Job Matching with Subsidy and Taxation

Fuhito Kojima [a], Ning Sun [b], and Ning Neil Yu [b]

[a] *Stanford University, Department of Economics, United States*
[b] *Nanjing Audit University, Institute for Social and Economic Research, China*

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Abstract

Governments often provide employers with financial incentives which depend on the sets of people they hire. This paper studies such fiscal policies (subsidy and taxation) in a Kelso-Crawford job matching framework, and characterizes which transfer functions preserve the substitutes condition (for all revenue functions that satisfies the condition), a condition crucial for guaranteeing the existence and other regularities of competitive equilibria. We find that a transfer function preserves the substitute condition if and only if it can be written as a sum of an additively separable transfer function and a cardinally concave transfer function. Then we characterize transfer functions that preserve the substitutes condition for revenue functions which are, respectively, group separable, group concave, cardinally concave, etc. The vectorial substitutes condition, a generalization of the substitutes condition, is also studied.

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* *E-mail addresses:* fkojima@stanford.edu (F. Kojima), sunning@nau.edu.cn (N. Sun), ningyu@stanford.edu (N. N. Yu).

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1 Introduction

Hiring entities often face various types of external financial incentives originating from government policies. The United States, through its immigration policy, essentially imposes a tax on each foreign worker hired by a company (Facchini, Mayda, and Mishra, 2011, Section 2); its states may provide firm-specific incentives to boost local employment (Slattery and Zidar, 2020, Page 98). China, through its Social Welfare Enterprise program, subsidizes a company if its workforce is more than 35% disabled (Chandra and Wong, 2016, Chapter 1). India's labor regulation imposes higher standards on an industrial establishment if it employs more than a certain number of workers; this amounts to taxation conditional on the total number of workers (Adhvaryu, Chari, and Sharma, 2012, Page 726). Globally, labor market interventions during the Covid-19 pandemic often embrace the fiscal approach.[1]

To study fiscal policy interventions in labor markets, we adopt the classical job matching model of Kelso and Crawford (1982), and investigate which policies interfere with the existence of competitive equilibrium and thus market stability.[2] The question bears theoretical significance: the existence of competitive equilibria is one central theme of economic theory (Walras, 1874; Arrow and Debreu, 1954; Kelso and Crawford, 1982; Gul and Stacchetti, 1999). It may have practical implications too: in different contexts, Roth (1986, 1991, 2018) observes that markets that are inconsistent with stability often fail.

It is well established that the (gross) *substitutes condition*, a condition on an employer's demand correspondence for workers, is sufficient and necessary (in a sense of maximal domain)[3] for the existence of competitive equilibria (Gul and Stacchetti, 1999, 2000; Milgrom, 2000; Hatfield et al., 2013; Yang, 2017). It is the requirement that, roughly speaking, a set of demanded workers should still be demanded after a rise of the salaries of other workers; and has a rich intellectual history (Arrow, Block, and Hurwicz, 1959; Kelso and Crawford, 1982; Milgrom, 2017). To study the existence of competitive equilibria under fiscal policy interventions, we can thus

[1] See the summary from https://www.oecd.org/coronavirus/en/.

[2] In their job matching model, the set of competitive equilibrium allocations equals the set of core allocations.

[3] A competitive equilibrium may or may not exist if one employer's demand correspondence violates the substitutes condition; but it is easy to construct others' demand correspondences so that none exists. Large markets may alleviate the concerns about equilibrium nonexistence in other settings (Kojima, Pathak, and Roth, 2013; Ashlagi, Braverman, and Hassidim, 2014; Azevedo and Hatfield, 2018; Che, Kim, and Kojima, 2019), but we are unaware of such results in the classical job matching setting of Kelso and Crawford (1982).

focus on the question of which policies interfere with the substitutes condition.

For this purpose, we model one hospital making employment decisions over a finite set of doctors. A *revenue function* is real-valued and defined on all possible sets of doctors, and specifies how much revenue the hospital generates if it employs each set; similarly, a *transfer function* specifies how much subsidy the hospital receives from the government (or how much taxation it pays if the value is negative). The profit of the hospital is the revenue plus the transfer minus total salaries paid to doctors it employs, and the demand correspondence is derived through profit maximization. We can accordingly say that a revenue function (or the sum of a revenue function and a transfer function) satisfies the substitutes condition.

This paper first provides a characterization of transfer functions which *preserve the substitutes condition* (for all revenue functions that satisfies the condition). In other words, we identify exactly which transfer functions can be added to any revenue function that satisfies the substitutes condition to produce demand correspondences that still satisfy the condition. When the government is uninformed of the details of the revenue function, a transfer policy which does not preserve the substitute condition could lead to the failure of the substitutes condition and thus market instability.

Our first main result states: a transfer function preserves the substitutes condition if and only if it is the sum of an *additively separable* transfer function and a *cardinally concave* transfer function. Roughly speaking, additive separability means that each doctor is assigned a real number and the total transfer is the sum of the assigned numbers of hired doctors; cardinal concavity means that the transfer is a concave-extensible function of the number of hired doctors. The characterized class of transfer functions is consistent with certain affirmative action policies, e.g., promising a fixed amount of subsidy for hiring a particular doctor. However, this class of transfer functions is still restrictive as, for instance, it rules out subsidizing the hospital for hiring at least a certain number of minority doctors.

There may be cases in which the government is better informed of the revenue function, especially regarding some distinct structures such as the aforementioned additive separability and cardinal concavity, as well as *group separability* and *group concavity*. As we define them, group separability roughly states that the revenue function is the sum of revenue functions (all of which satisfy the substitutes condition) of the hospital's subsidiaries/departments which hire from disjoint groups of doctors; group concavity says that the revenue function treats doctors within each group as homogeneous and satisfies the substitutes condition.

We analyze which transfer functions preserve the substitute condition for these subclasses of revenue functions.[4] These additional results help us shed light on how the policymaker's knowledge about revenue functions (such as group separability and group concavity) leads to a wider class of permitted transfer functions, which in turn translates into a larger class of policies that do not jeopardize market stability.

We also characterize which *vectorial functions* preserve the *vectorial substitutes condition*, a classical condition of substitutability when workers/goods are homogeneous within groups (Milgrom and Strulovici, 2009). The condition is also equivalent to M$^\natural$-concavity, the central concept of discrete convex analysis (see Murota, 2003, 2016, and our Appendix A). It will become clear that our results, especially the system of characterizations, contribute substantially to the mathematics of discrete convex analysis.

We further study an environment where the revenue function can be written as the sum of a group separable revenue function and a group concave one. We show that profit maximization in this case is consistent with the hospital delegating specific hiring decisions to the department level while only deciding the "headcount" for each department.

The paper unfolds as follows. Section 2 introduces our model. Section 3 characterizes which transfer functions preserve the substitutes condition. Section 4 characterizes which transfer functions preserve the substitutes condition for all group separable and group concave revenue functions, respectively. Section 5 focuses on the vectorial substitutes condition. Lastly, in Section 6, we explain how to derive new characterizations from existing ones, discuss future research, and conclude.

Related Literature

Governments are omnipresent in marketplaces, for the good or the bad (Roth, 2018). Although we are unaware of any market design research that uses our approach to study fiscal policy interventions in the form of subsidy and taxation, there is a large related literature on compulsory policy interventions.

In the job matching framework, Kojima, Sun, and Yu (2020*b*, henceforth, KSY) study constraints on the sets of workers an employer is allowed to hire.

[4] This is similar to how Kojima, Sun, and Yu (2020*b*) study compulsory policy interventions or constraints in labor markets; that paper covers group separability, but not group concavity. Kojima, Sun, and Yu (2020*c*) characterize which constraints preserve the substitutes condition for all group concave revenue functions; the proof strategies and conclusions are distinctively different.

KSY identifies exactly which constraints preserve the substitutes condition for two classes of revenue functions corresponding to the substitutes condition and group separability. The main connection between KSY and this paper is that both of them characterize classes of policy interventions that preserve the substitutes condition.[5] The main difference is in the kinds of interventions considered: KSY considers compulsory constraints, while this paper considers pecuniary transfer policies. Due to the modeling difference, the results of our paper do not follow from KSY.

KSY is most related to the current paper. But in transferable utility settings, it is one among many which study various types of constraints: e.g., Bing, Lehmann, and Milgrom (2004), Milgrom (2009), Biró et al. (2010), Abizada (2016), Hatfield, Plott, and Tanaka (2016), Echenique, Miralles, and Zhang (2019), and Gul, Pesendorfer, and Zhang (2019). It is a surprise for us that fiscal policy interventions have been largely overlooked by this literature.

In non-transferable utility settings, fiscal policies are naturally absent. An incomplete list of compulsory policy research includes studies of floor constraints (Biró et al., 2010; Huang, 2010), ceiling constraints (Abdulkadiroğlu and Sönmez, 2003; Fragiadakis and Troyan, 2017; Kamada and Kojima, 2018), type-specific constraints (Hafalir, Yenmez, and Yildirim, 2013; Ehlers et al., 2014; Ellison and Pathak, 2016; Kominers and Sönmez, 2016; Goto et al., 2017; Dur et al., 2018; Aygün and Turhan, 2020), proportionality constraints (Nguyen and Vohra, 2019), multidimensional resource constraints (Delacrétaz, Kominers, and Teytelboym, 2016; Noda, 2018), and joint constraints imposed on multiple entities (Kamada and Kojima, 2015, 2017, 2018). Although these studies and this paper share high-level interest in policy interventions in matching markets, there is no meaningful overlap given differences of the models (e.g., regarding utility transferability) as well as differences in the types of policy interventions (i.e., constraints versus transfers).

Methodologically, our paper is related to a large literature that examines various aspects of the substitutes condition (Kelso and Crawford, 1982; Gul and Stacchetti, 1999; Fujishige and Yang, 2003; Murota, 2003; Hatfield and Milgrom, 2005; Ostrovsky and Paes Leme, 2015; Shioura and Tamura, 2015; Murota, 2016, 2019, 2020, among many others). Some of our results are linked to early studies in a surprising way. For example, the characterized class of transfer functions which *preserve* the substitutes condition has long been recognized to *satisfy* the substitutes condition by Beviá, Quinzii, and Silva (1999), and often introduced as a tractable

[5] Kojima, Sun, and Yu (2020*a*) offer the full job matching model and welfare analysis.

subclass (Gul and Stacchetti, 1999). We review related results from discrete convex analysis (Murota, 2016) in Appendix A; it will become apparent that they are indispensable for our proofs.

2 The Model

There is a finite set of **doctors** D with cardinality $|D| = M$. We study the demand of a single **hospital** for doctors: investigating one hospital is sufficient for all the analysis of this paper while simplifying notation.[6] A **salary schedule** is a real-valued function $\mathbf{s} : D \to \mathbb{R}$ that specifies a salary for each doctor; for each $d \in D$, s_d is short for $\mathbf{s}(d)$. A **revenue function** $R : 2^D \to \mathbb{R}$ maps each subset of doctors to the revenue of the hospital if it hires them.

An entity called government regulates the labor market using fiscal incentives. A **transfer function** $T : 2^D \to \mathbb{R}$ maps each subset of doctors to the amount of government transfer a hospital receives if it hires them. Naturally, for $A \subset D$, the two cases of $T(A) > 0$ and $T(A) < 0$ correspond to government subsidy and taxation, respectively. Note that there is no mathematical distinction between a transfer function and a revenue function, so we can use the same taxonomy for both of them. For example, we say that a revenue or transfer function $S : 2^D \to \mathbb{R}$ is **additively separable** if for each $A \subset D$, $S(A) = \sum_{d \in A} S(\{d\})$.

