Weight-Ranked Divide-and-Conquer Contracts*

Lester T. Chan

Department of Economics

Boston University

lctlabc@bu.edu

July 2020

Abstract

This paper studies a large class of multi-agent bilateral contracting models with the property that agents' payoffs constitute a weighted potential game. I fully characterize a contracting scheme that is optimal for a large set of equilibrium selection criteria and implementation requirements. This scheme ranks agents in ascending order of their weights in the weighted potential game and induces them to accept their offers in a dominance-solvable way, starting from the first agent. I apply the general results to networks and pure/impure public goods/bads.

JEL Classification: C72, D85, D86, H41

Keywords: contracting with externalities, divide and conquer, networks, public goods, potential maximization

*I am grateful to many people for useful comments and discussions, especially Marina Halac, Hsueh-Ling Huynh, Duozhe Li, Barton Lipman, Albert Ma, Chiara Margaria, Dilip Mookherjee, Stephen Morris, Juan Ortner, Fernando Payro, Alexander White, Beixi Zhou, and seminar participants at Boston University.

1 Introduction

Many contracting situations involve multiple agents, and in most of these situations, an agent's payoff depends on other agents' actions. For example, the value of joining a platform increases with the number of users; the return from an investment is affected by others' investment decisions; the incentive to work changes with colleagues' efforts. A natural question arises: how does the principal's optimal contracting scheme take into account these (potentially very complex) externalities? Moreover, externalities among agents often lead to multiple equilibria. All the above examples may have (at least) a high- and a low-participation/investment/effort equilibrium. The principal's payoff may differ significantly across equilibria. This raises a more fundamental problem: how should we define the optimality of a contracting scheme when there are multiple equilibria? Ultimately, what contracts should the principal offer when there are multiple agents?

The most common approach to deal with the fundamental problem is to specify an equilibrium selection criterion (or, more generally, an implementation requirement)[1] and get rid of multiple equilibria. However, this approach does not fully resolve the problem because it replaces the issue of multiple equilibria with the issue of multiple equilibrium selection criteria. To see this, the optimal contracts for the best-case scenario are likely rejected by agents in less favorable scenarios. On the other hand, the optimal contracts for the worst-case scenario likely forgo huge profits in more favorable scenarios. If the principal (or we, as researchers) does not know the underlying equilibrium selection criterion, can we still recommend (or reasonably predict) what contracts the principal should (or would) offer?

This paper shows that the answer to the above question is "yes" for a large class of contracting models with complete information. The timing is standard: the principal offers

[1] As I will formally show on p. 11, any equilibrium selection criterion can be expressed as an implementation requirement, but not vice versa. For example, the best-case (worst-case) equilibrium selection criterion is equivalent to partial (robust) implementation, but any requirement stronger than robust implementation (e.g., unique, dominance-solvable, or dominant-strategy implementation) is not an equilibrium selection criterion.

each agent a menu of publicly observable bilateral[2] contracts in stage 1, and each agent simultaneously chooses a contract (or the outside option of rejecting all contracts) in stage 2. Each agent's set of possible actions can be any compact set. Regarding the externalities among agents, some agents' actions can be strategic complements while some others can be strategic substitutes. The principal can be self-interested or benevolent. The key assumption is that agents' payoffs constitute a *weighted potential game*. The solution concept is subgame-perfect Nash equilibrium with an equilibrium selection criterion. One equilibrium selection criterion is said to be *more pessimistic* than the other if, in every subgame, the selected equilibrium gives the principal a weakly lower expected payoff. Among all equilibrium selection criteria, one of them—*potential maximization*—plays a pivotal role in the analysis.

The main result of this paper is that under any equilibrium selection criterion that is more pessimistic than potential maximization, the principal's optimal contracting scheme is to offer *weight-ranked divide-and-conquer* (**w**-DC) *contracts*. The **w**-DC contracts rank all agents in increasing order of their *weights* in the weighted potential game and offer each agent *one*[3] contract that asks him to take a specified action. The associated contract prices/subsidies make the first agent have a weakly dominant strategy to accept the offer; given the first agent accepts the offer, the second agent has an (iterated) weakly dominant strategy to accept the offer as well, and so on. Thus, the **w**-DC contracts induce all agents to accept their offers as a dominance-solvable equilibrium. Moreover, I show that the **w**-DC contracts may be suboptimal if the underlying equilibrium selection criterion is *not more pessimistic* than potential maximization. Hence, I have identified the complete set of equilibrium selection criteria in which the principal always offers the **w**-DC contracts. I further extend the main result by showing that the **w**-DC contracts are optimal for a large set of implementation

[2]As pointed out by the literature (e.g., Bernstein and Winter 2012; Halac et al. 2020), the principal can only rely on bilateral contracts in many real-life contracting situations. If the principal is allowed to offer multilateral contracts (i.e., contracts that can condition on others' actions), she can easily induce a unique equilibrium that extracts all agents' surplus in most such models.

[3]Note that the principal can offer each agent a menu of contracts, but it turns out she just needs to offer one contract to each agent.

requirements, including robust, unique, and dominance-solvable implementation.

Section 4 of this paper applies the general results to three special cases: networks, public goods/bads (henceforth goods for simplicity), and impure public goods. In the undirected network model, agents are heterogeneous in many respects, including their (i) sets of connected agents, (ii) valuations of network benefits, and (iii) importance to their connected agents regarding network benefits. The **w**-DC contracts rank agents in increasing valuation-to-importance ratio. Contrary to the conventional wisdom in the economics of networks literature (see Section 7 of Jackson et al. 2017 for a survey) that the principal should offer more favorable contracts to agents with high centrality (i.e., those at central positions in the network), the network structure plays no role in agents' ranking. I further derive a natural network formation process and show that under this process, highly connected agents are those who value the network a lot and those with either high or low importance.

In the public good model, agents are heterogeneous in many respects, including their (i) valuations of the public good and (ii) importance of their contributions to the public good. In contrast to the network model, the **w**-DC contracts rank agents in increasing valuation regardless of their importance. As the impure public good model clarifies, the reason for these opposing results is that the public good is non-excludable whereas the "network good" is excludable. In that model, the **w**-DC contracts always prioritize agents with low valuations; they also prioritize those with high importance if and only if the good is sufficiently excludable.

This paper makes two general contributions. First, it extends our understanding of multi-agent contracting. Various contracting schemes are derived under different implementation requirements in the literature. For example, the seminal papers by Segal (1999, 2003) study the same model and derive very different contracting schemes under partial and unique implementation respectively. This paper shows that one contracting scheme—the **w**-DC contracts—is particularly robust because it is optimal for a large set of equilibrium selection criteria and implementation requirements. This result helps us make better predictions and policy advice on multi-agent contracting problems especially when we, as researchers, do not know the underlying equilibrium selection criterion or what implementation requirement the principal aims to meet.

In the literature, specific divide-and-conquer (DC) contracts are derived (e.g., Segal 2003; Winter 2004; Bernstein and Winter 2012; Halac et al. 2020) under (i) binary/one-dimensional actions for agents, (ii) strategic complementarities among all agents, and (iii) the requirement of unique implementation. This paper uncovers the generality and robustness of the DC contracts by relaxing all three restrictions substantially and deriving the general form of the DC contracts. Furthermore, this paper is the first to derive the optimal ranking for the DC contracts in a general setting and, thus, fully characterize the optimal contracting scheme: the **w**-DC contracts.

Second, this paper advances the analysis of multi-agent contracting problems. Although its primary focus is the **w**-DC contracts, the general framework and tools developed are applicable to all such problems. In particular, the novel externalities structure among agents, which is derived from two binary relations, is considerably more flexible than the usual strategic complementarity/substitutability structure. In addition, a methodological contribution of this paper is to incorporate the theory of potential games into multi-agent contracting. The concept of potential games was introduced by Rosenthal (1973) and formalized by Monderer and Shapley (1996). Potential maximization refines Nash equilibrium in (weighted) potential games, and this refinement is justified by many theoretical and experimental studies.[4] I will explain both concepts in the next section. As Section 4 reveals, agents' payoffs constitute a weighted potential game in many contracting models. By exploiting this useful property, one may be able to derive stronger results as this paper does.

Beyond the above general contributions, Section 4 contributes to the literature on the economics of networks, public goods, and impure public goods.[5] One contribution common

[4]See Chan (2019, Related Literature) for a summary of established justifications. In particular, it coincides with global-game selection in supermodular weighted potential games (Frankel et al. 2003).

[5]For the recently growing literature on contracting in networks, see, for example, Shi and Xing (2018), Belhaj and Deroian (2019), Jadbabaie and Kakhbod (2019), Bloch and Shabayek (2020), and Zhang and Chen (2020). See Bergstrom et al. (1986) and Cornes and Sandler (1984, 1994) for the seminal works on

to all three strands of literature is to show the optimality and robustness of the corresponding $\mathbf{w}$-DC contracts in each of the environments. A closely related paper to all three applications is Sakovics and Steiner (2012). They analyze a binary-action complete network under *global-game selection* (which is equivalent to potential maximization; see footnote 4) and show that the principal's optimal contracting scheme is to rank agents in increasing valuation-to-importance ratio and offer divide-and-conquer contracts. My network model shows that their main finding is robust to (i) more general actions, (ii) arbitrary undirected network structures, and (iii) all equilibrium selection criteria that are more pessimistic than global-game selection. However, my (impure) public good model shows that their finding is not robust to (partially) non-excludable externalities. Hence, my results urge caution in applying their finding to their leading applications—economic development and financial fragility—which are largely public goods/bads in nature or at least partially non-excludable. My impure public good model provides refined guidance on vertical/sectoral industrial policies (a.k.a. "picking winners") for the former and financial policies for the latter.

2 Model

A principal ("she") contracts with N agents ("he" for an agent). With a slight abuse of notation, let $N \equiv \{1, \ldots, N\}$ also denote the set of agents. Let $x_i \in X_i$ denote the action of agent $i \in N$, where X_i is a compact set with $o_i \in X_i$ denoting the outside option of rejecting the principal's offers. Let $\mathbf{x} \in X \equiv \prod_i X_i$ denote agents' action profile, and $\mathbf{x}_{-i} \in X_{-i} \equiv \prod_{j \neq i} X_j$ and $\mathbf{x}_{-ij} \in X_{-ij} \equiv \prod_{k \notin \{i,j\}} X_k$ are defined in the usual way. The game has two stages. In stage 1, the principal sets a price function $p_i \in P_i \equiv \{p_i : X_i \to \mathbb{R} | p_i(o_i) = 0\}$ for each agent i. This is equivalent to offering each agent a menu of bilateral contracts $(x_i, p_i(x_i))$ given that she can always prevent an agent from taking an action $x_i \in X_i \backslash \{o_i\}$ by charging an arbitrarily high price $p_i(x_i)$ for that action.[6] Let $\mathbf{p} \in P \equiv \prod_i P_i$ denote the menu profile

public goods and impure public goods respectively.

