theta^*, L) \) is decreasing in \( L \), \( \theta^{**}(\theta^*) \) is decreasing in \( L \). That is, the curve \( \theta^{**}(\theta^*) \) shifts down when \( L \) increases. So in Figure 12 the intersection between the curve \( \theta^{**}(\theta^*) \) and the 45 degree line decreases in \( L \). That is, the equilibrium threshold \( \theta^* \) is decreasing in \( L \).

\[^{35}\hat{\theta}(P) \text{ is decreasing in } P. \text{ So its inverse function exists and is decreasing.}\]


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