If the hospital hires $A \subset D$ while facing a salary schedule $\mathbf{s}$, a revenue function R, and a transfer function T, its **profit** is $V(A; \mathbf{s}, R+T) = R(A) + T(A) - \sum_{d \in A} s_d$, that is, its revenue plus the government transfer minus the sum of salaries paid to the doctors. We define the **maximal profit function** $\Pi(\,\cdot\,; R+T) : \mathbb{R}^D \to \mathbb{R}$ and the **demand correspondence** $X(\,\cdot\,; R+T) : \mathbb{R}^D \to \{\mathcal{A} : \mathcal{A} \subset 2^D \text{ and } \mathcal{A} \neq \emptyset\}$ so that for each salary schedule $\mathbf{s}$,

$$\begin{aligned} \Pi(\mathbf{s}; R+T) \;&=\; \max\{V(A; \mathbf{s}, R+T) : A \subset D\}; \\ X(\mathbf{s}; R+T) \;&=\; \{A \subset D : V(A; \mathbf{s}, R+T) = \Pi(\mathbf{s}; R+T)\}. \end{aligned}$$

Each element of $X(\mathbf{s}; R+T)$ is referred to as a **demand set**.

Note that we intentionally leave out the possibility of expanding the range of a revenue or transfer function to $\mathbb{R} \cup \{-\infty\}$ (see Hatfield et al., 2013; Fleiner et al., 2019; Hatfield et al., 2019, among others). In other words, all sets of doctors are assumed to be feasible. For results of the more general case in which some sets may

[6]For a model of multiple hospitals competing for doctors, see Kojima, Sun, and Yu (2020*a*) which builds upon the results of this paper.

be designated infeasible by the hospital itself or the government, it is straightforward to combine findings in KSY with those in the current paper. We work with the less general case for expositional simplicity.

The **substitutes condition** is the requirement that whenever a set of doctors is demanded (included in a demand set) given a price schedule, then, after a rise in others' salaries, this set must still be demanded.

Definition 1. A demand correspondence $X(\,\cdot\,; R)$ satisfies **the substitutes condition** if for any two salary schedules $\mathbf{s}$ and $\mathbf{s}'$ with $\mathbf{s}' \geq \mathbf{s}$, and any $A \in X(\mathbf{s}; R)$, there exists $A' \in X(\mathbf{s}'; R)$ such that $\{d \in A : s_d = s'_d\} \subset A'$. A revenue function R satisfies the substitutes condition if the demand correspondence $X(\cdot\,; R)$ satisfies it.

The substitutes condition is a natural assumption in modeling many real-life economic environments, where complements may be ruled out. One example is a **unit-demand** revenue function R, where for each $A \subset D$, $R(A) = \max_{d\in A} R(\{d\})$; this is natural when only one doctor/unit is needed. The substitutes condition is more general and is satisfied by other applications as well.

The substitutes condition is commonly assumed also because of its critical role in obtaining the existence and other regularity properties of competitive equilibria; in a maximal domain sense, it is both sufficient and necessary (see Gul and Stacchetti, 1999, 2000; Milgrom, 2000; Hatfield et al., 2013; Yang, 2017, among others). As Kelso and Crawford (1982) point out, nonexistence of equilibria is equivalent to the nonexistence of stable outcomes (the core is empty), which is linked to undesirable real-life consequences (see Roth, 1984, 1991, 2018, for example).

Given an environment in which all revenue functions satisfy the substitute condition, a policymaker may worry that some fiscal policy interventions in the form of transfer functions can cause the condition to fail – a case where R satisfies it but $R + T$ does not. To address such a concern, we investigate which transfer functions preserve the substitutes condition.

Definition 2. A transfer function T **preserves the substitutes condition** if for each revenue function R that satisfies the condition, $R+T$ satisfies it. Moreover, T **preserves the substitutes condition for a class of revenue functions** if for each function R in the class, $R+T$ satisfies the substitute condition.

It is well known that given two revenue functions that satisfy the substitutes condition, their sum may not. We are unaware of earlier attempts at characterizing the class of functions that, when added to any function which satisfies the substitutes condition, preserve the condition.[7]

Our aim is to study fiscal policy interventions, especially how they interfere with market stability. Given that the substitutes condition may be viewed as a necessary and sufficient condition for market equilibrium existence, it is important to understand exactly which transfer functions preserve the substitutes condition for different classes of revenue functions (which correspond to scenarios in which the government has different partial knowledge about hospital revenue functions). The ensuing sections address this question and provide characterizations of transfer functions that preserve the substitute condition for various classes of revenue functions.

3 Preserving the Substitutes Condition

This section provides a characterization of all transfer functions that preserve the substitutes condition. First, it is easy to see from the definition of the substitutes condition that it is invariant to the addition of an additively separable transfer function, so an additively separable transfer function preserves the substitutes condition.

Proposition 1. *Given an additively separable transfer function T, a revenue function R satisfies the substitutes condition if and only if $R+T$ satisfies it.*

Accordingly, the government can subsidize or tax the hospital for hiring individual doctors without causing the failure of the substitutes condition or market stability.

It is worth mentioning that using an additively separable transfer function is equivalent to directly subsidizing or taxing doctors (rather than the hospital) if they work for the hospital. In real life, we can observe both ways of implementation: e.g., In India, women empowerment subsidies go through firms via programs such as subsidized training (Rotemberg, 2019, Section I);[8] in China, subsidies to rural teachers are part of their salaries (Lin and Wong, 2012, Page 38).

How about other types of fiscal incentives? For example, the government may consider subsidizing a hospital when the number or proportion of minorities meets a certain criterion. Before providing a definitive answer for all transfer functions, we

[7] Murota (2019) provides an excellent summary of operations that preserve discrete convexity, and thus the substitutes condition, known before KSY and the current paper.

[8] See https://msme.gov.in/sites/default/files/MSME_Schemes_English_0.pdf for details.

need to define some classes of transfer functions. Let us denote the set of integers between m and m', where $m, m' \in \mathbb{Z}$ and $m \le m'$, by $[m, m']_{\mathbb{Z}} := \{m'' \in \mathbb{Z} : m \le m'' \le m'\}$.

A transfer function T is **cardinal** if there exists an associated function $f : [0, M]_{\mathbb{Z}} \to \mathbb{R}$ such that $T(A) = f(|A|)$ for each $A \subset D$. In other words, the transfer is said to be cardinal if it depends solely on the number of doctors the hospital hires. Given f and each $m \in [1, M]_{\mathbb{Z}}$, we denote by $\alpha_m^f := f(m) - f(m-1)$ the incremental transfer from hiring a doctor in addition to the $m - 1$ doctors already hired, so $T(A) = f(0) + \sum_{m \le |A|} \alpha_m^f$ for each $A \subset D$.

A transfer function T is **cardinally concave** if it is cardinal and the associated function f is extensible to a concave function on $\mathbb{R}$. Equivalently, the requirement is that the corresponding finite sequence $(a_1^f, a_2^f, \ldots, a_M^f)$ be weakly decreasing. Note that it is possible that entries of the sequence are positive at the beginning of the sequence and negative toward the end. Such a transfer policy seems to "softly" discourage the hospital from hiring too few or too many doctors, a feature reminiscent of a hard interval constraint studied by KSY but more flexible.

Our first main result demonstrates that sums of additively separable transfer functions and cardinally concave ones form the entire class that preserve the substitutes condition.

Theorem 1. *A transfer function preserves the substitutes condition if and only if it is the sum of an additively separable transfer function and a cardinally concave transfer function.*

All nontrivial proofs of our results are relegated to the Appendix.

In addition to additively separable transfer functions, Theorem 1 further supports the idea of basing fiscal policy interventions on cardinally concave ones. Such flexibility may be useful in applications. For instance, it seems well-suited for tackling the rural hospital problem, the problem of rural hospitals having persistent difficulties in recruitment (Roth, 1986). Subsidizing a rural hospital according to how many doctors it hires, regardless of whom, can be more reasonable than mandating a minimum number (a policy suggested by KSY) in many real-life environments. Subsidizing hiring is also a common way of boosting employment in a society. Theorem 1 guarantees that these policies will not destroy substitutability, which in turn guarantees the existence of stable outcomes.

A simple corollary of Theorem 1 is that transfer functions that satisfy the substitutes condition is exactly the class that preserve the substitutes condition for all cardinally concave revenue functions.

Corollary 1. *A transfer function preserves the substitutes condition for all cardinally concave revenue functions if and only if it satisfies the substitutes condition.*

A reader might wonder whether the conclusion of Theorem 1, especially the necessity of the characterized class of transfer functions, depends on the strong requirement of preserving the substitutes condition *for all revenue functions that satisfy the substitutes condition.* In the rest of this section, we show that this conclusion can in fact be obtained even for a substantially smaller class of revenue functions, strengthening the necessity part of the theorem.

A unit-demand revenue function R is **binary unit-demand** if there exist $d, d' \in D$ and $\alpha > 0$ such that $R(A) = \alpha \min\{1, |A \cap \{d, d'\}|\}$ for each $A \subset D$.[9] In other words, the revenue is $\alpha > 0$ if one of the two doctors d and d' is hired, and 0 otherwise. The following proposition shows that preserving the substitutes condition for all binary unit-demand revenue functions implies that a transfer function can be decomposed into an additively separable part and a cardinally concave part.

Proposition 2. *If a transfer function preserves the substitutes condition for all binary unit-demand revenue functions, then it is the sum of an additively separable transfer function and a cardinally concave transfer function.*

As cardinally concave transfer functions are nondiscriminatory toward doctors, Proposition 2 highlights the difficulty of designing affirmative action policies beyond additively separable transfer functions. The class of binary unit-demand revenue functions is small, and preserving the substitutes condition for them rules out all transfer functions outside the characterized class. In particular, some of the policies we mentioned before – such as subsidizing the hospital only when it hires at least a certain number of minority doctors – are ruled out.

It must be emphasized that the same conclusion as Proposition 2 can also be established for other small classes of revenue functions as well. A trivial example is a class obtained from adding an additively separable revenue function to all binary unit-demand revenue functions; a simple application of Proposition 1 proves the result. Such flexibility suggests that it is not easy to overcome the necessity part of Theorem 1 by excluding a subclass of revenue functions from consideration.

[9] A binary disjunctive revenue function defined in KSY is binary unit-demand with $\alpha = 1$.

This section assumes that the government has no knowledge about revenue functions beyond substitutability. In some situations, the government may possess further knowledge. The next section analyzes several such settings and characterizes transfer functions that preserve the substitute condition for different classes of revenue functions.

4 Subclasses of Revenue Functions

Fix a **partition** of D, which we denote by $\mathcal{P} \subset 2^D \setminus \{\emptyset\}$, so that $\cup\mathcal{P} := \cup_{P\in\mathcal{P}} P = D$, and for any $P, P' \in \mathcal{P}$ with $P \neq P'$, $P \cap P' = \emptyset$. Each member of $\mathcal{P}$ is called a **group**. In practice, a group may form based on a gender, an age group, an ethnic group, a specialty, a qualification, a location, an educational background, or a combination of several individual characteristics.

Given $\mathcal{P}$, we investigate which transfer functions preserve the substitutes condition for all "group separable" revenue functions in the first subsection, and for all "group concave" revenue functions in the second subsection.

4.1 Group Separability

A revenue function R is **group separable** if there is a family of functions $\{R_P\}_{P\in\mathcal{P}}$ such that each $R_P : 2^P \to \mathbb{R}$ satisfies the substitutes condition, and for each $A \subset D$, $R(A) = \sum_{P\in\mathcal{P}} R_P(A \cap P)$.[10]

A revenue function may be group separable, for example, because the hospital owns several departments that hire from disjoint pools of doctors corresponding to different specialties (so each group consists of all doctors of one specialty), or because it owns several branches that hire from disjoint pools of doctors corresponding to different geographical locations. It is plausible that the government may learn these facts but not further details of the revenue function.