[6] In addition to notational convenience, this formulation also has a technical advantage of implicitly ruling out the use of non-compact menus. See Segal (2003, p. 161–162) for why these menus are problematic, leading

offered to agents. In stage 2, all agents observe $\mathbf{p}$ and simultaneously choose from $\Delta(X_i)$, i.e., mixed strategies are allowed.[7] Agent i's payoff is linear in money, i.e., $u_i(\mathbf{x}) - p_i(x_i)$, where the continuous function $u_i : X \to \mathbb{R}$ measures his intrinsic utility. The principal's payoff is $U(\mathbf{x}, \sum_i p_i(x_i))$, where the upper semicontinuous function $U : X \times \mathbb{R} \to \mathbb{R}$ is weakly increasing in her total revenue $\sum_i p_i(x_i)$. The function U is sufficiently general to represent a self-interested (e.g., $U = \sum_i p_i(x_i)$) or benevolent (e.g., $U = \sum_i u_i(\mathbf{x})$) principal. All players are expected utility maximizers.

I make three assumptions on agents' (intrinsic) utilities $\mathbf{u} \equiv (u_i)_i$.

Assumption 1 (Weighted potential game) *There exists a (weight) vector* $\mathbf{w} \equiv (w_i)_i \in \mathbb{R}^N_{++}$ *and a (potential) function* $\Phi : X \to \mathbb{R}$ *such that for all* $i \in N$*,*

$$u_i(x_i, \mathbf{x}_{-i}) - u_i(x_i', \mathbf{x}_{-i}) = w_i[\Phi(x_i, \mathbf{x}_{-i}) - \Phi(x_i', \mathbf{x}_{-i})] \quad \textit{for all } x_i, x_i' \in X_i \textit{ and } \mathbf{x}_{-i} \in X_{-i}. \quad (1)$$

Assumption 1 (henceforth A1; similarly for A2 and A3) states that agents' utilities $\mathbf{u}$ constitute a *weighted potential game*. Verbally, there exists a real-valued function Φ defined on the set of action profiles such that the change in any agent's utility by unilaterally switching actions is proportional (with proportion w_i for agent i) to the corresponding change in Φ. Observe that (1) holds if and only if there exists a (pure externality) function $\xi_i : X_{-i} \to \mathbb{R}$ such that

$$u_i(\mathbf{x}) = w_i \Phi(\mathbf{x}) + \xi_i(\mathbf{x}_{-i}) \quad \text{for all } \mathbf{x} \in X.^{8} \quad (2)$$

Many contracting models satisfy A1. For example, let X_i be any compact subset of $\mathbb{R}_+$ with $o_i = 0 \in X_i$ and u_i take the following form:

$$u_i(\mathbf{x}) = b_i(x_i) + v_i x_i \sum_{j \in E_i} \theta_j x_j, \quad (3)$$

him to explicitly rule them out.

[7] Most of this literature does not consider mixed strategies, but this is with loss of generality. For example, in a subgame, a mixed-strategy equilibrium may be the unique equilibrium giving the principal the highest expected payoff. Therefore, if the principal can select her most preferred equilibrium as in Segal (1999), she will select the mixed-strategy equilibrium.

[8] The "if" part is trivial. For the "only if" part, the function $\xi_i(\mathbf{x}_{-i}) \equiv u_i(\mathbf{x}) - w_i \Phi(\mathbf{x})$ is well-defined because, by (1), $u_i(x_i, \mathbf{x}_{-i}) - w_i \Phi(x_i, \mathbf{x}_{-i}) = u_i(x_i', \mathbf{x}_{-i}) - w_i \Phi(x_i', \mathbf{x}_{-i})$ for all $x_i, x_i' \in X_i$.

where $b_i : X_i \to \mathbb{R}$ measures agent i's stand-alone benefit/cost, $E_i \subseteq N \backslash \{i\}$ is the subset of agents who interact with agent i (interactions are two-way, i.e., $j \in E_i$ iff $i \in E_j$), $v_i \in \mathbb{R}_{++}$ measures his valuation of interaction benefits, and $\theta_j \in \mathbb{R}_{++}$ measures the relative importance of agent j's actions to his interacting agents. Each agent can differ in five dimensions $(X_i, b_i, E_i, v_i, \theta_i)$ in this general example, which in turn covers a wide variety of contracting situations. Section 4.1 analyzes this example and (Lemma 3) shows that it satisfies A1 (and A2 and A3). Sections 4.2 and 4.3 study two other examples that also satisfy A1–A3.

In order to state the other two assumptions, I first define two binary relations C and S between any two distinct agents' action sets X_j and X_i.

Definition 1 *The expression $x_j C x_i$ ($x_j S x_i$) stands for*

$$u_i(x_i, x_j, \mathbf{x}_{-ij}) - u_i(o_i, x_j, \mathbf{x}_{-ij}) \geq (\leq)\ u_i(x_i, o_j, \mathbf{x}_{-ij}) - u_i(o_i, o_j, \mathbf{x}_{-ij}) \quad \forall \mathbf{x}_{-ij} \in X_{-ij}. \quad (4)$$

In words, $x_j C x_i$ ($x_j S x_i$) means x_j always strategically complements (substitutes) x_i relative to the outside option. The second assumption is stated as follows.

Assumption 2 (Sign independence of others' actions) *For each $\mathbf{x} \in X$ and distinct $i, j \in N$, $x_j C x_i$ or $x_j S x_i$.*

To better understand A2, consider a scenario where agents i and j only choose between a particular action (say, x_i for i and x_j for j) and the outside option, and all other agents choose the outside option. In this scenario, x_j either strategically complements or substitutes x_i because $u_i(x_i, x_j, \mathbf{o}_{-ij}) - u_i(o_i, x_j, \mathbf{o}_{-ij})$ is either greater or less than $u_i(x_i, o_j, \mathbf{o}_{-ij}) - u_i(o_i, o_j, \mathbf{o}_{-ij})$. A2 states that if x_j strategically complements (substitutes) x_i in this scenario, then x_j always strategically complements (substitutes) x_i regardless of others' actions $\mathbf{x}_{-ij}$.

Observe from (1) and (4) that A1 implies C and S are symmetric, i.e., $x_j C x_i$ ($x_j S x_i$) if and only if $x_i C x_j$ ($x_i S x_j$).[9] In other words, any two agents' actions either strategically

[9] To see this more clearly, by (1), $u_i(x_i, x_j, \mathbf{x}_{-ij}) - u_i(o_i, x_j, \mathbf{x}_{-ij}) \geq u_i(x_i, o_j, \mathbf{x}_{-ij}) - u_i(o_i, o_j, \mathbf{x}_{-ij})$ iff $\Phi(x_i, x_j, \mathbf{x}_{-ij}) - \Phi(o_i, x_j, \mathbf{x}_{-ij}) \geq \Phi(x_i, o_j, \mathbf{x}_{-ij}) - \Phi(o_i, o_j, \mathbf{x}_{-ij})$ iff $u_j(x_i, x_j, \mathbf{x}_{-ij}) - u_j(x_i, o_j, \mathbf{x}_{-ij}) \geq u_j(o_i, x_j, \mathbf{x}_{-ij}) - u_j(o_i, o_j, \mathbf{x}_{-ij})$.

complement or substitute each other relative to the outside option. I write $x_j \bar{C} x_i$ if $x_j C x_i$ but not $x_j S x_i$. Clearly, $\bar{C}$ is also symmetric. The last assumption is stated as follows.

Assumption 3 (Weak transitivity for C) *For each $\mathbf{x} \in X$ and distinct $i_1, i_2, \ldots, i_n \in N$ ($n \leq N$), if $x_{i_1} \bar{C} x_{i_2} \bar{C} \cdots \bar{C} x_{i_n}$ then $x_{i_1} C x_{i_n}$.*

Observe that A3 is weaker than assuming C ($\bar{C}$) is transitive, which replaces all $\bar{C}$ (C) in A3 with C ($\bar{C}$). A2 and A3 are rather weak. They are vacuous if there are only two agents. For more agents, they impose no restrictions on any two actions $x_i, x_i' \in X_i \backslash \{o_i\}$ from the same agent. In particular, they allow $x_j C x_i$ for some x_i but $x_j S x_i'$ for some other x_i'. Even if we strengthen A2 and A3 to $x_j C x_i$ (similarly for $x_j S x_i$) for all $i, j \in N$ and $\mathbf{x} \in X$, this is still much weaker than the following strategic complementarity (similarly for substitutability) assumption imposed by most of the literature.

Condition 1 (Strategic complementarities) *For all $i \in N$, $o_i = 0 \in X_i \subseteq \mathbb{R}_+$ and for all $x_i, x_i' \in X_i$ with $x_i > x_i'$, $u_i(x_i, \mathbf{x}_{-i}) - u_i(x_i', \mathbf{x}_{-i})$ is weakly increasing in $\mathbf{x}_{-i} \in X_{-i}$.*

Observe that Condition 1 (henceforth C1) restricts agents' actions to a single dimension and imposes restrictions (strategic complementarities) on every two pairs of actions (x_i, x_i') and (x_j, x_j') from any two agents. Unlike A2 and A3, C1 is far from vacuous when $N = 2$. The extra flexibility of A2 and A3 allows us to study many more contracting situations. For example, in the context of public good provision, building a public facility involves careful planning, huge capital, on-site construction, and so on. These actions differ in nature and therefore cannot be meaningfully compared along a single dimension. Furthermore, actions of the same nature may be strategic substitutes (e.g., the provider may need only one good planner) whereas those of different nature may be strategic complements. Section 4.2 formalizes and analyzes this example.

In the case of binary-action games (i.e., $|X_i| = 2$ for all i), A1–A3 imply that agents are partitioned into several groups, in which agents' actions strategically complement each other within the same group and substitute each other across different groups.[10] In the general case, A1–A3 imply that for each action profile $\mathbf{x} \in X$, agents are partitioned analogously, i.e., $x_j C x_i$ if i and j belong to the same group and $x_j S x_i$ otherwise. Nevertheless, different action profiles can lead to different partitions.

Solution Concept The solution concept is subgame-perfect Nash equilibrium with an equilibrium selection criterion. First, I introduce a notion that allows us to compare two equilibrium selection criteria.

Definition 2 *One equilibrium selection criterion is more pessimistic than the other if, in every subgame, the equilibrium selected by the former gives the principal a weakly lower expected payoff than that by the latter.*

One equilibrium selection criterion, *potential maximization*, plays a pivotal role in the analysis. For a weighted potential game, it is well known that the maximizer of the potential function always exists, is generically unique, and is a (generically pure-strategy) Nash equilibrium.[11] Hence, potential maximization is simply to select the potential maximizer. As I will prove shortly in the next section, under A1, every subgame is a weighted potential game. By selecting the potential maximizer in each subgame, we resolve the multiple equilibria issue. In the analysis, I first derive the equilibrium outcome under potential maximization, and then show that all equilibrium selection criteria that are (not) more pessimistic than

[10] See the proof of Theorem 1 (p. 26) for the exact way to partition the agents. Bernstein and Winter (2012, Section 3.D) study a similar externalities structure in their binary-action model.