Define a **vectorization transformation** $\tau : 2^D \to \mathbb{Z}^{\mathcal{P}}$ so that for each $A \subset D$, $\tau(A)$ is an integer-valued function on $\mathcal{P}$ with $\tau(A)(P) = |A \cap P|$ for each $P \in \mathcal{P}$. For each set of doctors A, $\tau(A)$ represents the number of doctors in A who belong to different groups. We call any $W : \tau(2^D) \to \mathbb{R}$ a **vectorial function**. A transfer function T is **group cardinal** if there exists an associated vectorial function $W : \tau(2^D) \to \mathbb{R}$ such that for each $A \subset D$, $T(A) = W(\tau(A))$. In this case, the transfer depends only on the number of doctors hired from each group, but not on

[10] KSY formally defines group separable revenue functions, and characterizes which constraints preserve the substitutes condition for that class of revenue functions.

within-group identities of the doctors. A group cardinal transfer function is **group concave** if it also satisfies the substitutes condition. A cardinally concave transfer function is obviously group concave.

The characterization theorem for preserving the substitutes condition for all group separable revenue functions is as follows.

Theorem 2. *A transfer function preserves the substitutes condition for all group separable revenue functions if and only if it is the sum of an additively separable transfer function and a group concave transfer function.*

This result suggests that, due to the knowledge of group separability of the revenue function, the government can utilize a broader class of transfer functions (including group concave ones) without causing the failure of the substitutes condition or market stability. Policy interventions based on group concave transfer functions can treat doctors in different groups differently while treating those in the same group equally (note that this is in a sharp contrast to cardinally concave transfer functions in Theorem 1). For example, the government can design the transfer function as a concave-extensible function of the number of doctors hired from one group or the union of several groups. If this group or union consists entirely of minorities (or majorities), the policy can be used to achieve diversity or balance of the workforce.

As in the last section, we end with a proposition which is stronger than the necessity part of the main theorem. To rule out triviality, we assume that there exists a group with at least two doctors. We say that a binary unit-demand revenue function is **within-group** if its two associated doctors are within the same group. A within-group binary unit-demand revenue function is group separable.

Proposition 3. *If a transfer function preserves the substitutes condition for all within-group binary unit-demand revenue functions, then it is the sum of an additively separable transfer function and a group concave transfer function.*

4.2 Group Concavity

Group concavity may be a reasonable simplifying assumption for the revenue function when doctors within each group are, with regard to generating revenues, approximately homogeneous or indistinguishable from each other before being hired.[11] Similar to group separability, group concavity of the revenue function may

[11] This class of revenue functions is absent from KSY.

be easy to observe for the government, while further knowledge may be difficult to come by.

The main theorem for group concavity is the following.

Theorem 3. *Suppose there are at least three groups. A transfer function preserves the substitutes condition for all group concave revenue functions if and only if it is the sum of a group separable transfer function and a cardinally concave transfer function.*

Group separability of a transfer function T is consistent with policy interventions that discriminate within groups. For instance, given a group $P^* \in \mathcal{P}$ and a set of minority doctors $A \subset D$ (which may be unrelated to $\mathcal{P}$), we can consider a concave-extensible function $f : [0, |P^* \cap A|]_{\mathbb{Z}} \to \mathbb{R}$, and a transfer function T satisfying $T(B) := f(|P^* \cap A \cap B|)$ for each $B \subset D$. Note that T is degenerately group separable. Hence, in the settings of Theorem 3, such within-group affirmative action, which is not allowed in the setting of Theorem 1, preserves the substitutes condition and thus does not threaten market stability.

There is also a strengthening of the necessity part of Theorem 3. A revenue function R is **spline concave** in $A \subset D$ if there exists a function $f : [0, |A|]_{\mathbb{Z}} \to \mathbb{R}$, $m^* \in [1, |A|-1]_{\mathbb{Z}}$, and $\alpha > 0$ such that $f(m) = \alpha \min\{m, m^*\}$ for each $m \in [0, |A|]_{\mathbb{Z}}$, and $R(B) = f(|A \cap B|)$ for each $B \subset D$.[12] In other words, if the revenue function is spline concave in a set, every additional doctor in the set generates the same positive revenue for the hospital until a quota is met. We say that a spline concave revenue function in some $P \in \mathcal{P}$ is **uni-group**; when it is in $D \setminus P$, the complement of some $P \in \mathcal{P}$, we say it is $\overline{\textbf{uni}}$**-group**.

Revenue functions that are uni- or $\overline{\text{uni}}$-group spline concave form a small subclass of group concave revenue functions. In general, preserving the substitutes condition for them dictates a transfer function to be the sum of a group separable transfer function and a cardinally concave transfer function.

Proposition 4. *Suppose there are at least three groups. If a transfer function preserves the substitutes condition for all uni- and $\overline{\textit{uni}}$-group spline concave revenue functions, then it is the sum of a group separable transfer function and a cardinally concave transfer function.*

For the case of $|\mathcal{P}| = 2$, the class of permitted transfer functions is larger: a

[12] In the definition, we rule out $m^* = 0$ and $m^* = |A|$: they correspond to degenerate cases of additive separability. Also, a spline concave revenue function in A is cardinal if and only if $A = D$.

group concave transfer function is allowed instead of merely a cardinally concave transfer function.

Proposition 5. *Suppose there are two groups. The sum of a group separable transfer function and a group concave transfer function preserves the substitutes condition for all group concave revenue functions.*[13]

5 The Vectorial Substitutes Condition

We say that a vectorial function $U : \tau(2^D) \to \mathbb{R}$ satisfies the **vectorial substitutes condition** if it is associated with a group concave revenue function R. The condition is commonly used to model substitutability given homogeneous goods within groups: it is called the "strong-substitutes valuation" by Milgrom and Strulovici (2009). Working with a standard object assignment model, they list many applications such as the allocation of airport landing slots, and show that the condition is crucial for guaranteeing that competitive equilibria exist, that the core contains the Vickrey outcomes, and that the "law of aggregate demand" holds. Analogous results are true for a job matching setting (Kojima, Sun, and Yu, 2020*a*).

Mathematically, the vectorial substitutes condition is a more general concept than the substitutes condition. To see this, we only need to adopt the finest partition $\mathcal{P} = \{\{d\} : d \in D\}$ and represent any $A \subset D$ by $\tau(A)$ (note that τ is a bijection in this case). Accordingly, a revenue function that satisfies the substitutes condition can be viewed as a vectorial function that satisfies the vectorial substitutes condition.

5.1 Preserving the Vectorial Substitutes Condition

We study which vectorial functions preserve the vectorial substitutes condition. In essence, answering this question tells us which group cardinal transfer functions preserve the substitutes condition for all group concave revenue functions.[14]

Definition 3. A vectorial function W **preserves the vectorial substitutes condition** if for each vectorial function U that satisfies it, $U + W$ satisfies it. Moreover, W **preserves the vectorial substitutes condition for a class of vectorial functions** if for each function U in the class, $U + W$ satisfies the vectorial substitute condition.

[13] We conjecture that this is a characterization result too.

[14] This subsection may be viewed as redundant given Subsection 4.2, but the prominence of the vectorial substitutes condition makes the exercise worthwhile. Another justification is that the analysis here is needed for the proof of Theorem 3.

A vectorial function W is **additively separable**[15] if there exists a family of functions $\{f_P\}_{P\in\mathcal{P}}$ such that $f_P : [0,|P|]_{\mathbb{Z}} \to \mathbb{R}$ is concave-extensible for each $P \in \mathcal{P}$, and $W(\mathbf{z}) = \sum_{P\in\mathcal{P}} f_P(\mathbf{z}(P))$ for each $\mathbf{z} \in \tau(2^D)$. When the partition is the finest, this concept is equivalent to the additive separability of transfer functions.

Abusing notation, for any $\mathbf{z} \in \mathbb{Z}^{\mathcal{P}}$ and any collection of groups $\mathcal{Q} \subset \mathcal{P}$, we denote the number of doctors in $\cup\mathcal{Q}$ by $|\mathbf{z}(\mathcal{Q})| := \sum_{P\in\mathcal{Q}} \mathbf{z}(P)$, and the total number by $|\mathbf{z}| := |\mathbf{z}(\mathcal{P})|$. A vectorial function $W : \tau(2^D) \to \mathbb{R}$ is **cardinally concave** if there exists a concave-extensible function $f : [0,M]_{\mathbb{Z}} \to \mathbb{R}$ such that $W(\mathbf{z}) = f(|\mathbf{z}|)$ for each $\mathbf{z} \in \tau(2^D)$.[16]

The main theorem tells us that the class identified above contains all vectorial functions that preserve the vectorial substitutes condition.

Theorem 4. *Suppose there are at least three groups. A vectorial function preserves the vectorial substitutes condition if and only if it is the sum of an additively separable vectorial function and a cardinally concave vectorial function.*

Analogous to Propositions 2-4, Proposition 6 highlights the restrictiveness of the requirement of preserving the vectorial substitutes condition by focusing on a small class of vectorial functions.[17] A vectorial function U is **$\overline{\text{uni}}$-group spline concave** if it is associated with a $\overline{\text{uni}}$-group spline concave revenue function. It is straightforward to show that additively separable, cardinally concave, and $\overline{\text{uni}}$-group spline concave vectorial functions all satisfy "M$^\natural$-concavity," which is equivalent to the vectorial substitutes condition (see Appendix A and Murota (2003)).

Proposition 6. *Suppose there are at least three groups. If a vectorial function preserves the vectorial substitutes condition for all $\overline{\text{uni}}$-group spline concave vectorial functions, then it is the sum of an additively separable vectorial function and a cardinally concave vectorial function.*

When $|\mathcal{P}| = 2$, the characterization for preserving the vectorial substitutes condition is simple.

[15] Additively separable vectorial functions should not be confused with additively separable revenue/transfer functions. We intentionally adopt the same wording to highlight the analogy. The same logic is behind our naming of cardinally concave vectorial functions.

[16] It is easy to see that a group concave transfer function is cardinally concave if and only if its associated vectorial function is cardinally concave.

[17] An interesting fact we note here is that the class of vectorial functions identified by Proposition 6 is closed under addition, similar to the class of transfer functions identified by Proposition 2.

Proposition 7. *Suppose there are two groups. A vectorial function preserves the vectorial substitutes condition if and only if it satisfies the vectorial substitutes condition.*

It has long been known that the class of all functions that satisfy the vectorial substitutes condition is not closed under addition, so the fact that it is for the case of $|\mathcal{P}| = 2$ may be surprising.

When the partition is the finest, i.e., every doctor forms a group, Theorem 4 (as well as Proposition 7) degenerates into Theorem 1, so it is mathematically more general.

5.2 Delegation of Hiring Decisions

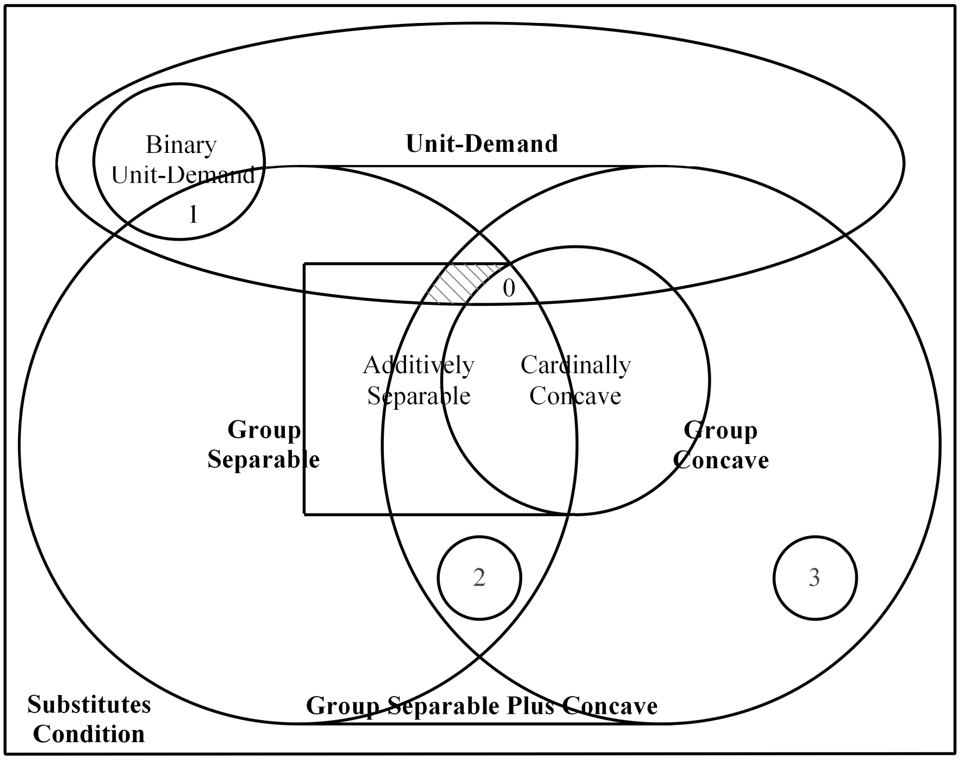
Large organizations often delegate hiring decisions to their departments, possibly because headquarters often lack the expertise and/or information to measure revenue impacts of particular candidates. Is there a scenario for the delegation to be compatible with profit maximization? We will present a tractable class of revenue functions for which the answer is positive.