[11] Given X is compact and all the u_i functions (and thus the potential function Φ) are continuous, the potential maximizer exists by the extreme value theorem. See Appendix B for generic uniqueness of the potential maximizer. The potential maximizer is a Nash equilibrium: if someone deviates from the potential maximizer, the potential will decrease, and, by (1), the deviator will have a lower payoff. The potential of a mixed-strategy equilibrium is a convex combination of the potentials defined on the set of pure-strategy action profiles. Therefore, a mixed-strategy equilibrium is a potential maximizer only when all the respective pure-strategy action profiles are also potential maximizers; this is highly non-generic. For more interpretations of weighted potential games, see Chan (2019, Section 2.3).

potential maximization (may not) yield the same outcome. Hence, potential maximization serves as the "cut-off" equilibrium selection criterion.

Before proceeding, we need to take care of two technical problems. To facilitate the discussion, I formally define an equilibrium selection criterion as a function $f : P \to \Delta(X)$ where $f(\mathbf{p})$ is a Nash equilibrium of the subgame with $\mathbf{p}$ set by the principal in stage 1. With a slight abuse of notation, define $F(\mathbf{x}) \equiv \{\mathbf{p} \in P | f(\mathbf{p}) = \mathbf{x}\}$ as the set of menu profiles that implement $\mathbf{x} \in \Delta(X)$ under f. Thus, an equilibrium selection criterion f is expressed and henceforth interpreted as an *implementation requirement*, which is fully characterized by $F \equiv (F(\mathbf{x}))_{\mathbf{x}\in\Delta(X)}$. The first problem is that an optimal contracting scheme may not exist because $F(\mathbf{x})$ is not always closed.[12] To guarantee existence, I slightly relax each implementation requirement F by enlarging $F(\mathbf{x})$ to its closure for all $\mathbf{x} \in \Delta(X)$. The second problem is that potential maximization fails to select a unique equilibrium when there are multiple potential maximizers in a non-generic subgame. Nevertheless, the above enlargement already solves this problem. Precisely, after the enlargement, the principal can select among potential maximizers.

3 Analysis

The analysis before Theorem 1 relies only on A1; the subsequent analysis relies on all three assumptions. First, I show that as long as agents' utilities $\mathbf{u}$ constitute a weighted potential game, any subgame with an arbitrary menu profile $\mathbf{p} \in P$ offered by the principal is also a weighted potential game. All proofs are in Appendix A.

Lemma 1 *Every subgame is a weighted potential game with the same weight vector* $\mathbf{w}$ *given*

[12] To illustrate, consider a one-agent example with $X_1 = \{o_1 = 0, 1\}$, $u_1(0) = 0$, $u_1(1) = 1$, and $U(x_1, p_1(x_1)) = p_1(x_1)$. Multiple equilibria exist only when $p_1(1) = 1$. If f selects $x_1 = 0$ when $p_1(1) = 1$ (and therefore $F(1) = \{p_1 \in P_1 | p_1(1) < 1\}$, which is not closed), then an optimal contracting scheme does not exist.

in A1 and the potential function

$$\Phi_{\mathbf{p}}(\mathbf{x}) = \Phi(\mathbf{x}) - \sum_i \frac{p_i(x_i)}{w_i}. \tag{5}$$

Under potential maximization, the principal's problem can be formulated as the following two-step optimization problem. In step 1, given a target action profile $\hat{\mathbf{x}} \in X$,[13] she chooses the optimal menu profile $\mathbf{p}^* \in P$ such that $\hat{\mathbf{x}}$ is a potential maximizer in the subgame, i.e.,

$$\max_{\mathbf{p} \in P} U(\hat{\mathbf{x}}, \sum_i p_i(\hat{x}_i)) \quad \text{s.t.} \quad \hat{\mathbf{x}} \in \arg\max_{\mathbf{x} \in X} \Phi_{\mathbf{p}}(\mathbf{x}). \tag{6}$$

In step 2, she chooses the optimal action profile $\mathbf{x}^* \in X$, i.e.,

$$\max_{\mathbf{x} \in X} U(\mathbf{x}, \sum_i p_i^*(x_i)). \tag{7}$$

I first analyze the step-1 problem (6). Observe that given a fixed $\hat{\mathbf{x}}$, the principal's objective is to maximize her total revenue $\sum_i p_i(\hat{x}_i)$. Moreover, the constraints can be simplified with (5). Thus, (6) is simplified to

$$\max_{\mathbf{p} \in P} \sum_i p_i(\hat{x}_i) \quad \text{s.t.} \quad \sum_i \frac{p_i(\hat{x}_i) - p_i(x_i)}{w_i} \leq \Phi(\hat{\mathbf{x}}) - \Phi(\mathbf{x}) \quad \text{for all } \mathbf{x} \in X. \tag{8}$$

Now observe that charging arbitrarily high prices $p_i(x_i)$ for all $x_i \notin \{o_i, \hat{x}_i\}$ of every agent $i \in N$ relaxes all constraints involving $x_i \notin \{o_i, \hat{x}_i\}$ and has no impact on $\sum_i p_i(\hat{x}_i)$. Therefore, we have the following lemma.

Lemma 2 *Under potential maximization, the principal can restrict herself to offer (at most) one contract to each agent without loss of optimality.*

Under this restriction, agent i only chooses from $\{o_i, \hat{x}_i\}$ if $\hat{x}_i \neq o_i$ and chooses the outside option otherwise. Let $\hat{N} \equiv \{i \in N | \hat{x}_i \neq o_i\}$ denote the set of agents whom the principal

[13] In the highly non-generic case in which a mixed-strategy equilibrium is a potential maximizer (see footnote 11), the principal can select among all the respective pure-strategy potential maximizers. Given that her expected payoff in the mixed-strategy potential maximizer is a convex combination of her payoffs in the respective pure-strategy potential maximizers, she can, without loss of optimality, neglect the mixed-strategy potential maximizer and select among the respective pure-strategy potential maximizers.

wants to contract with, $\hat{p}_i \equiv p_i(\hat{x}_i)$ denote the respective contract price, and $\hat{\mathbf{p}} \equiv (\hat{p}_i)_{i\in\hat{N}}$ denote the price vector. The step-1 problem (8) is further simplified as follows.

Proposition 1 *For any target action profile $\hat{\mathbf{x}} \in X$, the principal's optimal contracts under potential maximization solve the following (finite) linear program:*

$$\max_{\hat{\mathbf{p}}\in\mathbb{R}^{|\hat{N}|}} \sum_{i\in\hat{N}} \hat{p}_i \quad s.t. \quad \sum_{i\in\{j\in\hat{N}|x_j=o_j\}} \frac{\hat{p}_i}{w_i} \leq \Phi(\hat{\mathbf{x}}) - \Phi(\mathbf{x}) \quad \textit{for all } \mathbf{x} \in \prod_i \{o_i, \hat{x}_i\}. \tag{9}$$

Without A2 or A3, the finite linear program (9) is still computationally very tractable. With A2 and A3, there is a closed-form expression for the optimal contracts.

Theorem 1 *Relabel agents such that $w_1 \leq \cdots \leq w_N$. For any target action profile $\hat{\mathbf{x}} \in X$, the principal's optimal contracts under potential maximization are*

$$\hat{p}_i^* = u_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, \hat{x}_i, \hat{y}_{i+1}, \ldots, \hat{y}_N) - u_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, o_i, \hat{y}_{i+1}, \ldots, \hat{y}_N) \quad \textit{for all } i \in N, \tag{10}$$

where $\hat{y}_j = o_j$ if $\hat{x}_j C \hat{x}_i$ and $\hat{y}_j = \hat{x}_j$ otherwise.[14] *If $w_1 < \cdots < w_N$, the above contracts are the unique optimal contracts.*

I call the above contracts (10) the *weight-ranked divide-and-conquer* ($\mathbf{w}$-DC) *contracts*. To better understand the $\mathbf{w}$-DC contracts, first consider the following special case.

Corollary 1 *If $\hat{x}_j C \hat{x}_i$ for all $i, j \in N$, the $\mathbf{w}$-DC contracts reduce to*

$$\hat{p}_i^* = u_i(\hat{x}_1, \ldots, \hat{x}_i, o_{i+1}, \ldots, o_N) - u_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, o_i, \ldots, o_N) \quad \textit{for all } i \in N. \tag{11}$$

I call (11) the *weight-ranked simple divide-and-conquer* ($\mathbf{w}$-SDC) *contracts*. In words, the $\mathbf{w}$-SDC contracts rank all agents in ascending order of weight w_i and offer each agent a price that would make him indifferent between accepting and rejecting the offer if all agents who precede him in the ranking accept their offers and all subsequent agents reject their offers. In the case of $\hat{x}_j C \hat{x}_i$ for all $i, j \in N$ (i.e., all agents' target actions strategically complement

[14] For expositional convenience, the optimal contracts also include agents with $\hat{x}_i = o_i$. For these agents, we have $\hat{p}_i^* = 0$, which coincides with the requirement that $p_i(o_i) = 0$.

each other relative to the outside option), the first agent has a weakly dominant strategy to accept the offer.[15] Given the first agent accepts the offer, the second agent has an (iterated) weakly dominant strategy to accept the offer as well, and so on. Thus, the $\mathbf{w}$-SDC contracts in fact implement $\hat{\mathbf{x}}$ as a dominance-solvable equilibrium in this special case.

The $\mathbf{w}$-DC contracts generalize the $\mathbf{w}$-SDC contracts by taking full account of the strategic substitutabilities from all subsequent agents. By the same token, the $\mathbf{w}$-DC contracts implement $\hat{\mathbf{x}}$ as a dominance-solvable equilibrium in the general case.[16] Note that this property has nothing to do with the underlying equilibrium selection criterion; the principal can always offer the $\mathbf{w}$-DC contracts and implement $\hat{\mathbf{x}}$ in the same dominance-solvable way.[17] Moreover, under any equilibrium selection criterion that is more pessimistic than potential maximization, the principal, by definition, cannot do better than she does under potential maximization. Therefore, the $\mathbf{w}$-DC contracts remain optimal for all these equilibrium selection criteria. A simple two-agent example suffices to show that the $\mathbf{w}$-DC contracts may be suboptimal if the underlying equilibrium selection criterion is *not more pessimistic* than potential maximization. Thus, I have identified the entire set of equilibrium selection criteria in which the principal always offers the $\mathbf{w}$-DC contracts. The following theorem summarizes the above discussion.