We say that a revenue function $S : 2^D \to \mathbb{R}$ is **group separable plus concave** if $S = S^1 + S^2$, where S^1 is group separable and S^2 is group concave. Group separable plus concave revenue functions form a large subclass of revenue functions that satisfy the substitutes condition, encompassing most subclasses mentioned in this paper as shown in Figure 1.[18] Such S may originate from adding a group concave transfer function to a group separable revenue function, or adding a group separable transfer function to a group concave revenue function. As seen in Theorems 2 and 3, those transfer policies preserve the substitutes condition in their respective settings.

Let the hospital maximize profit $V(\,\cdot\,; \mathbf{s}, S)$, where $\mathbf{s}$ is a salary schedule and S is group separable plus concave as defined above. There is a sense in which the hospital can delegate most hiring decisions to the department level: each department hires doctors from a particular group, while it only decides how many doctors each department should hire, based on an optimization problem in the domain of $\tau(2^D)$.

Let $\{S^1_P\}_{P\in\mathcal{P}}$ be associated with S^1 and W be the vectorial function associated with S^2. Given $\mathbf{s}$, imagine a department that hires doctors from $P \in \mathcal{P}$ solving the

[18]Not every revenue function that satisfies the substitutes condition is a group separable plus concave revenue function. An ingenious example due to Ostrovsky and Paes Leme (2015) is not.



Notes: It is assumed that $|\mathcal{P}| > 2$ and for every $P \in \mathcal{P}$, $|P| > 2$. Each class occupies the smallest convex area that covers its name. For example, group separable plus concave revenue functions occupy the convex hull of the circle for group separable revenue functions and the circle for group concave revenue functions. Areas 0, 1, 2, and 3 represent "Zero" (the zero revenue/transfer function), "Within-Group Binary Unit-Demand," "Uni-Group Spline Concave," and "$\overline{\text{Uni}}$-Group Spline Concave," respectively. Every area corresponds to a nonempty class, except for the shaded area besides Area 0.

Figure 1: Venn Diagram for Representative Classes of Revenue/Transfer Functions

following constrained optimization problem for each $m \in [0, |P|]_{\mathbb{Z}}$:

$$\Pi_P(m) \coloneqq \max\Big\{S_P^1(A) - \sum_{d\in A} s_d : A \subset P \text{ and } |A| = m\Big\};$$

$$X_P(m) \coloneqq \Big\{A \subset P : S_P^1(A) - \sum_{d\in A} s_d = \Pi_P(m) \text{ and } |A| = m\Big\}.$$

In words, each department figures out, for every possible exact quota imposed on them, its profit and demand sets.

The optimization problem at the hospital level can now be written:

$$\max_{\mathbf{z}\in\tau(2^D)} W(\mathbf{z}) + \sum_{P\in\mathcal{P}} \Pi_P(\mathbf{z}(P)).$$

The maximizer $\mathbf{z}^*$ can then be used to dictate the exact quota for each group P as $\mathbf{z}^*(P)$, while the department associated with P employs any set in $X_P(\mathbf{z}^*(P))$ accordingly. The union of such sets across $\mathcal{P}$ is an optimal solution to the hospital's profit maximization problem.

It is quite appealing that the hospital can achieve optimality without knowing $\{S^1_P\}_{P\in\mathcal{P}}$ or $\mathbf{s}$. Through delegation of specific hiring decisions to the department level, it only needs to optimize based on W and $\{\Pi_P\}_{P\in\mathcal{P}}$, and each Π_P can be reported by the associated department. This process may provide a suitable model of hiring decisions of a large organization, where the top-level management determines "headcounts" for its units but not whom to fill the positions with.

6 Discussion and Conclusion

In a doctor-hospital job matching setting, this paper studies fiscal policy interventions in the form of transfer functions, each of which specifies amounts of subsidy or taxation for all possible sets of doctors to be hired by a hospital. The current paper investigates which transfer functions preserve the substitutes condition in several practical settings, and thus preserve the existence and other regularities of competitive equilibria. Our results are summarized in Table 1.

It is easy to derive some new results using our analysis. First, by Proposition 1, translating a class of revenue functions by additively separable revenue functions (potentially applying different translations to different revenue functions) never changes the set of transfer functions that preserve the substitutes condition for the class. Second, if one class of revenue functions is the subclass of another, then preserving the substitutes condition for the former is less restrictive than for the latter. For instance, the class of unit-demand revenue functions is between the class of binary unit-demand revenue functions and the class of revenue functions that preserve the substitutes condition (Figure 1), so, by Theorem 1 and Proposition 2, every transfer function that preserves the substitutes condition for all unit-demand revenue functions is the sum of an additively separable transfer function and a cardinally concave one.

Table 1: Preserving the Substitutes Condition for Different Classes of Revenue Functions

Revenue Functions	Transfer Functions	Reference
Substitutes Condition	Additively Separable + Cardinally Concave	Theorem 1
Binary Unit-Demand	Additively Separable + Cardinally Concave	Theorem 1 Proposition 2
Group Separable	Additively Separable + Group Concave	Theorem 2
Within-Group Binary Unit-Demand	Additively Separable + Group Concave	Theorem 2 Proposition 3
Group Concave	Group Separable + Cardinally Concave	Theorem 3
Uni- & $\overline{\text{Uni}}$-Group Spline Concave	Group Separable + Cardinally Concave	Theorem 3 Proposition 4
Cardinally Concave	Substitutes Condition	Corollary 1
Additively Separable	Substitutes Condition	Proposition 1

Notes: In each row, the class of transfer functions in the second column is exactly the class that preserve the substitutes condition for the class of revenue functions in the first column. The "+" sign represents the fact that a function in the class can be written as the sum of a function in the class before "+" and one after. For uni- & $\overline{\text{uni}}$-group spline concave revenue functions and group concave revenue functions, the results hold when there are more than two groups. All classes of revenue/transfer functions in the table are shown in Figure 1.

We also characterize exactly which vectorial functions preserve the vectorial substitutes condition, a prominent condition for modeling substitutability when doctors/goods may be homogeneous within groups. We further present a scenario in which the hospital can delegate specific hiring decisions to the department level, but still achieve optimality.

All of our results are readily applicable to models of object assignments, or multi-item auctions, because the substitutes condition is the central assumption in those models too (see Gul and Stacchetti, 1999, 2000; Ausubel, 2006; Cramton et al., 2010; Milgrom, 2017, for example). Governments often facilitate transactions of objects: troubled assets among financial institutions (Ausubel and Cramton, 2013); radio spectrums owned by TV stations and demanded by wireless broadband services (Milgrom and Segal, 2020); etc. They may let transfers depend on the sets of objects market participants obtain (see Milgrom, 2017, Page 62, for example).

Most of our characterization results,[19] when translated into the language of discrete convex analysis, are new (Murota, 2019). Appendix A explains the connections. In essence, we systematically answer the natural mathematical question of which M$^\natural$-concave functions can be added to interesting classes of M$^\natural$-concave functions and preserve M$^\natural$-concavity. The question happens to bear economic policy relevance, analogous to how KSY's study of constraints is both mathematically and economically relevant.

There are still unanswered questions. Despite our best effort, other interesting classes of revenue functions may still be found and considered. Further, it is an open question whether generalizing the substitutes condition (Sun and Yang, 2006; Shioura and Yang, 2015; Baldwin and Klemperer, 2019) alters our conclusions. Finally, the class of all possible fiscal policy interventions is strictly larger than those captured by transfer functions (as we define them). For instance, subsidy or taxation may depend on the salaries of doctors or the states of other hospitals. We leave these possibilities for future research.

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[19] For example, the only known result in Table 1 is the trivial one in the last row.

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APPENDIX

A The Substitutes Condition and M$^\natural$-Concavity

This section briefly reviews the connection between the substitutes condition (which is central to economic theory) and M$^\natural$-concavity (which is the mainstay of discrete convex analysis), and relevant results used in this paper (see Murota, 1996; Murota and Shioura, 1999; Fujishige and Yang, 2003; Murota, 2003; Milgrom and Strulovici, 2009; Shioura and Tamura, 2015; Murota, 2016, 2019, for more details).

Denote the **indicator function for** $P \in \mathcal{P}$ by $\mathbf{i}^P \in \mathbb{R}^{\mathcal{P}}$, that is, $\mathbf{i}^P(P) = 1$ while $\mathbf{i}^P(P') = 0$ for each $P' \neq P$. A vectorial function U is **M$^\natural$-concave** if for any $\mathbf{z}, \mathbf{z}' \in \tau(2^D)$ and $P \in \mathcal{P}$ such that $\mathbf{z}(P) > \mathbf{z}'(P)$, either $U(\mathbf{z}) + U(\mathbf{z}') \leq U(\mathbf{z} - \mathbf{i}^P) + U(\mathbf{z}' + \mathbf{i}^P)$, or there exists $P' \in \mathcal{P}$ such that $\mathbf{z}'(P') > \mathbf{z}(P')$ and $U(\mathbf{z}) + U(\mathbf{z}') \leq U(\mathbf{z} - \mathbf{i}^P + \mathbf{i}^{P'}) + U(\mathbf{z}' + \mathbf{i}^P - \mathbf{i}^{P'})$, or both.

There is an analogous concept for revenue functions. Given any set of doctors $A \subset D$ and $d \in D$, we write $A + d := A \cup \{d\}$ and $A - d := A \setminus \{d\}$, and follow this convention for other sets too. A revenue function R is **discrete concave** if for any $A, B \subset D$ and $d \in A \setminus B$, either $R(A) + R(B) \leq R(A - d) + R(B + d)$, or there exists $d' \in B \setminus A$ such that $R(A) + R(B) \leq R(A - d + d') + R(B + d - d')$, or both.[20]

The following characterization of the substitutes condition by Fujishige and Yang (2003) brings together the economics literature on the substitutes condition and discrete convex analysis.

Lemma 1 (Fujishige and Yang (2003))**.** *A revenue function R satisfies the substitutes condition if and only if it is discrete concave.*

Given Lemma 1, it is straightforward to prove the following known equivalence between the vectorial substitutes condition and M$^\natural$-concavity (Shioura and Tamura, 2015, Theorem 4.1).