Theorem 2 *The $\mathbf{w}$-DC contracts are optimal for all games if and only if the underlying equilibrium selection criterion is more pessimistic than potential maximization.*

[15] Formally, we have $u_1(\hat{x}_1, \mathbf{x}_{-1}) - \hat{p}_1^* \geq u_1(o_1, \mathbf{x}_{-1})$ for all $\mathbf{x}_{-1} \in \prod_{i\neq 1}\{o_i, \hat{x}_i\}$. This is because $\hat{p}_1^* = u_1(\hat{x}_1, \mathbf{o}_{-1}) - u_1(\mathbf{o})$ by (11) and $\hat{x}_i C \hat{x}_1$ for all $i \neq 1$ implies $u_1(\hat{x}_1, \mathbf{x}_{-1}) - u_1(o_1, \mathbf{x}_{-1}) \geq u_1(\hat{x}_1, \mathbf{o}_{-1}) - u_1(\mathbf{o})$ for all $\mathbf{x}_{-1} \in \prod_{i\neq 1}\{o_i, \hat{x}_i\}$.

[16] Similar to footnote 15, we have $u_1(\hat{x}_1, \mathbf{x}_{-1}) - \hat{p}_1^* \geq u_1(o_1, \mathbf{x}_{-1})$ for all $\mathbf{x}_{-1} \in \prod_{i\neq 1}\{o_i, \hat{x}_i\}$. This is because $\hat{p}_1^* = u_1(\hat{x}_1, \hat{\mathbf{y}}_{-1}) - u_1(o_1, \hat{\mathbf{y}}_{-1})$ by (10) and $\hat{y}_i = o_i$ if $\hat{x}_i C \hat{x}_1$ and $\hat{y}_i = \hat{x}_i$ otherwise imply $u_1(\hat{x}_1, \mathbf{x}_{-1}) - u_1(o_1, \mathbf{x}_{-1}) \geq u_1(\hat{x}_1, \hat{\mathbf{y}}_{-1}) - u_1(o_1, \hat{\mathbf{y}}_{-1})$ for all $\mathbf{x}_{-1} \in \prod_{i\neq 1}\{o_i, \hat{x}_i\}$.

[17] Note that $\hat{\mathbf{x}}$ becomes the unique and strict Nash equilibrium if the principal charges each agent a price slightly lower than that of the $\mathbf{w}$-DC contracts. The enlargement of $F(\hat{\mathbf{x}})$ to its closure on p. 11 guarantees she can implement $\hat{\mathbf{x}}$ with the $\mathbf{w}$-DC contracts under any equilibrium selection criterion.

I now restrict attention to the case where the principal always offers the $\mathbf{w}$-DC contracts and analyze her step-2 problem (7). By Theorem 1, her equilibrium payoff given a target action profile $\hat{\mathbf{x}}$ is

$$
\begin{aligned}
& U(\hat{\mathbf{x}}, \sum_i [u_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, \hat{x}_i, \hat{y}_{i+1}, \ldots, \hat{y}_N) - u_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, o_i, \hat{y}_{i+1}, \ldots, \hat{y}_N)]) \\
= \;& U(\hat{\mathbf{x}}, \sum_i w_i [\Phi(\hat{x}_1, \ldots, \hat{x}_{i-1}, \hat{x}_i, \hat{y}_{i+1}, \ldots, \hat{y}_N) - \Phi(\hat{x}_1, \ldots, \hat{x}_{i-1}, o_i, \hat{y}_{i+1}, \ldots, \hat{y}_N)]). \quad \text{(by (1))}
\end{aligned}
$$

She then chooses the optimal action profile $\mathbf{x}^* \in X$ to maximize the above payoff function. If $\hat{x}_j C \hat{x}_i$ for all $i, j \in N$ and all agents have the same weight of w, the payoff function is simplified to $U(\hat{\mathbf{x}}, w[\Phi(\hat{\mathbf{x}}) - \Phi(\mathbf{o})])$. Hence, we have the following corollary.

Corollary 2 *If $x_j C x_i$ and $w_i = w$ for all $i, j \in N$ and $\mathbf{x} \in X$, the principal's optimal action profile is*

$$
\mathbf{x}^* \in \arg\max_{\mathbf{x} \in X} U(\mathbf{x}, w[\Phi(\mathbf{x}) - \Phi(\mathbf{o})]).
$$

Recall from p. 9 that $x_j C x_i$ for all $i, j \in N$ and $\mathbf{x} \in X$ is much weaker than C1. In addition, all agents having the same weight does not imply agents are identical. The next section demonstrates how agents can be heterogeneous in many aspects while having the same weight.

3.1 Discussion

Optimal Ranking Theorem 1 shows that the optimal ranking to offer divide-and-conquer contracts is to rank agents in order of increasing weight w_i. To understand the rationale of this ranking, first consider the extreme case where $\hat{x}_j C \hat{x}_i$ for all $i, j \in N$. This, together with (1), implies $\Phi(\hat{x}_i, \mathbf{x}_{-i}) - \Phi(o_i, \mathbf{x}_{-i})$ is increasing in $\mathbf{x}_{-i} \in \prod_{j \neq i} \{o_j, \hat{x}_j\}$ if we view o_j as 0 and $\hat{x}_j$ as 1. In other words, the potential function Φ captures the strategic complementarities among agents' target actions. Observe from (2) that agents with higher weights care more about Φ, i.e., they care more about the strategic complementarities. Therefore, by placing them at lower ranks in the (simple) divide-and-conquer contracts, the principal can extract more surplus. Now consider the other extreme case where $\hat{x}_j S \hat{x}_i$ for all $i, j \in N$. Observe from (10) that each agent's price is $u_i(\hat{\mathbf{x}}) - u_i(o_i, \hat{\mathbf{x}}_{-i})$ no matter how we relabel the agents, i.e., the ranking does not matter at all. In other words, the strategic substitutabilities among agents play no role in the ranking decision. The insights from these two extreme cases carry over to the general case, and therefore the principal optimally ranks agents in increasing weight. A1 is based on agents' utilities $\mathbf{u}$, and therefore so is the weight vector $\mathbf{w}$.[18] The next section studies three different examples of $\mathbf{u}$ and demonstrates how $\mathbf{w}$ differs in different contexts.

Exact Potential Games If all agents have the same weight of w, we can normalize the weight to one by rescaling the potential function Φ to $w\Phi$; the resulting potential game is called an *exact potential game*. For exact potential games, Theorem 1 shows that the principal can rank agents arbitrarily. However, every agent prefers to be ranked higher because he would be offered a lower price, or in other words, a more favorable contract. In particular, observe from (10) that the lowest ranked agent N faces a price of $u_N(\hat{\mathbf{x}}) - u_N(o_N, \hat{\mathbf{x}}_{-N})$, i.e., the principal fully extracts his surplus, making him indifferent between accepting and rejecting the offer in equilibrium. Hence, the ranking serves as a measure of prioritization of agents.

Implementation Requirements The previous analysis focuses on equilibrium selection criteria, which is one type of implementation requirements as explained on p. 11. I now extend Theorem 2 if we are also interested in other implementation requirements. Recall that an implementation requirement is fully characterized by $F \equiv (F(\mathbf{x}))_{\mathbf{x}\in\Delta(X)}$ where $F(\mathbf{x}) \subseteq P$. I say one implementation requirement F is *stronger* than the other F' if $F(\mathbf{x}) \subseteq F'(\mathbf{x})$ for

[18] Clearly, $\mathbf{w}$ is independent of the principal's payoff U. Shrinking agents' action sets X_i also has no impact on $\mathbf{w}$. In the context of capital raising, this implies the optimal ranking to offer divide-and-conquer contracts is independent of agents' capital endowments; this is opposite to the main finding of Halac et al. (2020). The difference in results is driven by the fact that their principal can make payments contingent on stochastic outcomes whereas mine cannot. To introduce stochastic outcomes into my model, one would need to make further assumptions on $\mathbf{u}$ and U.

all $\mathbf{x} \in \Delta(X)$. Note that "stronger" and "more pessimistic" are different concepts.[19] Let $\mathcal{F}$ denote the set of equilibrium selection criteria that are more pessimistic than potential maximization and F^* be an implementation requirement where $F^*(\mathbf{x}) = \bigcup_{F \in \mathcal{F}} F(\mathbf{x})$. By construction, F^* is *weaker* than each $F \in \mathcal{F}$. Yet, the fact that the $\mathbf{w}$-DC contracts are optimal for all $F \in \mathcal{F}$ implies they remain optimal under F^*. In addition, the $\mathbf{w}$-DC contracts implement the target action profile as a dominance-solvable equilibrium, and therefore they are feasible under the very strong requirement of *dominance-solvable implementation*. The above two findings imply the following corollary.

Corollary 3 *The $\mathbf{w}$-DC contracts are optimal for all implementation requirements that are stronger than F^* and weaker than dominance-solvable implementation.*

Divide and Conquer Although the term "divide and conquer" is not formally defined in the literature, a coherent definition is "to implement a target action profile as a dominance-solvable equilibrium." Hence, the use of a divide-and-conquer contracting scheme *per se* is not particularly interesting because one can always rationalize it with the requirement of dominance-solvable implementation. An interesting result in the literature (see p. 5) is that oftentimes it can also be rationalized by the weaker requirement (but still stronger than every equilibrium selection criterion) of *unique implementation*, i.e., to implement a target action profile as the unique Nash equilibrium in the subgame. Theorem 2 and Corollary 3 show that, for a large class of contracting models, we can actually rationalize a specific divide-and-conquer contracting scheme—the $\mathbf{w}$-DC contracts—with a large set of equilibrium selection criteria and implementation requirements. This allows us to advise/predict with confidence that the principal should/would offer the $\mathbf{w}$-DC contracts even if we, as researchers, do not know the underlying equilibrium selection criterion or what implementation requirement the principal wants to meet.

[19] In fact, before the enlargement (see p. 11), one equilibrium selection criterion F is either not stronger than or equivalent to the other F' because both $\{F(\mathbf{x})\}_{\mathbf{x} \in \Delta(X)}$ and $\{F'(\mathbf{x})\}_{\mathbf{x} \in \Delta(X)}$ are partitions of P. This also implies that any implementation requirement strictly stronger than an equilibrium selection criterion rules out the use of certain menu profiles $\mathbf{p} \in P$.

Single Contract Lemma 2, Theorem 2, and Corollary 3 together imply the principal can, without loss of optimality, offer one contract to each agent for a large set of equilibrium selection criteria and implementation requirements. This result is non-trivial because a major contribution of Segal (2003, Lemma 3) is to show that, under C1 and the requirement of unique implementation, the principal is generally better off offering multiple contracts to each agent. It turns out that just imposing A1–A3 already rules out this possibility.

4 Applications

This section applies the previous results to (i) networks, (ii) public goods, and (iii) impure public goods, and derives novel implications for each application. These applications together demonstrate how seemingly contradictory implications across applications are reconciled with the general theories developed in the previous section.

4.1 Networks

I now revisit the general example (3). Observe that it covers the two most popular forms of network games: the binary-action form if $X_i = \{0, 1\}$ and the linear-quadratic form if $b_i(x_i) = \alpha_i x_i - \beta_i x_i^2$. In the former, the principal's contracting scheme reduces to charging each agent a participation fee $p_i(1)$. Agents' utilities $\mathbf{u}$ clearly satisfy C1 (which implies A2 and A3); they also satisfy A1.