Lemma 2. *A vectorial function satisfies the vectorial substitutes condition if and only if it is M$^\natural$-concave.*

Due to this lemma, we use the concept of M$^\natural$-concavity interchangeably with the vectorial substitutes condition, and are allowed to utilize powerful results in discrete convex analysis in our proofs. For example, the combination of the following two conditions characterize M$^\natural$-concavity and thus the vectorial substitutes condition.[21]

Theorem 5 (Murota and Shioura (2018))**.** *A vectorial function U is M$^\natural$-concave if and only the following two conditions are both satisfied:*

- *for any $\mathbf{z}, \mathbf{z}' \in \tau(2^D)$ such that $|\mathbf{z}| > |\mathbf{z}'|$, there exists $P \in \mathcal{P}$ such that $\mathbf{z}(P) > \mathbf{z}'(P)$ and $U(\mathbf{z}) + U(\mathbf{z}') \leq U(\mathbf{z} - \mathbf{i}^P) + U(\mathbf{z}' + \mathbf{i}^P)$;*
- *for any $\mathbf{z}, \mathbf{z}' \in \tau(2^D)$ and $P \in \mathcal{P}$ such that $|\mathbf{z}| \leq |\mathbf{z}'|$ and $\mathbf{z}(P) > \mathbf{z}'(P)$, there exists $P' \in \mathcal{P}$ such that $\mathbf{z}'(P') > \mathbf{z}(P')$ and $U(\mathbf{z}) + U(\mathbf{z}') \leq U(\mathbf{z} - \mathbf{i}^P + \mathbf{i}^{P'}) + U(\mathbf{z}' + \mathbf{i}^P - \mathbf{i}^{P'})$.*

[20] A discrete concave revenue function is called an M$^\natural$-concave set function by Murota (2016). We rename it to avoid ambiguity.

[21] The exact statement of the theorem is absent from Murota and Shioura (2018); it combines their Theorems 1.1 and 2.1.

There is obviously a revenue function version of Theorem 5 (again, consider the finest partition); we refer to it still by Theorem 5.

The combination of the following three local conditions characterize discrete concavity and thus the substitutes condition. The following statement is Theorem 3.2 of Murota (2016).[22]

Theorem 6. *A revenue function R is discrete concave if and only if for any $A \subset D$, the following three conditions are all satisfied:*

- *for any distinct $d, d' \in D \setminus A$, $R(A + d + d') + R(A) \leq R(A + d) + R(A + d')$;*
- *for any distinct $d, d', d'' \in D \setminus A$,*

$$R(A + d + d') + R(A + d'') \leq \max\{R(A + d + d'') + R(A + d'), \\ R(A + d) + R(A + d' + d'')\};$$

- *for any distinct $d, d', d'', d''' \in D \setminus A$,*

$$R(A + d + d') + R(A + d'' + d''') \leq \max\{R(A + d + d'') + R(A + d' + d'''), \\ R(A + d + d''') + R(A + d' + d'')\}.$$

B Main Proofs

B.1 Proofs of Propositions 2 and 3

In preparation for the main proofs, we establish an equation that a transfer function must satisfy if it preserves the substitutes condition for all spline concave revenue functions in a set of doctors (defined in Section 4.2).[23]

Lemma 3. *If a transfer function T preserves the substitutes condition for all spline concave revenue functions in $A \subset D$, then for any $B \subset D$ not containing $\overline{d}, \underline{d} \in A$, and any $d^* \in D \setminus (A \cup B)$, we have $T(B+\overline{d}+d^*)-T(B+\overline{d}) = T(B+\underline{d}+d^*)-T(B+\underline{d})$.*

Proof. Suppose, for contradiction, that

$$T(B + \overline{d} + d^*) - T(B + \overline{d}) > T(B + \underline{d} + d^*) - T(B + \underline{d}). \tag{1}$$

(The "<" case is symmetric to this ">" case.) In what follows, we will construct a spline concave revenue function R in A such that $R + T$ does not satisfy the substitutes condition.

[22] We are unable to track down its first appearance.

[23] To keep the appendix within reasonable length, we have to constantly invoke mathematical objects defined in later sections to prove results in earlier sections.

Let $\alpha \coloneqq \max\{T(C) : C \subset D\} - \min\{T(C) : C \subset D\} + 1 > 0$. We define f so that $f(m) = 3\alpha \min\{m, |A \cap B| + 1\}$ for each $m \in [0, |A|]_{\mathbb{Z}}$, and, accordingly, R so that $R(C) = f(|C \cap A|)$ for each $C \subset D$. Note that because $B \subset D$ does not contain $\overline{d}, \underline{d} \in A$, we can infer that $1 \leq |A \cap B| + 1 \leq |A| - 1$. So R is spline concave in A.

Let $\Delta \coloneqq \frac{1}{2}[(T(B + \overline{d} + d^*) - T(B + \overline{d})) - (T(B + \underline{d} + d^*) - T(B + \underline{d}))] > 0$. Consider a salary schedule $\mathbf{s}$, where $s_{\overline{d}} = \alpha + T(B + \overline{d})$, $s_{\underline{d}} = \alpha + T(B + \underline{d})$, $s_{d^*} = (T(B + \overline{d} + d^*) - T(B + \overline{d})) - \Delta$, $s_d = -4\alpha$ for each $d \in B$, and $s_d = 4\alpha$ for each $d \notin B \cup \{\overline{d}, \underline{d}, d^*\}$.

When the hospital faces $\mathbf{s}$, R, and T, we first note that α is so large that all doctors in B are demanded, that no doctor outside $B \cup \{\overline{d}, \underline{d}, d^*\}$ is demanded, and that either $\overline{d}$ or $\underline{d}$ is demanded (but not both). This leaves four possibilities for demand sets: $B + \overline{d}$, $B + \underline{d}$, $B + \overline{d} + d^*$, and $B + \underline{d} + d^*$. But

$$\begin{aligned} V(B + \overline{d}; \mathbf{s}, R + T) &= 3\alpha|B \cap A| + 4\alpha|B| + 2\alpha; \\ V(B + \underline{d}; \mathbf{s}, R + T) &= 3\alpha|B \cap A| + 4\alpha|B| + 2\alpha; \\ V(B + \overline{d} + d^*; \mathbf{s}, R + T) &= 3\alpha|B \cap A| + 4\alpha|B| + 2\alpha + \Delta; \\ V(B + \underline{d} + d^*; \mathbf{s}, R + T) &= 3\alpha|B \cap A| + 4\alpha|B| + 2\alpha - \Delta. \end{aligned}$$

There is a unique demand set $B + \overline{d} + d^*$.

Now, raise the salary of doctor $\overline{d}$ to 4α to obtain a new salary schedule $\mathbf{s}'$. Under $\mathbf{s}'$, the cost of $\overline{d}$ is so large that $B + \overline{d}$ and $B + \overline{d} + d^*$ can never be optimal. We can compare the profits of $B + \underline{d}$ and $B + \underline{d} + d^*$ (which remain the same as before), and conclude that there is a unique demand set $B + \underline{d}$. Therefore, raising the salary of doctor $\overline{d}$ excludes doctor d^* from the demand set, a violation of the substitutes condition. This contradicts our assumption. □

Under the assumption of Lemma 3, for any $B \subset D$ such that $\overline{d}, \underline{d} \in A \setminus B$, adding an arbitrary $d^* \in D \setminus (A \cup B)$ to $B + \overline{d}$ brings about the same change in the value of the transfer function T as adding d^* to $B + \underline{d}$. Under the same assumption, we now show that for any $C \subset D$ with $\overline{d} \in A \cap C$ and $\underline{d} \in A \setminus C$, swapping $\overline{d}$ in C for $\underline{d}$ brings about the same change in the value of the transfer function T as swapping $\overline{d}$ in $A \cap C$ for $\underline{d}$.

Lemma 4. *If a transfer function T preserves the substitutes condition for all spline concave revenue functions in $A \subset D$, then for any $C \subset D$, $\overline{d} \in A \cap C$, and $\underline{d} \in A \setminus C$,*

$$T(C - \overline{d} + \underline{d}) - T(C) = T((A \cap C) - \overline{d} + \underline{d}) - T(A \cap C). \tag{2}$$

Proof. We carry out mathematical induction on $|C \setminus A|$. First, note that the statement is trivially true for any C with $|C \setminus A| = 0$; i.e., when $C = A \cap C$. Assume that the statement is true for all C such that $|C \setminus A| \leq m$ with m being a nonnegative integer less than $|D \setminus A|$. Consider $C \subset D$ with $|C \setminus A| = m + 1$. There exists $d^* \in C \setminus A$. By the induction hypothesis,

$$T(C - d^* - \overline{d} + \underline{d}) - T(C - d^*) = T((A \cap C) - \overline{d} + \underline{d}) - T(A \cap C).$$

By Lemma 3, we have $T(C - \overline{d} + \underline{d}) - T(C - d^* - \overline{d} + \underline{d}) = T(C) - T(C - d^*)$ by setting $B = C - d^* - \overline{d}$, or, equivalently,

$$T(C - \overline{d} + \underline{d}) - T(C) = T(C - d^* - \overline{d} + \underline{d}) - T(C - d^*).$$

We can connect the two equalities above to obtain Equation (2). $\square$

Proof of Proposition 3. Take a transfer function T that preserves the substitutes condition for all within-group binary unit-demand revenue functions. Define an additively separable transfer function T^1 such that $T^1(A) = \sum_{d \in A} T(\{d\})$ for each $A \subset D$. Let $T^2 := T - T^1$. We only need to show that T^2 is group concave.

For any $P \in \mathcal{P}$, $d, d' \in P$, and $A \subset D$ such that $d \in A$ and $d' \notin A$, we have

$$\begin{aligned} T^2(A - d + d') - T^2(A) &= (T(A - d + d') - T^1(A - d + d')) - (T(A) - T^1(A)) \\ &= (T(A - d + d') - T(A)) - (T^1(A - d + d') - T^1(A)) \\ &= (T(\{d'\}) - T(\{d\})) - (T(\{d'\}) - T(\{d\})) \\ &= 0, \end{aligned}$$

where the third equality follows from Lemma 4 and the definition of T^1. To see this, we only need to recognize that the class of all binary unit-demand revenue functions associated with $\overline{d}, \underline{d} \in D$ is the same as the class of all spline concave revenue functions in $\{\overline{d}, \underline{d}\}$, so we can apply Lemma 4 by setting $A = \{\overline{d}, \underline{d}\}$.

As a result, for any $A, B \subset D$ with $\tau(A) = \tau(B)$, we can turn A into B by swapping within-group elements one at a time, without changing the transfer specified by T^2, so $T^2(A) = T^2(B)$. In other words, T^2 is group cardinal; i.e., there exists a function $W : \tau(2^D) \to \mathbb{R}$ such that $T^2(A) = W(\tau(A))$ for each $A \subset D$.

Suppose T^2 is not group concave. Then, W is not M$^\natural$-concave by Lemma 2. There exist $\overline{\mathbf{z}}, \underline{\mathbf{z}} \in \tau(2^D)$ and $P^* \in \mathcal{P}$ such that $\overline{\mathbf{z}}(P^*) > \underline{\mathbf{z}}(P^*)$, $W(\overline{\mathbf{z}}) + W(\underline{\mathbf{z}}) > W(\overline{\mathbf{z}} - \mathbf{i}^{P^*}) + W(\underline{\mathbf{z}} + \mathbf{i}^{P^*})$, and $W(\overline{\mathbf{z}}) + W(\underline{\mathbf{z}}) > W(\mathbf{z} - \mathbf{i}^{P^*} + \mathbf{i}^{P'^*}) + W(\mathbf{z}' + \mathbf{i}^{P^*} - \mathbf{i}^{P'^*})$

for each $P'^* \in \mathcal{P}$ with $\mathbf{z}'(P'^*) > \mathbf{z}(P'^*)$. Define

$$\alpha := \frac{1}{4}\min\Big\{(W(\overline{\mathbf{z}}) + W(\underline{\mathbf{z}}) - W(\overline{\mathbf{z}} - \mathbf{i}^{P^*}) - W(\mathbf{z}' + \mathbf{i}^{P^*})),$$
$$\min_{P'^* \in \mathcal{P} \text{ with } \underline{\mathbf{z}}(P'^*) > \overline{\mathbf{z}}(P'^*)} (W(\overline{\mathbf{z}}) + W(\mathbf{z}') - W(\mathbf{z} - \mathbf{i}^{P^*} + \mathbf{i}^{P'^*}) - W(\mathbf{z}' + \mathbf{i}^{P^*} - \mathbf{i}^{P'^*}))\Big\},$$

and consider, for two same-group doctors $\overline{d}$ and $\underline{d}$, a binary unit-demand revenue function R with $R(C) = \alpha \min\{1, |C \cap \{\overline{d}, \underline{d}\}|\}$ for each $C \subset D$. We now show that $R + T$ fails the substitutes condition.