Lemma 3 *Agents' utilities constitute a weighted potential game with* $\mathbf{w} = \left(\frac{v_i}{\theta_i}\right)_i$ *and*

$$\Phi(\mathbf{x}) = \sum_i \frac{\theta_i b_i(x_i)}{v_i} + \frac{1}{2} \sum_i \sum_{j \in E_i} \theta_i \theta_j x_i x_j. \tag{12}$$

Therefore, all previous results apply to this example. Theorem 2 and Corollary 1 imply the following corollary.

Corollary 4 *For any target action profile* $\hat{\mathbf{x}} \in X$ *and any equilibrium selection criterion more pessimistic than potential maximization, the principal's optimal contracting scheme is to offer the* $\mathbf{w}$*-SDC contracts where* $\mathbf{w} = \left(\frac{v_i}{\theta_i}\right)_i$.

Agents' ranking in the $\mathbf{w}$-SDC contracts is based solely on their valuation-to-importance ratios $\frac{v_i}{\theta_i}$ but not X_i, b_i, or E_i. This generalizes the main finding of Sakovics and Steiner (2012, Proposition 2) who study a binary-action complete network (i.e., $X_i = \{0, 1\}$ and $E_i = N \backslash \{i\}$) under *global-game selection* (which is equivalent to potential maximization as stated in footnote 4). They show that the optimal ranking depends only on $\frac{v_i}{\theta_i}$ but not b_i. Corollary 4 shows that the ranking is also independent of agents' action sets X_i and sets of interacting agents E_i, and this result is robust to all equilibrium selection criteria that are more pessimistic than global-game selection. Contrary to the conventional wisdom that the principal should prioritize agents with high centrality, the entire network structure actually plays no role in the optimal ranking. Furthermore, as we will see more clearly in (13), an agent faces a higher, not lower, price if he interacts with more agents.

When all agents have the same weight (i.e., they have the same valuation-to-importance ratio), they can still differ in all other three dimensions (X_i, b_i, E_i). The literature on the economics of networks often assumes $v_i = \theta_i = 1$ for all i. This assumption implies all agents have the same weight, and therefore the principal's optimal action profile $\mathbf{x}^*$ is characterized by Corollary 2. Moreover, recall from p. 16 that the principal can rank agents arbitrarily in this case. This echoes the previous finding: the principal indeed has no strict incentive to prioritize and offer more favorable contracts to agents with high centrality.

I now discuss further implications of this network model. To facilitate the discussion, assume $X_i = \{0, 1\}$ for all i, $\frac{v_1}{\theta_1} < \cdots < \frac{v_N}{\theta_N}$ (which implies $w_1 < \cdots < w_N$), and $\hat{\mathbf{x}} = \mathbf{x}^* = \mathbf{1}$. The $\mathbf{w}$-SDC contracts (11) are given by

$$p_i^*(1) = b_i(1) - b_i(0) + v_i \sum_{j \in \{k \in E_i | w_k < w_i\}} \theta_j \quad \text{for all } i \in N. \tag{13}$$

Hence, agent i's equilibrium payoff is

$$u_i(\mathbf{1}) - p_i^*(1) = b_i(0) + v_i \sum_{j \in \{k \in E_i | w_k > w_i\}} \theta_j.$$

Now consider a situation where agent i interacts with an additional agent $j \notin E_i$. Agent i is strictly better off if $w_j > w_i$: he pays the same price but derives additional interaction benefits. Conditional on $w_j > w_i$, he most prefers the additional agent with the highest importance θ_j. By contrast, agent i is just as well off if $w_j < w_i$: the principal raises his price by an amount equal to his additional interaction benefits. In any case, agent i does not mind interacting with more agents. Therefore, if the network is endogenously formed in stage 0, a natural formation process is that each agent unilaterally enables a few interactions. Under this process, agents with high weights $\frac{v_i}{\theta_i}$ and/or importance θ_i end up interacting with many agents in equilibrium. In other words, popular agents are those who value the network a lot and those with either high or low importance. If all agents have the same valuation but different importance and can only enable one interaction, an assortative line network is formed in which agent i chooses agent $i+1$ (agent N is indifferent between choosing any agent). To the best of my knowledge, these findings are novel to the literature on network formation.

4.2 Public Goods

I now formalize the public good example described on p. 9. Each agent's action set X_i remains the most general form, i.e., it can be any compact set. Agent i's utility takes the following form:

$$u_i(\mathbf{x}) = b_i(x_i) + v_i g(\mathbf{x}), \tag{14}$$

where $b_i : X_i \to \mathbb{R}$ measures his stand-alone benefit/cost, $v_i \in \mathbb{R}_{++}$ measures his valuation of the public good, and $g : X \to \mathbb{R}$ measures the size of the public good. Agents differ in four dimensions: (X_i, b_i, v_i) and how each agent's actions affect the size of the public good g. The function g can be very general (but not arbitrary as I will explain shortly), and it captures the nature (in particular, the importance) of each agent's contribution to the public good. For example, if $o_i = 0 \in X_i \subseteq \mathbb{R}_+$ and $g(\mathbf{x}) = (\sum_j \theta_j x_j)^2$, then $\theta_j \in \mathbb{R}_{++}$ measures the relative importance of agent j's actions as in the previous network model. The following lemma shows that agents' utilities satisfy A1.

Lemma 4 *Agents' utilities constitute a weighted potential game with* $\mathbf{w} = (v_i)_i$ *and*

$$\Phi(\mathbf{x}) = \sum_i \frac{b_i(x_i)}{v_i} + g(\mathbf{x}). \tag{15}$$

To state the condition for agents' utilities to satisfy A2 and A3, I first define the modified binary relations C_g and S_g as follows.

Definition 3 *The expression* $x_j C_g x_i$ $(x_j S_g x_i)$ *stands for*

$$g(x_i, x_j, \mathbf{x}_{-ij}) - g(o_i, x_j, \mathbf{x}_{-ij}) \geq (\leq)\ g(x_i, o_j, \mathbf{x}_{-ij}) - g(o_i, o_j, \mathbf{x}_{-ij}) \quad \forall \mathbf{x}_{-ij} \in X_{-ij}.$$

We can easily verify that $x_j C x_i$ $(x_j S x_i)$ if and only if $x_j C_g x_i$ $(x_j S_g x_i)$. Therefore, A2 and A3 hold if and only if they remain true when C and S are replaced by C_g and S_g respectively. Suppose they indeed remain true. Then all results in Section 3 apply to this model.

Corollary 5 *For any target action profile* $\hat{\mathbf{x}} \in X$ *and any equilibrium selection criterion more pessimistic than potential maximization, the principal's optimal contracting scheme is to offer the* $\mathbf{w}$*-DC contracts where* $\mathbf{w} = (v_i)_i$.

Agents' ranking in the $\mathbf{w}$-DC contracts is based solely on their valuations of the public good v_i but not X_i, b_i, or g. In contrast to the network model where the optimal ranking depends crucially on agents' importance θ_i by Corollary 4, the ranking is now independent of their importance to the public good (as captured by g). The reason for these opposing results is that the public good is *non-excludable* whereas the "network good" is *excludable*, i.e., an agent derives zero interaction benefit whenever he rejects the offer. In other words, the principal's optimal contracting scheme depends critically on the excludability of externalities. The next subsection formalizes the above argument. Recall from p. 19 that the main finding of Sakovics and Steiner (2012) is a special case of Corollary 4 and, therefore, does not apply to public goods or non-excludable externalities. This urges caution in applying their proposed contracting scheme to their leading applications: economic development and financial fragility; both are largely public goods/bads in nature or at least partially non-excludable. The next subsection proposes a refined contracting scheme.

4.3 Impure Public Goods

Consider a hybrid of the previous network and public good models. Agent i's action set X_i is a compact subset of $\mathbb{R}_+$ with $o_i = 0 \in X_i$. Agent i's utility takes the following form:

$$u_i(\mathbf{x}) = b_i(x_i) + \underbrace{\delta v_i x_i \sum_j \theta_j x_j}_{\text{excludable externalities}} + \underbrace{\frac{(1-\delta)v_i}{2}(\sum_j \theta_j x_j)^2}_{\text{public good externalities}} + \underbrace{(1-\delta)\xi_i(\mathbf{x}_{-i})}_{\text{pure externalities}},$$

where $\delta \in [0,1]$ measures the degree of excludability, $\xi_i : X_{-i} \to \mathbb{R}$ measures the pure externalities generated by others' actions, and $b_i : X_i \to \mathbb{R}$, $v_i \in \mathbb{R}_{++}$, and $\theta_j \in \mathbb{R}_{++}$ are interpreted the same way as before. Both public good and pure externalities are non-excludable, but the latter play no strategic role. In fact, adding arbitrary pure externalities to agents' utilities in the previous models makes them more general and has no impact on subsequent results. If $\delta = 1$, this model reduces to the network model with a complete network. If $\delta = 0$, it reduces to the public good model with $o_i = 0 \in X_i \subseteq \mathbb{R}_+$ and $g(\mathbf{x}) = \frac{1}{2}(\sum_j \theta_j x_j)^2$. Agents' utilities clearly satisfy C1; they also satisfy A1.[20]

Lemma 5 *Agents' utilities constitute a weighted potential game with* $\mathbf{w} = \left(\delta \frac{v_i}{\theta_i} + (1-\delta)v_i\right)_i$ *and*

$$\Phi(\mathbf{x}) = \sum_i \frac{b_i(x_i) + \frac{1}{2}\delta v_i \theta_i x_i^2}{w_i} + \frac{1}{2}(\sum_i \theta_i x_i)^2. \tag{16}$$

Therefore, Theorem 2 and Corollary 1 imply the following corollary.

Corollary 6 *For any target action profile* $\hat{\mathbf{x}} \in X$ *and any equilibrium selection criterion more pessimistic than potential maximization, the principal's optimal contracting scheme is to offer the* $\mathbf{w}$*-SDC contracts where* $\mathbf{w} = \left(\delta \frac{v_i}{\theta_i} + (1-\delta)v_i\right)_i$.

[20] The fact that this model is a special case of a convex combination of the previous two models does not imply it satisfies A1 because a convex combination of two weighted potential games need not be a weighted potential game. Moreover, even if the combination turns out to be a weighted potential game, the corresponding weights need not be a convex combination of the previous weights. Therefore, what the following lemma shows are specific to this model rather than some universal facts.

In this model, the optimal ranking is determined by the weighted average of each agent's valuation-to-importance ratio $\frac{v_i}{\theta_i}$ (which is the only determinant in the network model) and his valuation v_i (which is the only determinant in the public good model), and the relative weight depends on the degree of excludability δ. In other words, the principal always prioritizes agents with low valuations, whereas she also prioritizes those with high importance if and only if the good is sufficiently excludable. As stated, economic development and financial fragility are partially non-excludable. Hence, Corollary 6 provides theoretical guidance on sectoral industrial policies for the former and financial policies for the latter. Note that agents' action sets X_i play no role in the optimal ranking for this and all previous models. In other words, the principal need not prioritize "large" agents (who can take large actions x_i and affect all agents significantly). This suggests that the well-known "too-big-to-fail" doctrine (i.e., bailing out large financial institutions) may be suboptimal for preventing financial crises.