Let us consider $A, B \subset D$ such that $\tau(A) = \overline{\mathbf{z}}$, $\tau(B) = \underline{\mathbf{z}}$, and $|A \cap B|$ is the largest possible among those pairs of sets satisfying the last two conditions. Since $\overline{\mathbf{z}}(P^*) > \underline{\mathbf{z}}(P^*)$, there exists $d^* \in P^*$ such that $d^* \in A \setminus B$. We have

$$\begin{aligned}
&(R+T)(A) + (R+T)(B) - (R+T)(A - d^*) - (R+T)(B + d^*) \\
=\,&(R(A) + R(B) - R(A - d^*) - R(B + d^*)) + \\
&(T^1(A) + T^1(B) - T^1(A - d^*) - T^1(B + d^*)) + \\
&(T^2(A) + T^2(B) - T^2(A - d^*) - T^2(B + d^*)),
\end{aligned}$$

where the first term of the last expression is bounded below by -2α by the construction of R, the second term is 0 by the definition of T^1, and the third term equals $(W(\overline{\mathbf{z}}) + W(\underline{\mathbf{z}}) - W(\overline{\mathbf{z}} - \mathbf{i}^{P^*}) - W(\mathbf{z}' + \mathbf{i}^{P^*}))$. So the expression is strictly positive. We can similarly prove that for any $d'^* \in B \setminus A$,

$$(R+T)(A) + (R+T)(B) > (R+T)(A - d^* + d'^*) + (R+T)(B + d^* - d'^*).$$

In conclusion, $R + T$ cannot be discrete concave. By Lemma 1, $R + T$ fails the substitutes condition, a contradiction. □

Proposition 2 is a corollary of Proposition 3.

Proof of Proposition 2. Take a transfer function T that preserves the substitutes condition for all binary unit-demand revenue functions. Let the partition be the coarsest: $\mathcal{P} = \{D\}$. Then the set of all within-group binary unit-demand revenue functions is the set of all binary unit-demand revenue functions. Applying Proposition 3 tells us that $T = T^1 + T^2$, where T^1 is additively separable, and T^2 is group concave. But given $\mathcal{P} = \{D\}$, group concavity is the same as cardinal concavity. □

B.2 Proof of Theorem 1

We first make a simple observation that the sum of two transfer functions that preserve the substitutes condition still preserves the substitutes condition.[24]

Lemma 5. *The class of transfer functions that preserve the substitutes condition is closed under addition.*

Proof. Consider any two transfer functions T^1 and T^2 that preserve the substitutes condition. We need to show that $T^1 + T^2$ preserves the substitutes condition. But given any revenue function R that satisfies the substitutes condition, by the definition of preserving the substitutes condition, $R+T^1$ satisfies the condition, and then $R + (T^1 + T^2) = (R + T^1) + T^2$ also satisfies the condition. $\square$

The proof of Theorem 1 utilizes results from KSY, so we introduce some of the concepts there. The hospital may be constrained to pick a set out of a nonempty collection $\mathcal{F} \subset 2^D$, called **its feasibility collection**. For example, for integers $0 \le f \le c \le M$, the feasibility collection $\mathcal{D}_{[f,c]} := \{A \subset D : f \le |A \cap D| \le c\}$ is said to be defined by an **interval constraint**; as special cases, $\mathcal{D}_{[f,M]}$, $\mathcal{D}_{[0,c]}$, and $\mathcal{D}_{[f,f]}$ are defined by a **floor constraint**, a **ceiling constraint**, and an **exact constraint**, respectively. Abusing notation, given $\mathcal{F}$, a revenue function R, and a transfer function T, we define the maximal profit function Π and demand correspondence X so that for each salary schedule $\mathbf{s}$,

$$\begin{aligned}\Pi(\mathbf{s}; R+T, \mathcal{F}) &= \max\{V(A; \mathbf{s}, R+T) : A \in \mathcal{F}\};\\ X(\mathbf{s}; R+T, \mathcal{F}) &= \{A \in \mathcal{F} : V(A; \mathbf{s}, R+T) = \Pi(\mathbf{s}; R+T, \mathcal{F})\}.\end{aligned}$$

Since the substitutes condition is defined on demand correspondences, it is straightforward to extend Definition 1 when constraints are allowed.

A classical characterization of the substitutes condition by Gul and Stacchetti (1999) generalizes to the case with constraints. A demand correspondence $X(\cdot; R, \mathcal{F})$ satisfies the **single-improvement property** if for any salary schedule $\mathbf{s}$ and $A \in \mathcal{F}$ such that $A \notin X(\mathbf{s}; R, \mathcal{F})$, there exists $A' \in \mathcal{F}$ such that $V(A; \mathbf{s}, R) < V(A'; \mathbf{s}, R)$, $|A \setminus A'| \le 1$, and $|A' \setminus A| \le 1$.

Lemma 6 (Gul and Stacchetti (1999))**.** *A demand correspondence $X(\,\cdot\,; R, \mathcal{F})$ satisfies the substitutes condition if and only if it satisfies the single-improvement property.*

[24] Note that the class of transfer functions that preserve the substitutes condition for a subclass of revenue functions may not be closed under addition.

A feasibility collection $\mathcal{F}$ **preserves the substitutes condition** if for each revenue function R that satisfies the condition, $X(\,\cdot\,; R, \mathcal{F})$ satisfies it. Theorem 1 of KSY implies that a feasibility collection defined by an interval constraint preserves the substitutes condition. Now we proceed to prove Theorem 1.

Proof of Theorem 1. The necessity part follows from Proposition 2. For sufficiency, due to Proposition 1 and Lemma 5, we only need to show that a cardinally concave transfer function T preserves the substitutes condition, i.e., given a revenue function R that satisfies the substitutes condition, $R + T$ satisfies it too. By Lemma 6, we can instead prove the single-improvement property of $R + T$ (when the hospital is unconstrained with $\mathcal{F} = 2^D$).

Consider a salary schedule $\mathbf{s}$ and a suboptimal $A \in 2^D$ such that $A \notin X(\mathbf{s}; R + T, 2^D)$; i.e., there exists $A^* \subset A$ such that

$$R(A^*) + T(A^*) - \sum_{d \in A^*} s_d > R(A) + T(A) - \sum_{d \in A} s_d. \tag{3}$$

We need to find $A' \subset D$ such that $|A \setminus A'| \leq 1$, $|A' \setminus A| \leq 1$, and $R(A') + T(A') - \sum_{d \in A'} s_d > R(A) + T(A) - \sum_{d \in A} s_d$.

Let f be the concave-extensible function on $[0, M]_{\mathbb{Z}}$ associated with T. Define $\overline{\alpha} := f(|A|) - f(|A| - 1)$ and $\underline{\alpha} := f(|A| + 1) - f(|A|)$. Construct transfer functions $\overline{T}$ and $\underline{T}$ such that $\overline{T}(B) = f(|A|) + \overline{\alpha}(|B| - |A|)$ and $\underline{T}(B) = f(|A|) + \underline{\alpha}(|B| - |A|)$ for each $B \subset D$. By the concave extensibility of f, we have $\overline{T} \geq T$ and $\underline{T} \geq T$.

Note that $\overline{T}$ and $\underline{T}$ are cardinal, each equal to a constant plus an additively separable transfer function (which assigns the same transfer amount to each doctor). Since a constant transfer function does not change demand correspondences, it preserves the substitutes condition. Hence, by Proposition 1 and Lemma 5, $\overline{T}$ and $\underline{T}$ both preserve the substitutes condition. In particular, $R + \overline{T}$ and $R + \underline{T}$ satisfy the substitutes condition.

When $|A^*| \leq |A|$, we have

$$\begin{aligned} R(A^*) + \overline{T}(A^*) - \sum_{d \in A^*} s_d &\geq R(A^*) + T(A^*) - \sum_{d \in A^*} s_d \\ &> R(A) + T(A) - \sum_{d \in A} s_d = R(A) + \overline{T}(A) - \sum_{d \in A} s_d, \end{aligned}$$

where the first inequality follows from $\overline{T} \geq T$, and the second from Inequality (3). So A is less profitable than A^* under $\mathbf{s}$ and $R + \overline{T}$. Theorem 1 of KSY implies that, as $R + \overline{T}$ satisfies the substitutes condition, $X(\,\cdot\,; R + \overline{T}, \mathcal{D}_{[0,|A|]})$ does too. But note that A^* is feasible under the ceiling constraint of $|A|$ by assumption. Lemma 6 and the suboptimality of A imply the existence of $A' \in \mathcal{D}_{[0,|A|]}$ such that $|A' \setminus A| \le 1$, $|A \setminus A'| \le 1$, and $R(A') + \overline{T}(A') - \sum_{d\in A'} s_d > R(A) + \overline{T}(A) - \sum_{d \in A} s_d$. But then

$$\begin{aligned} R(A') + T(A') - \sum_{d\in A'} s_d &= R(A') + \overline{T}(A') - \sum_{d\in A'} s_d \\ &> R(A) + \overline{T}(A) - \sum_{d\in A} s_d = R(A) + T(A) - \sum_{d\in A} s_d, \end{aligned}$$

where the first equality follows from the fact that $|A'|$ equals $|A|$ or $|A|-1$.

The case of $|A^*| > |A|$ is analogous. Theorem 1 of KSY implies that as $R + \underline{T}$ satisfies the substitutes condition, $X(\,\cdot\,; R + \underline{T}, \mathcal{D}_{[|A|,M]})$ does too. A single-improvement opportunity A' over A under the floor constraint is a single-improvement opportunity without the constraint. We skip the details. $\square$

B.3 Proof of Proposition 6

We first introduce a simple consequence of Lemma 3.

Lemma 7. *If a vectorial function W preserves the vectorial substitutes condition for all spline concave vectorial functions in $Q = \cup\mathcal{Q}$, where $\mathcal{Q} \subsetneq \mathcal{P}$, then for any $\mathbf{z}, \mathbf{z}' \in \tau(2^D)$ and $P^* \in \mathcal{P} \setminus \mathcal{Q}$ such that $|\mathbf{z}(\mathcal{Q})| = |\mathbf{z}'(\mathcal{Q})|$, $\mathbf{z}(P) = \mathbf{z}'(P)$ for each $P \in \mathcal{P} \setminus \mathcal{Q}$, and $\mathbf{z}(P^*) < |P^*|$, we have*

$$W(\mathbf{z} + \mathbf{i}^{P^*}) - W(\mathbf{z}) = W(\mathbf{z}' + \mathbf{i}^{P^*}) - W(\mathbf{z}'). \tag{4}$$

Proof. Let W be associated with a group concave transfer function T. Then T preserves the substitutes condition for all spline concave revenue functions in Q. Because $|\mathbf{z}(\mathcal{Q})| = |\mathbf{z}'(\mathcal{Q})|$ and $\mathbf{z}(P) = \mathbf{z}'(P)$ for each $P \in \mathcal{P} \setminus \mathcal{Q}$, we can find $A, A' \subset D$ such that $\tau(A) = \mathbf{z}$, $\tau(A') = \mathbf{z}'$, $|A \cap Q| = |A' \cap Q|$, and $A \setminus Q = A' \setminus Q$. Let $d^* \in P^* \setminus A$.

Lemma 3 tells us that in the case of $|A \setminus A'| = |A' \setminus A| = 1$, $T(A + d^*) - T(A) = T(A' + d^*) - T(A')$, which implies Equation (4). In the case of $|A \setminus A'| = |A' \setminus A| > 1$, we can turn A into A' one element at a time, obtain a series of equations, and connect them to obtain $T(A + d^*) - T(A) = T(A' + d^*) - T(A')$. So Equation (4) holds. $\square$

Equation (4) helps us dissect W.