5 Conclusion

This paper carefully answers the ultimate question of what contracts the principal should or would offer when there are multiple agents. For a large class of contracting models, I show that the $\mathbf{w}$-DC contracts are optimal for a large set of equilibrium selection criteria and implementation requirements. This result provides robust predictions and policy guidance for a wide variety of applications. Finally, the general framework, newly developed tools, and the incorporation of potential game theory promise to open a wide range of new research opportunities in multi-agent contracting.

References

[1] Belhaj, M., & Deroian, F. (2019). Group targeting under networked synergies. *Games and Economic Behavior*, *118*, 29–46. [5]

[2] Bergstrom, T., Blume, L., & Varian, H. (1986). On the private provision of public goods. *Journal of Public Economics*, *29*, 25–49. [5]

[3] Bernstein, S., & Winter, E. (2012). Contracting with heterogeneous externalities. *American Economic Journal: Microeconomics*, *4*, 50–76. [3, 5, 10]

[4] Bloch, F., & Shabayek, S. (2020). Targeting in social networks with anonymized information. *arXiv preprint arXiv:2001.03122*. [5]

[5] Chan, L. (2019). Divide and conquer in two-sided Markets: A potential-game approach. *Available at SSRN 3498394*. [5, 10]

[6] Cornes, R., & Sandler, T. (1984). Easy riders, joint production, and public goods. *Economic Journal*, *94*, 580–598. [5]

[7] Cornes, R., & Sandler, T. (1994). The comparative static properties of the impure public good model. *Journal of Public Economics*, *54*, 403–421. [5]

[8] Frankel, D., Morris, S., & Pauzner, A. (2003). Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory*, *108*, 1–44. [5]

[9] Halac, M., Kremer, I., & Winter, E. (2020). Raising capital from heterogeneous investors. *American Economic Review*, *110*, 889–921. [3, 5, 16]

[10] Jackson, M., Rogers, B., & Zenou, Y. (2017). The economic consequences of social-network structure. *Journal of Economic Literature*, *55*, 49–95. [4]

[11] Jadbabaie, A., & Kakhbod, A. (2019). Optimal contracting in networks. *Journal of Economic Theory*, *183*, 1094–1153. [5]

[12] Mangasarian, O. (1979). Uniqueness of solution in linear programming. *Linear Algebra and its Applications*, *25*, 151–162. [29]

[13] Monderer, D., & Shapley, L. (1996). Potential games. *Games and Economic Behavior*, *14*, 124–143. [5]

[14] Rosenthal, R. (1973). A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, *2*, 65–67. [5]

[15] Sakovics, J., & Steiner, J. (2012). Who matters in coordination problems?. *American Economic Review*, *102*, 3439–61. [6, 19, 21]

[16] Segal, I. (1999). Contracting with externalities. *Quarterly Journal of Economics*, *114*, 337–388. [4, 7]

[17] Segal, I. (2003). Coordination and discrimination in contracting with externalities: Divide and conquer?. *Journal of Economic Theory*, *113*, 147–181. [4, 5, 6, 18]

[18] Shi, F., & Xing, Y. (2018). Screening with network externalities. *Available at SSRN 3286427*. [5]

[19] Winter, E. (2004). Incentives and discrimination. *American Economic Review*, *94*, 764–773. [5]

[20] Zhang, Y., & Chen, Y. (2020). Optimal nonlinear pricing in social networks under asymmetric network information. *Operations Research*, *68*, 818–833. [5]

Appendix

A Proofs

Proof of Lemma 1 For all $\mathbf{p} \in P$, $i \in N$, and $\mathbf{x} \in X$,

$$\begin{aligned} u_i(\mathbf{x}) - p_i(x_i) &= w_i\Phi(\mathbf{x}) + \xi_i(\mathbf{x}_{-i}) - p_i(x_i) \quad \text{(by A1 and (2))} \\ &= w_i\Phi_{\mathbf{p}}(\mathbf{x}) + w_i \sum_{j \neq i} \frac{p_j(x_j)}{w_j} + \xi_i(\mathbf{x}_{-i}) \quad \text{(by (5))} \\ &= w_i\Phi_{\mathbf{p}}(\mathbf{x}) + \xi'_i(\mathbf{x}_{-i}). \quad (\xi'_i(\mathbf{x}_{-i}) \equiv w_i \sum_{j \neq i} \frac{p_j(x_j)}{w_j} + \xi_i(\mathbf{x}_{-i})) \end{aligned}$$

Proof of Lemma 2 It follows from the discussion in the text.

Proof of Proposition 1 It follows from the discussion in the text.

Proof of Theorem 1 The target action profile $\hat{\mathbf{x}}$ is fixed throughout this proof. For expositional convenience, the optimal contracts (10) include all agents but only those with $\hat{x}_i \neq o_i$ (i.e., those belong to $\hat{N}$) matter in the linear program (9); see also footnote 14. Without loss of generality, assume $\hat{N} = \{1, \ldots, |\hat{N}|\}$ and $w_1 \leq \cdots \leq w_{|\hat{N}|}$.

To prove this theorem, I first introduce some notation. Recall from p. 9 that $\bar{C}$ is symmetric. Therefore, agents can be partitioned into several groups so that for any two group members i and j, there exist mutual group members $k_1, \ldots, k_n \in \hat{N}$ $(0 \leq n \leq |\hat{N}| - 2)$ such that $\hat{x}_j \bar{C} \hat{x}_{k_1} \bar{C} \hat{x}_{k_2} \bar{C} \cdots \bar{C} \hat{x}_{k_n} \bar{C} \hat{x}_i$.[21] Let $G_i \subseteq \hat{N}$ denote the set of agent i's group members together with agent i himself. Clearly, $G_i = G_j$ for all $j \in G_i$. A3 implies $\hat{x}_j C \hat{x}_i$ for all $j \in G_i \backslash \{i\}$. A2 implies $\hat{x}_j S \hat{x}_i$ for all $j \notin G_i$. For notational convenience, define $H_i \equiv \hat{N} \backslash G_i$, $\widetilde{\Phi} : 2^{\hat{N}} \to \mathbb{R}$ where $\widetilde{\Phi}(Z) = \Phi(\mathbf{x})$ with $x_i = \hat{x}_i$ if $i \in Z$ $(Z \subseteq \hat{N})$ and $x_i = o_i$ otherwise, and $\widetilde{u}_i : 2^{\hat{N}} \to \mathbb{R}$ is defined analogously. We can easily show that A1 and the above derived properties imply the following:

[21] In graph theory language, each vertex represents an agent, and agents i and j are linked iff $\hat{x}_j \bar{C} \hat{x}_i$. Every undirected graph can be decomposed into several connected components, which are the groups I have described.

B1. for all $i \in \hat{N}$ and $Z \subseteq \hat{N} \backslash \{i\}$, $\widetilde{u}_i(Z \cup \{i\}) - \widetilde{u}_i(Z) = w_i[\widetilde{\Phi}(Z \cup \{i\}) - \widetilde{\Phi}(Z)]$, and

B2. for all $i \in \hat{N}$, $\widetilde{\Phi}(G \cup H \cup \{i\}) - \widetilde{\Phi}(G \cup H)$ is increasing in $G \subseteq G_i \backslash \{i\}$ and decreasing in $H \subseteq H_i$.

The linear program (9) is now re-expressed as

$$\max_{\hat{\mathbf{p}} \in \mathbb{R}^{|\hat{N}|}} \sum_{i \in \hat{N}} \hat{p}_i \quad \text{s.t.} \quad \sum_{i \in Z} \frac{\hat{p}_i}{w_i} \leq \widetilde{\Phi}(\hat{N}) - \widetilde{\Phi}(\hat{N} \backslash Z) \quad \text{for all } Z \subseteq \hat{N}. \tag{17}$$

The contracts (10) are re-expressed as

$$\hat{p}_i^* = \widetilde{u}_i(\{1, \ldots, i\} \cup H_i) - \widetilde{u}_i(\{1, \ldots, i-1\} \cup H_i) \quad \text{for all } i \in \hat{N}. \tag{18}$$

The rest of the proof is standard in linear programming which consists of three steps. First, I show that $\hat{\mathbf{p}}^* \equiv (\hat{p}_i^*)_{i \in \hat{N}}$ is feasible. Next, I show that $\hat{\mathbf{p}}^*$ is optimal. Last, I show that $\hat{\mathbf{p}}^*$ is the unique optimal solution if $w_1 < \cdots < w_{|\hat{N}|}$.

Feasibility For $\hat{\mathbf{p}}^*$ to be feasible, we need to show

$$\sum_{i \in Z} \frac{\widetilde{u}_i(\{1, \ldots, i\} \cup H_i) - \widetilde{u}_i(\{1, \ldots, i-1\} \cup H_i)}{w_i} \leq \widetilde{\Phi}(\hat{N}) - \widetilde{\Phi}(\hat{N} \backslash Z) \quad \text{for all } Z \subseteq \hat{N}.$$

By B1, the above inequalities become

$$\sum_{i \in Z} [\widetilde{\Phi}(\{1, \ldots, i\} \cup H_i) - \widetilde{\Phi}(\{1, \ldots, i-1\} \cup H_i)] \leq \widetilde{\Phi}(\hat{N}) - \widetilde{\Phi}(\hat{N} \backslash Z) \quad \text{for all } Z \subseteq \hat{N}.$$

Note that any set $Z \subseteq \hat{N}$ takes the form of $\{i_1, \ldots, i_n\}$ for some $i_1, \ldots, i_n \in \hat{N}$ and $i_1 < \cdots < i_n$ ($n \leq |\hat{N}|$). Thus, the respective inequality for the set $Z = \{i_1, \ldots, i_n\}$ is re-expressed as

$$\sum_{m=1}^{n} [\widetilde{\Phi}(\{1, \ldots, i_m\} \cup H_{i_m}) - \widetilde{\Phi}(\{1, \ldots, i_m - 1\} \cup H_{i_m})] \leq \sum_{m=1}^{n} [\widetilde{\Phi}(\hat{N} \backslash \{i_{m+1}, \ldots, i_n\}) - \widetilde{\Phi}(\hat{N} \backslash \{i_m, \ldots, i_n\})].$$

For the above inequality to hold, it suffices to show that for all $m = 1, \ldots, n$,

$$\widetilde{\Phi}(\{1, \ldots, i_m - 1\} \cup H_{i_m} \cup \{i_m\}) - \widetilde{\Phi}(\{1, \ldots, i_m - 1\} \cup H_{i_m}) \leq \widetilde{\Phi}(\hat{N} \backslash \{i_m, \ldots, i_n\} \cup \{i_m\}) - \widetilde{\Phi}(\hat{N} \backslash \{i_m, \ldots, i_n\}).$$

Observe that $\{1, \ldots, i_m - 1\} \cup H_{i_m} = (G_{i_m} \cap \{1, \ldots, i_m - 1\}) \cup H_{i_m}$ and $\hat{N} \backslash \{i_m, \ldots, i_n\} = (G_{i_m} \backslash \{i_m, \ldots, i_n\}) \cup (H_{i_m} \backslash \{i_m, \ldots, i_n\})$. Hence, the above inequality holds by B2 because $G_{i_m} \cap \{1, \ldots, i_m - 1\} \subseteq G_{i_m} \backslash \{i_m, \ldots, i_n\}$ and $H_{i_m} \backslash \{i_m, \ldots, i_n\} \subseteq H_{i_m}$.

Optimality The proof of the optimality of $\hat{\mathbf{p}}^*$ consists of three steps. First, I derive the dual problem of (17). Next, I construct a feasible solution $\boldsymbol{\lambda}^*$ for the dual. Last, I show that the objective function value of the dual at $\boldsymbol{\lambda}^*$ is equal to that of the primal at $\hat{\mathbf{p}}^*$. The weak duality theorem states that the objective function value of the dual at any feasible solution is weakly greater than that of the primal at any feasible solution. This implies $\hat{\mathbf{p}}^*$ is optimal for the primal.

The dual problem of (17) is

$$\min_{\lambda(Z)\geq 0, Z\subseteq \hat{N}} \sum_{Z\subseteq \hat{N}} \lambda(Z)[\widetilde{\Phi}(\hat{N}) - \widetilde{\Phi}(\hat{N}\backslash Z)] \quad \text{s.t.} \quad \sum_{i\in Z\subseteq \hat{N}} \lambda(Z) = w_i \quad \text{for all } i \in \hat{N}.$$

Define $w_0 \equiv 0$, $i' \equiv \max\{j \in G_i \cup \{0\} | j < i\}$, and $\boldsymbol{\lambda}^* \equiv (\lambda^*(Z))_{Z\subseteq \hat{N}}$ where $\lambda^*(Z) = w_i - w_{i'}$ if $Z = G_i \cap \{i, i+1, \dots, |\hat{N}|\}$ $(i \in \hat{N})$ and $\lambda^*(Z) = 0$ otherwise. The solution $\boldsymbol{\lambda}^*$ is feasible because for all $i \in \hat{N}$,

$$\begin{aligned}
\sum_{i\in Z\subseteq \hat{N}} \lambda^*(Z) &= \sum_{i\geq j\in G_i} \lambda^*(G_j \cap \{j, \dots, |\hat{N}|\}) = \sum_{i\geq j\in G_i} (w_j - w_{j'}) \\
&= w_i - w_{i'} + \sum_{i'\geq j\in G_{i'}} (w_j - w_{j'}) \quad (G_i = G_{i'}) \\
&= w_i - w_{i'} + w_{i'} - w_{i''} + \cdots - w_0 = w_i - w_0 = w_i.
\end{aligned}$$

Denote $\bar{G} \equiv \bigcup_{i\in\hat{N}} \max G_i$. The objective function value of the dual at $\boldsymbol{\lambda}^*$ is

$$\begin{aligned}
& \sum_{Z\subseteq \hat{N}} \lambda^*(Z)[\widetilde{\Phi}(\hat{N}) - \widetilde{\Phi}(\hat{N}\backslash Z)] \\
= \; & \sum_{i\in \hat{N}} \lambda^*(G_i \cap \{i, \dots, |\hat{N}|\})[\widetilde{\Phi}(\hat{N}) - \widetilde{\Phi}(\hat{N}\backslash (G_i \cap \{i, \dots, |\hat{N}|\}))] \\
= \; & \sum_{j\in\bar{G}} \sum_{i\in G_j} (w_i - w_{i'})[\widetilde{\Phi}(\hat{N}) - \widetilde{\Phi}(\{1, \dots, i-1\} \cup H_i)] \\
= \; & \widetilde{\Phi}(\hat{N}) \sum_{j\in\bar{G}} \sum_{i\in G_j} (w_i - w_{i'}) - \sum_{j\in\bar{G}} \sum_{i\in G_j} w_i \widetilde{\Phi}(\{1, \dots, i-1\} \cup H_i) + \sum_{j\in\bar{G}} \sum_{i\in G_j} w_{i'} \widetilde{\Phi}(\{1, \dots, i-1\} \cup H_i) \\
= \; & \widetilde{\Phi}(\hat{N}) \sum_{i\in\bar{G}} w_i - \sum_{i\in\bar{G}} w_i \widetilde{\Phi}(\{1, \dots, i-1\} \cup H_i) - \sum_{j\in\bar{G}} \sum_{i\in G_j\backslash\bar{G}} w_i \widetilde{\Phi}(\{1, \dots, i-1\} \cup H_i) \\
+ \; & \sum_{j\in\bar{G}} \sum_{i\in G_j} w_{i'} \widetilde{\Phi}(\{1, \dots, i'\} \cup H_{i'}) \quad (\{i'+1, \dots, i-1\} \subseteq H_i = H_{i'})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in\bar{G}} w_i[\widetilde{\Phi}(\{1,\ldots,i\}\cup H_i) - \widetilde{\Phi}(\{1,\ldots,i-1\}\cup H_i)] \quad (\{1,\ldots,i\}\cup H_i = \hat{N} \text{ for all } i\in\bar{G}) \\
&+ \sum_{j\in\bar{G}}\sum_{i\in G_j\setminus\bar{G}} w_i[\widetilde{\Phi}(\{1,\ldots,i\}\cup H_i) - \widetilde{\Phi}(\{1,\ldots,i-1\}\cup H_i)] \\
&= \sum_{j\in\bar{G}}\sum_{i\in G_j} w_i[\widetilde{\Phi}(\{1,\ldots,i\}\cup H_i) - \widetilde{\Phi}(\{1,\ldots,i-1\}\cup H_i)] \\
&= \sum_{i\in\hat{N}}[\widetilde{u}_i(\{1,\ldots,i\}\cup H_i) - \widetilde{u}_i(\{1,\ldots,i-1\}\cup H_i)] = \sum_{i\in\hat{N}}\hat{p}_i^*, \quad \text{(by B1 and (18) respectively)}
\end{aligned}
$$

which is equal to the objective function value of the primal at $\hat{\mathbf{p}}^*$.

Uniqueness Denote $\widetilde{p}_i \equiv \frac{\hat{p}_i}{w_i}$ and re-express (17) as

$$
\max_{\mathbf{p}'\in\mathbb{R}^{|\hat{N}|}} \sum_{i\in\hat{N}} w_i\widetilde{p}_i \quad \text{s.t.} \quad \sum_{i\in Z}\widetilde{p}_i \le \widetilde{\Phi}(\hat{N}) - \widetilde{\Phi}(\hat{N}\setminus Z) \quad \text{for all } Z\subseteq\hat{N}. \tag{19}
$$

For $\widetilde{\mathbf{p}}^* \equiv \left(\frac{\hat{p}_i^*}{w_i}\right)_{i\in\hat{N}}$ (and thus $\hat{\mathbf{p}}^*$) to be the unique optimal solution, the necessary and sufficient condition provided by Mangasarian (1979, Theorem 1) is that $\widetilde{\mathbf{p}}^*$ remains an optimal solution for all linear programs obtained from (19) by an arbitrary but sufficiently small perturbation of the (cost) vector $(w_i)_{i\in\hat{N}}$. This condition is satisfied if $w_1 < \cdots < w_{|\hat{N}|}$ because any sufficiently small perturbation does not alter the ranking of w_i and, therefore, $\widetilde{\mathbf{p}}^*$ remains optimal.

Proof of Corollary 1 It is a direct application of Theorem 1.

Proof of Theorem 2 It remains to prove the "only if" part by contrapositive. Consider a two-agent game with $X_1 = X_2 = \{0,1\}$ ($o_i = 0$), $u_1(\mathbf{x}) = u_2(\mathbf{x}) = 1$ if $\mathbf{x} = (1,1)$ and $u_1(\mathbf{x}) = u_2(\mathbf{x}) = 0$ otherwise, and $U(\mathbf{x}, p_1(x_1) + p_2(x_2)) = V(\mathbf{x}) + p_1(x_1) + p_2(x_2)$ where $V(1,0) = V(0,1) = 1$ and $V(0,0) = V(1,1) = 0$. Clearly, u_1 and u_2 satisfy C1 (which implies A2 and A3); they also satisfy A1 with $w_1 = w_2 = 1$ and $\Phi = u_1 = u_2$. For notational convenience, denote $p_1 \equiv p_1(1)$ and $p_2 \equiv p_2(1)$.

There are exactly three types of contracts (p_1, p_2) that lead to multiple equilibria in stage 2: (i) $p_i \le 0$ and $p_j = 1$ ($i, j = 1, 2$; $i \ne j$), (ii) $p_i \ge 1$ and $p_j = 0$, and (iii)

$(p_1, p_2) \in [0,1] \times [0,1]$. For the first type, there is a continuum of mixed-strategy equilibria in which $x_i = 1$ with probability 1 and $x_j = 1$ with any probability. Similarly, for the second type, there is a continuum of mixed-strategy equilibria in which $x_i = 0$ with probability 1 and $x_j = 1$ with any probability. For the third type, there are three equilibria: (i) $\mathbf{x} = (1,1)$, (ii) $\mathbf{x} = (0,0)$, and (iii) the mixed-strategy equilibrium in which $x_i = 1$ with probability p_j.

For the first (second) type, all equilibria have the same potential of $-p_i$ (0). Recall from p. 11 that the principal is allowed to select among potential maximizers under potential maximization. Therefore, she can always select the equilibrium that gives her the highest payoff for both types.[22] For the third type, we can easily show that the principal's expected payoffs in those three equilibria are $p_1 + p_2$, 0, and $p_1 + p_2$ respectively, and the potential maximizer is $\mathbf{x} = (1,1)$ if $p_1 + p_2 \leq 1$ and $\mathbf{x} = (0,0)$ if $p_1 + p_2 \geq 1$. Observe that potential maximization selects the equilibrium that gives the principal the highest payoff if and only if $p_1 + p_2 \leq 1$.

If the underlying equilibrium selection criterion is not more pessimistic than potential maximization, there exists a non-empty subset of the third type of contracts $\Gamma \subseteq \{(p_1, p_2) \in (0,1] \times (0,1] | p_1 + p_2 > 1\}$ in which either $\mathbf{x} = (1,1)$ or the mixed-strategy equilibrium is selected; both equilibria give the principal the same expected payoff of $p_1 + p_2$, which is strictly greater than 1. If instead the principal offers the $\mathbf{w}$-DC contracts, the optimal action profiles are $(1,1)$, $(1,0)$, and $(0,1)$ by Corollary 2; in either case her payoff is only 1.

Proof of Corollary 2 It follows from the discussion in the text.