Lemma 8. *When $|\mathcal{P}| > 2$, for a vectorial function W, if Equation (4) holds for any $\mathbf{z}, \mathbf{z}' \in \tau(2^D)$ and any $P^* \in \mathcal{P}$ such that $|\mathbf{z}(\mathcal{P} - P^*)| = |\mathbf{z}'(\mathcal{P} - P^*)|$, $\mathbf{z}(P^*) = \mathbf{z}'(P^*) < |P^*|$, then there exists a family of real-valued functions on $[0, M]_{\mathbb{Z}}$, $\{f_P\}_{P\in\mathcal{P}}$, and $f :\to \mathbb{R}$ such that $W(\mathbf{z}) = \sum_{P\in\mathcal{P}} f_P(\mathbf{z}(P)) + f(|\mathbf{z}|)$ for each $\mathbf{z} \in \tau(2^D)$.*

Proof. We assign values to $\{f_P\}_{P\in\mathcal{P}}$ and f by induction. Let $f(0) = W(\tau(\emptyset))$, $f(1) = 0$, and for each $P \in \mathcal{P}$, $f_P(0) = 0$ and $f_P(1) = W(\mathbf{i}^P)$. Further, for $p \in [2, |\mathcal{P}|]_{\mathbb{Z}}$, set $f(p) = W(\mathbf{z}) - \sum_{P\in\mathcal{P}} f_P(\mathbf{z}(P))$ for any $\mathbf{z} \in \tau(2^D)$ with $|\mathbf{z}| = p$ and $\mathbf{z} \leq \sum_{P\in\mathcal{P}} \mathbf{i}^P$. We need to show that this way of assigning values to f is independent of the chosen $\mathbf{z}$. We carry out induction on $|\mathbf{z}|$.

The base cases of $|\mathbf{z}| = 1$ is trivial. Assume that for some $p \in [1, |\mathcal{P}| - 1]_{\mathbb{Z}}$, $f(p) = W(\mathbf{z}) - \sum_{P\in\mathcal{P}} f_P(\mathbf{z}(P))$ is independent of the chosen $\mathbf{z} \in \tau(2^D)$ with $|\mathbf{z}| = p$ and $\mathbf{z} \leq \sum_{P\in\mathcal{P}} \mathbf{i}^P$. Consider $\overline{\mathbf{z}}, \underline{\mathbf{z}} \in \tau(2^D)$ with $|\overline{\mathbf{z}}| = |\underline{\mathbf{z}}| = p+1$ and $\overline{\mathbf{z}}, \underline{\mathbf{z}} \leq \sum_{P\in\mathcal{P}} \mathbf{i}^P$. First, impose the additional assumption that there exists $P^* \in \mathcal{P}$ with $\overline{\mathbf{z}}(P^*) = \underline{\mathbf{z}}(P^*) = 1$. Hence, we have

$$\begin{aligned}
&\Big(W(\overline{\mathbf{z}}) - \sum_{P\in\mathcal{P}} f_P(\overline{\mathbf{z}}(P))\Big) - \Big(W(\underline{\mathbf{z}}) - \sum_{P\in\mathcal{P}} f_P(\underline{\mathbf{z}}(P))\Big)\\
=&\,(W(\overline{\mathbf{z}}) - W(\underline{\mathbf{z}})) - \sum_{P\in\mathcal{P}} (f_P(\overline{\mathbf{z}}(P)) - f_P(\underline{\mathbf{z}}(P)))\\
=&\,(W(\overline{\mathbf{z}} - \mathbf{i}^{P^*}) - W(\underline{\mathbf{z}} - \mathbf{i}^{P^*})) - \sum_{P\in\mathcal{P}} (f_P(\overline{\mathbf{z}}(P)) - f_P(\underline{\mathbf{z}}(P))) = 0,
\end{aligned}$$

where the second equality follows from Equation (4) and the third follows from the inductive hypothesis. So the assignment of $f(p+1)$ is the same based on any two $\overline{\mathbf{z}}$ and $\underline{\mathbf{z}}$ that share a positive dimension. But any $\overline{\mathbf{z}}, \underline{\mathbf{z}} \in \tau(2^D)$ with $|\overline{\mathbf{z}}| = |\underline{\mathbf{z}}| > 1$ and $\overline{\mathbf{z}}, \underline{\mathbf{z}} \leq \sum_{P\in\mathcal{P}} \mathbf{i}^P$ are connected through a series of such $\mathbf{z} \in \tau(2^D)$, where any two neighbors share a positive dimension. We can conclude that the assignment of $f(p+1)$ is the same based on any two $\overline{\mathbf{z}}$ and $\underline{\mathbf{z}}$, and finish the induction.

Given the above, we can start another induction with the base case as follows: $W(\mathbf{z}) = \sum_{P\in\mathcal{P}} f_P(\mathbf{z}(P)) + f(|\mathbf{z}|)$ for each $\mathbf{z} \in \tau(2^D)$ with $\mathbf{z} \leq \sum_{P\in\mathcal{P}} \mathbf{i}^P$. Assume that we have assigned values so that $W(\mathbf{z}) = \sum_{P\in\mathcal{P}} f_P(\mathbf{z}(P)) + f(|\mathbf{z}|)$ for each $\mathbf{z} \in \tau(2^D)$ with $\mathbf{z} \leq \mathbf{y}$, where $\mathbf{y} \in \tau(2^D)$, $\mathbf{y} \geq \sum_{P\in\mathcal{P}} \mathbf{i}^P$, and $\mathbf{y}(\overline{P}) < |\overline{P}|$ for some $\overline{P} \in \mathcal{P}$. Assign new values so that

$$\begin{aligned}
f_{\overline{P}}(\mathbf{y}(\overline{P}) + 1) &= W((\mathbf{y}(\overline{P}) + 1)\mathbf{i}^{\overline{P}}) - f(\mathbf{y}(\overline{P}) + 1);\\
f(|\mathbf{y}| + 1) &= W(\mathbf{y} + \mathbf{i}^{\overline{P}}) - \sum_{P\in\mathcal{P}} f_P((\mathbf{y} + \mathbf{i}^{\overline{P}})(P)).
\end{aligned}$$

We only need to show that $W(\mathbf{z}) = \sum_{P\in\mathcal{P}} f_P(\mathbf{z}(P)) + f(|\mathbf{z}|)$ for each $\mathbf{z} \in \tau(2^D)$ with $\tau(\emptyset) \leq \mathbf{z} \leq \mathbf{y} + \mathbf{i}^{\overline{P}}$ and $\mathbf{z}(\overline{P}) = \mathbf{y}(\overline{P}) + 1$. We do this by strong induction on $m := |\mathbf{z}(\mathcal{P} - \overline{P})|$.[25] The case of $m = 0$ follows from the assignment of $f_{\overline{P}}(\mathbf{y}(\overline{P}) + 1)$

[25]Note that this is an induction argument within an induction argument.

and $f_P(0)$ above. The case of $m = |\mathbf{y}(\mathcal{P} - \overline{P})|$ follows from the assignment of $f(|\mathbf{y}| + 1)$.

Assume that for some $m \in [0, |\mathbf{y}(\mathcal{P} - \overline{P})| - 2]_{\mathbb{Z}}$, $W(\mathbf{z}) = \sum_{P \in \mathcal{P}} f_P(\mathbf{z}(P)) + f(|\mathbf{z}|)$ for each $\mathbf{z} \in \tau(2^D)$ satisfying $\mathbf{z}(\overline{P}) = \mathbf{y}(\overline{P}) + 1$, $\mathbf{z}(P) \leq \mathbf{y}(P)$ for each $P \in \mathcal{P} - \overline{P}$, and $|\mathbf{z}(\mathcal{P} - \overline{P})| \leq m$. Consider $\mathbf{x} \in \tau(2^D)$ that satisfies $\mathbf{x}(\overline{P}) = \mathbf{y}(\overline{P}) + 1$, $\mathbf{x}(P) \leq \mathbf{y}(P)$ for each $P \in \mathcal{P} - \overline{P}$, and $|\mathbf{x}(\mathcal{P} - \overline{P})| = m + 1$. Since $m + 1 \in [1, |\mathbf{y}(\mathcal{P} - \overline{P})| - 1]_{\mathbb{Z}}$, $|\mathcal{P}| > 2$, and $\mathbf{y} \geq \sum_{P \in \mathcal{P}} \mathbf{i}^P$, there exists distinct $P^*, \underline{P} \in \mathcal{P} - \overline{P}$ such that $\mathbf{x}(P^*) \geq 1$ and $\mathbf{x}(\underline{P}) < \mathbf{y}(\underline{P})$. Hence, we have

$$\begin{aligned} W(\mathbf{x}) &= W(\mathbf{x} - \mathbf{i}^{P^*}) + W(\mathbf{x} - \mathbf{i}^{\overline{P}} + \mathbf{i}^{\underline{P}}) - W(\mathbf{x} - \mathbf{i}^{\overline{P}} - \mathbf{i}^{P^*} + \mathbf{i}^{\underline{P}}) \\ &= \sum_{P \in \mathcal{P}} f_P(\mathbf{x}(P)) + f(|\mathbf{x}|), \end{aligned}$$

where the first equality follows from Equation (4), and the second follows from the induction hypotheses, i.e., $W(\mathbf{z}) = \sum_{P \in \mathcal{P}} f_P(\mathbf{z}(P)) + f(|\mathbf{z}|)$ holds for all three terms.

In conclusion, we have $W(\mathbf{z}) = \sum_{P \in \mathcal{P}} f_P(\mathbf{z}(P)) + f(|\mathbf{z}|)$ for each $\mathbf{z} \in \tau(2^D)$ satisfying $\mathbf{z}(\overline{P}) = \mathbf{y}(\overline{P}) + 1$, $\mathbf{z}(P) \leq \mathbf{y}(P)$ for $P \in \mathcal{P} - \overline{P}$, and $|\mathbf{z}(\mathcal{P} - \overline{P})| \leq |\mathbf{y}(\mathcal{P} - \overline{P})| - 1$. Combining this with earlier results, we are done. □

Proof of Proposition 6. Take a vectorial function W on $\tau(2^D)$ that preserves the vectorial substitutes condition for all $\overline{\text{uni}}$-group spline concave vectorial functions. By Lemma 7, we know the conditions of Lemma 8 are satisfied. We only need to show that all obtained functions, $\{f_P\}_{P \in \mathcal{P}}$ and f, are concave-extensible.

Let $\alpha_P(m+1) = f_P(m+1) - f_P(m)$ for any $m \in [0, |P| - 1]_{\mathbb{Z}}$ and $P \in \mathcal{P}$. Lemma 8 says that for any distinct $P, P' \in \mathcal{P}$ and $m \in [1, |P| - 1]_{\mathbb{Z}}$,

$$W((m+1)\mathbf{i}^P) - W(m\mathbf{i}^P + \mathbf{i}^{P'}) = f_P(m+1) - f_P(m) - f_{P'}(1) = \alpha_P(m+1) - f_{P'}(1);$$
$$W(m\mathbf{i}^P) - W((m-1)\mathbf{i}^P + \mathbf{i}^{P'}) = f_P(m) - f_P(m-1) - f_{P'}(1) = \alpha_P(m) - f_{P'}(1).$$

Using a "small-α" argument analogous to the one in the proof of Proposition 3, we can show that $W : \tau(2^D) \to \mathbb{R}$ must be M$^\natural$-concave. Thus, by the first condition in Theorem 5, we have

$$\begin{aligned} &\alpha_P(m+1) - \alpha_P(m) \\ &= (W((m+1)\mathbf{i}^P) + W((m-1)\mathbf{i}^P + \mathbf{i}^{P'})) - (W(m\mathbf{i}^P + \mathbf{i}^{P'}) + W(m\mathbf{i}^P)) \\ &\leq (W(m\mathbf{i}^P) + W(m\mathbf{i}^P + \mathbf{i}^{P'})) - (W(m\mathbf{i}^P + \mathbf{i}^{P'}) + W(m\mathbf{i}^P)) \;=\; 0. \end{aligned}$$

Therefore, f_P is concave-extensible.