Proof of Corollary 3 It follows from the discussion in the text.

[22] For the first type, all equilibria give her the same expected payoff of $1 + p_i$. For the second type, the pure-strategy equilibrium in which $x_i = 0$ and $x_j = 1$ gives her the highest payoff of 1.

Proof of Lemma 3 First, note that

$$
\begin{aligned}
&\frac{1}{2}\sum_j \sum_{k\in E_j} \theta_j\theta_k x_j x_k = \frac{1}{2}\sum_j\left(\theta_j x_j \sum_{k\in E_j}\theta_k x_k\right) = \frac{1}{2}\left[\theta_i x_i \sum_{k\in E_i}\theta_k x_k + \sum_{j\neq i}\left(\theta_j x_j \sum_{k\in E_j}\theta_k x_k\right)\right] \\
= &\ \frac{1}{2}\left[\theta_i x_i \sum_{k\in E_i}\theta_k x_k + \sum_{j\neq i}\left(\theta_j x_j \theta_i x_i \cdot 1_{i\in E_j} + \theta_j x_j \sum_{k\in E_j\setminus\{i\}}\theta_k x_k\right)\right] \\
= &\ \frac{1}{2}\left[\theta_i x_i \sum_{k\in E_i}\theta_k x_k + \theta_i x_i \sum_{j\in E_i}\theta_j x_j + \sum_{j\neq i}\left(\theta_j x_j \sum_{k\in E_j\setminus\{i\}}\theta_k x_k\right)\right] \quad (i\in E_j \text{ iff } j\in E_i) \\
= &\ \theta_i x_i \sum_{j\in E_i}\theta_j x_j + \frac{1}{2}\sum_{j\neq i}\sum_{k\in E_j\setminus\{i\}}\theta_j\theta_k x_j x_k.
\end{aligned}
$$

For all $i \in N$ and $\mathbf{x} \in X$,

$$
\begin{aligned}
&w_i\Phi(\mathbf{x}) = \frac{v_i}{\theta_i}\left(\sum_j \frac{\theta_j b_j(x_j)}{v_j} + \frac{1}{2}\sum_j\sum_{k\in E_j}\theta_j\theta_k x_j x_k\right) \quad (\text{by } (12)) \\
= &\ b_i(x_i) + \frac{v_i}{\theta_i}\sum_{j\neq i}\frac{\theta_j b_j(x_j)}{v_j} + v_i x_i \sum_{j\in E_i}\theta_j x_j + \frac{v_i}{2\theta_i}\sum_{j\neq i}\sum_{k\in E_j\setminus\{i\}}\theta_j\theta_k x_j x_k \\
= &\ u_i(\mathbf{x}) - \xi_i(\mathbf{x}_{-i}). \quad \left(\text{by } (3) \text{ and } \xi_i(\mathbf{x}_{-i}) \equiv -\frac{v_i}{\theta_i}\sum_{j\neq i}\frac{\theta_j b_j(x_j)}{v_j} - \frac{v_i}{2\theta_i}\sum_{j\neq i}\sum_{k\in E_j\setminus\{i\}}\theta_j\theta_k x_j x_k\right)
\end{aligned}
$$

Proof of Corollary 4 It is a direct application of Theorem 2, Corollary 1, and Lemma 3.

Proof of Lemma 4 For all $i \in N$ and $\mathbf{x} \in X$,

$$
\begin{aligned}
w_i\Phi(\mathbf{x}) &= v_i\left(\sum_j \frac{b_j(x_j)}{v_j} + g(\mathbf{x})\right) \quad (\text{by } (15)) \\
&= b_i(x_i) + v_i\sum_{j\neq i}\frac{b_j(x_j)}{v_j} + v_i g(\mathbf{x}) \\
&= u_i(\mathbf{x}) - \xi_i(\mathbf{x}_{-i}). \quad \left(\text{by } (14) \text{ and } \xi_i(\mathbf{x}_{-i}) \equiv -v_i\sum_{j\neq i}\frac{b_j(x_j)}{v_j}\right)
\end{aligned}
$$

Proof of Corollary 5 It is a direct application of Theorem 2 and Lemma 4.

Proof of Lemma 5 For all $i \in N$ and $\mathbf{x} \in X$,

$$
\begin{aligned}
w_i\Phi(\mathbf{x}) &= b_i(x_i) + \frac{1}{2}\delta v_i\theta_i x_i^2 + w_i\sum_{j\neq i}\frac{b_j(x_j)+\frac{1}{2}\delta v_j\theta_j x_j^2}{w_j} + \frac{w_i}{2}(\sum_j \theta_j x_j)^2 \quad \text{(by (16))} \\
&= b_i(x_i) + \frac{1}{2}\delta v_i\theta_i x_i^2 + \frac{\delta v_i}{2\theta_i}(\theta_i x_i + \sum_{j\neq i}\theta_j x_j)^2 + \frac{(1-\delta)v_i}{2}(\sum_j \theta_j x_j)^2 + w_i\sum_{j\neq i}\frac{b_j(x_j)+\frac{1}{2}\delta v_j\theta_j x_j^2}{w_j} \\
&= b_i(x_i) + \frac{1}{2}\delta v_i\theta_i x_i^2 + \frac{1}{2}\delta v_i\theta_i x_i^2 + \delta v_i x_i\sum_{j\neq i}\theta_j x_j + \frac{\delta v_i}{2\theta_i}(\sum_{j\neq i}\theta_j x_j)^2 \\
&+ \frac{(1-\delta)v_i}{2}(\sum_j \theta_j x_j)^2 + w_i\sum_{j\neq i}\frac{b_j(x_j)+\frac{1}{2}\delta v_j\theta_j x_j^2}{w_j} \\
&= u_i(\mathbf{x}) - \xi_i'(\mathbf{x}_{-i}). \quad (\xi_i'(\mathbf{x}_{-i}) \equiv (1-\delta)\xi_i(\mathbf{x}_{-i}) - \frac{\delta v_i}{2\theta_i}(\sum_{j\neq i}\theta_j x_j)^2 - w_i\sum_{j\neq i}\frac{b_j(x_j)+\frac{1}{2}\delta v_j\theta_j x_j^2}{w_j})
\end{aligned}
$$

Proof of Corollary 6 It is a direct application of Theorem 2, Corollary 1, and Lemma 5.

B Generic Uniqueness of the Weighted Potential Maximizer

Suppose a game $\mathcal{G} \equiv (N, X, \mathbf{u})$ is a weighted potential game. Given a potential function Φ (together with a weight vector $\mathbf{w} \in \mathbb{R}^N_{++}$) of $\mathcal{G}$, it is clear that the maximizer of Φ is generically unique. However, it is not clear whether another potential function Φ' (together with another weight vector $\mathbf{w}' \in \mathbb{R}^N_{++}$) of $\mathcal{G}$ has the same maximizer(s). Therefore, the exact statement to prove is as follows. To the best of my knowledge, this paper is the first to give a direct proof of this statement.

Lemma 6 *The set of potential maximizers of a weighted potential game is independent of the choice of the potential function.*

Proof. Suppose $(\mathbf{w}, \Phi)$ and $(\mathbf{w}', \Phi')$ are two different choices of "weight-potential" pairs. By the definition of weighted potential games (A1), for all $i \in N$, $x_i, x_i' \in X_i$, and $\mathbf{x}_{-i} \in X_{-i}$,

$$
u_i(x_i, \mathbf{x}_{-i}) - u_i(x_i', \mathbf{x}_{-i}) = w_i[\Phi(x_i, \mathbf{x}_{-i}) - \Phi(x_i', \mathbf{x}_{-i})] = w_i'[\Phi'(x_i, \mathbf{x}_{-i}) - \Phi'(x_i', \mathbf{x}_{-i})]. \quad (20)
$$

Denote $\widetilde{w}_i \equiv \frac{w_i'}{w_i}$. Clearly, $\widetilde{w}_i > 0$ for all $i \in N$. Without loss of generality, assume $\widetilde{w}_1 \leq \cdots \leq \widetilde{w}_N$. It remains to show that $\mathbf{x}^* \in \arg\max_{\mathbf{x}\in X} \Phi'(\mathbf{x})$ implies $\mathbf{x}^* \in \arg\max_{\mathbf{x}\in X} \Phi(\mathbf{x})$.

To see this, for all $\mathbf{x} \in X$,

$$
\begin{aligned}
& \Phi(\mathbf{x}^*) - \Phi(\mathbf{x}) \\
= \;& \sum_{i=1}^{N} [\Phi(x_1^*, \ldots, x_i^*, x_{i+1}, \ldots, x_N) - \Phi(x_1^*, \ldots, x_{i-1}^*, x_i, \ldots, x_N)] \\
= \;& \sum_{i=1}^{N} \widetilde{w}_i [\Phi'(x_1^*, \ldots, x_i^*, x_{i+1}, \ldots, x_N) - \Phi'(x_1^*, \ldots, x_{i-1}^*, x_i, \ldots, x_N)] \quad (\text{by } (20)) \\
\geq \;& \sum_{i=1}^{N-1} \widetilde{w}_i [\Phi'(x_1^*, \ldots, x_i^*, x_{i+1}, \ldots, x_N) - \Phi'(x_1^*, \ldots, x_{i-1}^*, x_i, \ldots, x_N)] \quad (\mathbf{x}^* \in \arg\max_{\mathbf{x} \in X} \Phi'(\mathbf{x})) \\
+ \;& \widetilde{w}_{N-1} [\Phi'(\mathbf{x}^*) - \Phi'(x_1^*, \ldots, x_{N-1}^*, x_N)] \\
= \;& \sum_{i=1}^{N-2} \widetilde{w}_i [\Phi'(x_1^*, \ldots, x_i^*, x_{i+1}, \ldots, x_N) - \Phi'(x_1^*, \ldots, x_{i-1}^*, x_i, \ldots, x_N)] \\
+ \;& \widetilde{w}_{N-1} [\Phi'(\mathbf{x}^*) - \Phi'(x_1^*, \ldots, x_{N-2}^*, x_{N-1}, x_N)] \\
\geq \;& \sum_{i=1}^{N-2} \widetilde{w}_i [\Phi'(x_1^*, \ldots, x_i^*, x_{i+1}, \ldots, x_N) - \Phi'(x_1^*, \ldots, x_{i-1}^*, x_i, \ldots, x_N)] \quad (\mathbf{x}^* \in \arg\max_{\mathbf{x} \in X} \Phi'(\mathbf{x})) \\
+ \;& \widetilde{w}_{N-2} [\Phi'(\mathbf{x}^*) - \Phi'(x_1^*, \ldots, x_{N-2}^*, x_{N-1}, x_N)] \\
\geq \;& \cdots \geq \widetilde{w}_1 [\Phi'(\mathbf{x}^*) - \Phi'(\mathbf{x})] \geq 0. \quad (\mathbf{x}^* \in \arg\max_{\mathbf{x} \in X} \Phi'(\mathbf{x})) \;\blacksquare
\end{aligned}
$$