Let $\beta(m+1) = f(m+1) - f(m)$ for $m \in [0, |D| - 1]_{\mathbb{Z}}$. For any $m \in [1, |D| - 1]_{\mathbb{Z}}$, find $\mathbf{z} \in \tau(2^D)$ and $\overline{P}, \underline{P} \in \mathcal{P}$ such that $\mathbf{z}(\overline{P}), \mathbf{z}(\underline{P}) \geq 1$ and $|\mathbf{z}| = m + 1 \geq 2$. By Lemma 8, we know

$$\begin{aligned}
W(\mathbf{z}) - W(\mathbf{z} - \mathbf{i}^{\overline{P}}) &= [f(m+1) - f(m)] + [f_{\overline{P}}(\mathbf{z}(\overline{P})) - f_{\overline{P}}(\mathbf{z}(\overline{P}) - 1)] \\
&= \beta(m+1) + [f_{\overline{P}}(\mathbf{z}(\overline{P})) - f_{\overline{P}}(\mathbf{z}(\overline{P}) - 1)]; \\
W(\mathbf{z} - \mathbf{i}^{\underline{P}}) - W(\mathbf{z} - \mathbf{i}^{\overline{P}} - \mathbf{i}^{\underline{P}}) &= [f(m) - f(m-1)] + [f_{\overline{P}}(\mathbf{z}(\overline{P})) - f_{\overline{P}}(\mathbf{z}(\overline{P}) - 1)] \\
&= \beta(m) + [f_{\overline{P}}(\mathbf{z}(\overline{P})) - f_{\overline{P}}(\mathbf{z}(\overline{P}) - 1)].
\end{aligned}$$

Thus, by the first condition in Theorem 5, we have

$$\begin{aligned}
\beta(m+1) - \beta(m) &= [W(\mathbf{z}) + W(\mathbf{z} - \mathbf{i}^{\overline{P}} - \mathbf{i}^{\underline{P}})] - [W(\mathbf{z} - \mathbf{i}^{\overline{P}}) + W(\mathbf{z} - \mathbf{i}^{\underline{P}})] \\
&\leq [W(\mathbf{z} - \mathbf{i}^{\overline{P}}) + W(\mathbf{z} - \mathbf{i}^{\underline{P}})] - [W(\mathbf{z} - \mathbf{i}^{\overline{P}}) + W(\mathbf{z} - \mathbf{i}^{\underline{P}})] = 0.
\end{aligned}$$

Therefore, f is concave-extensible. $\square$

B.4 Proof of Proposition 4

Lemma 4 easily implies the following results.

Lemma 9. *If a transfer function T preserves the substitutes condition for all spline concave revenue functions in $A \subset D$, then for any $B \subset D \setminus A$ and $\overline{C}, \underline{C} \subset A$ such that $|\overline{C}| = |\underline{C}|$,*

$$T(B \cup \overline{C}) - T(B \cup \underline{C}) = T(\overline{C}) - T(\underline{C}). \tag{5}$$

Proof. We can start with setting the C in Equation (2) to be $B \cup \underline{C}$, and swapping one element in $\underline{C} \setminus \overline{C}$ – say $\underline{d}$ – for one element in $\overline{C} \setminus \underline{C}$ – say $\overline{d}$. This gives us one equation. Continue the swapping until $\underline{C}$ becomes $\overline{C}$, and connect all obtained equations to get Equation (5). $\square$

Proof of Proposition 4. Take a transfer function T that preserves the substitutes condition for all uni-group and $\overline{\text{uni}}$-group spline concave revenue functions. Define a transfer function T^1 such that $T^1(A) = \sum_{P \in \mathcal{P}} T(A \cap P)$ for each $A \subset D$, and let $T^2 := T - T^1$.

For any $P \in \mathcal{P}$, $d, d' \in P$, and $A \subset D$ such that $d \in A$ and $d' \notin A$, we have

$$\begin{aligned}
&T^2(A - d + d') - T^2(A) \\
&= (T(A - d + d') - T^1(A - d + d')) - (T(A) - T^1(A)) \\
&= (T(A - d + d') - T(A)) - (T^1(A - d + d') - T^1(A)) \\
&= (T(A - d + d') - T(A)) - (T((A \cap P) - d + d') - T(A \cap P)) = 0,
\end{aligned}$$

where the third equality follows from the definition of T^1, and fourth follows from Lemma 9. As a result, T^2 is group cardinal; there exists a vectorial function W on $\tau(2^D)$ such that for each $A \subset D$, $T^2(A) = W(\tau(A))$. We have $T(A) = \sum_{P \in \mathcal{P}} T(A \cap P) + W(\tau(A))$ for each $A \subset D$.

We now show that W satisfies the conditions of Lemma 8. Pick any $\mathbf{z}, \mathbf{z}' \in \tau(2^D)$ and $P^* \in \mathcal{P}$ such that $|\mathbf{z}(\mathcal{P} - P^*)| = |\mathbf{z}'(\mathcal{P} - P^*)|$, $\mathbf{z}(P^*) = \mathbf{z}'(P^*) < |P^*|$. We can find $A, A' \subset D$ such that $\tau(A) = \mathbf{z}$, $\tau(A') = \mathbf{z}'$, and $A \cap P^* = A' \cap P^*$. Because T preserves the substitutes condition for all $\overline{\text{uni}}$-group spline concave revenue functions, by Lemma 9, for any $d \in P^* \setminus A$, we can derive

$$T(A + d) - T(A) = T(A' + d) - T(A'). \tag{6}$$

We obtain Equation (4) through

$$\begin{aligned}
W(\mathbf{z} + \mathbf{i}^{P^*}) - W(\mathbf{z}) &= W(\tau(A + d)) - W(\tau(A)) \\
&= (T(A + d) - T^1(A + d)) - (T(A) - T^1(A)) \\
&= (T(A + d) - T(A)) - (T((A + d) \cap P^*) - T(A \cap P^*)) \\
&= (T(A' + d) - T(A')) - (T((A' + d) \cap P^*) - T(A' \cap P^*)) \\
&= (T(A' + d) - T^1(A' + d)) - (T(A') - T^1(A')) \\
&= W(\tau(A' + d)) - W(\tau(A')) \\
&= W(\mathbf{z}' + \mathbf{i}^{P^*}) - W(\mathbf{z}'),
\end{aligned}$$

where the fourth equality follows from Equation (6). Therefore, by Lemma 8, there exists $\{f_P\}_{P \in \mathcal{P}}$ and f such that $W(\mathbf{z}) = \sum_{P \in \mathcal{P}} f_P(\mathbf{z}(P)) + f(|\mathbf{z}|)$ for each $\mathbf{z} \in \tau(2^D)$. For each $P \in \mathcal{P}$, define $T_P : 2^P \to \mathbb{R}$ so that $T_P(A) = T(A) + f_P(|A|)$ for each $A \subset P$. We have $T(A) = \sum_{P \in \mathcal{P}} T_P(A \cap P) + f(|A|)$ for each $A \subset D$.

Next, we show that each T_P satisfies the substitutes condition on 2^P, through proving the three conditions of Theorem 6. Let $\Psi \coloneqq \sum_{P' \in \mathcal{P} - P} T_{P'}(\emptyset))$. Pick any $A, B \subset P$ with $|A| = |B|$ and any $d \in A \setminus B$. Using a "small-α" argument analogous to the one in the proof of Proposition 3, we can prove that T must satisfy the substitutes condition. By the second condition in Theorem 5, there exists $d' \in B \setminus A$ such that

$$T(A) + T(B) \leq T(A - d + d') + T(B + d - d').$$

Then we know the third condition of Theorem 6 holds as a special case of the following (i.e., when $|A \setminus B| = 2$):

$$\begin{aligned}
& T_P(A) + T_P(B) \\
= & (T(A) - f(|A|) - \Psi) + (T(B) - f(|B|) - \Psi) \\
\leq & (T(A - d + d') - f(|A - d + d'|) - \Psi) \ + \ (T(B + d - d') - f(|B + d - d'|) - \Psi) \\
= & T_P(A - d + d') + T_P(B + d - d').
\end{aligned}$$

Pick any $A \subset P$ and distinct $d, d', d'' \in P \setminus A$. By the first condition in Theorem 5, we have

$$T(A+d+d')+T(A+d'') \leq \max\{T(A+d)+T(A+d'+d''),\, T(A+d')+T(A+d+d'')\}.$$

Without loss of generality, assume $T(A+d+d')+T(A+d'') \leq T(A+d)+T(A+d'+d'')$. Then we know the second condition of Theorem 6 holds:

$$\begin{aligned}
& T_P(A + d + d') + T_P(A + d'') \\
= & (T(A + d + d') - f(|A| + 2) - \Psi) \ + \ (T(A + d'') - f(|A| + 1) - \Psi) \\
\leq & (T(A + d' + d'') - f(|A| + 2) - \Psi) \ + \ (T(A + d) - f(|A| + 1) - \Psi) \\
= & T_P(A + d' + d'') + T_P(A + d).
\end{aligned}$$

Pick any $A \subset P$ and any distinct $d, d' \in P \setminus A$. Choose any two distinct $\bar{P}, \hat{P} \in \mathcal{P} - P$, any $d'' \in \bar{P}$, and $d''' \in \hat{P}$. By the second condition in Theorem 5, we see that

$$\begin{aligned}
& T(A + d + d') + T(A + d'' + d''') \\
& \quad \leq \max\{T(A + d + d'') + T(A + d' + d'''),\, T(A + d' + d'') + T(A + d + d''')\}.
\end{aligned}$$

Without loss of generality, assume $T(A + d + d') + T(A + d'' + d''') \leq T(A + d + d'') + T(A + d' + d''')$. Then we know the first condition of Theorem 6 holds:

$$\begin{aligned}
& T_P(A + d + d') + T_P(A) \\
= \ & (T(A + d + d') - f(|A| + 2) - \Psi) \\
& \quad + \ (T(A + d'' + d''') - f(|A| + 2) - \Psi + T_{\bar{P}}(\emptyset) + T_{\hat{P}}(\emptyset) - T_{\bar{P}}(d'') - T_{\hat{P}}(d''')) \\
\leq \ & (T(A + d + d'') - f(|A| + 2) - \Psi + T_{\bar{P}}(\emptyset) - T_{\bar{P}}(d'')) \\
& \quad + (T(A + d' + d''') - f(|A| + 2) - \Psi + T_{\hat{P}}(\emptyset) - T_{\hat{P}}(d''')) \\
= \ & T_P(A + d) + T_P(A + d').
\end{aligned}$$

Finally, we prove f is concave-extensible. For any $m \in [1, |D| - 1]_{\mathbb{Z}}$, pick $A \subset D$ so that $|A| = m - 1$ and there exists $d' \in P'$ and $d'' \in P''$ for distinct $P', P'' \in \mathcal{P}$.

Let $\Phi \coloneqq \sum_{P \in \mathcal{P}} T_P(A \cap P)$. Note that $T(A+d'+d'')+T(A) \leq T(A+d')+T(A+d'')$ by the first condition of Theorem 6. Thus we know

$$\begin{aligned}
&f(m+1) - f(m) \\
&= (T(A+d'+d'') - \Phi + T_{P'}(A \cap P') + T_{P''}(A \cap P'') - T_{P'}(A \cap P' + d') \\
&\qquad - T_{P''}(A \cap P'' + d'')) \; - \; (T(A+d') - \Phi + T_{P'}(A \cap P') - T_{P'}(A \cap P' + d')) \\
&\leq (T(A+d'') - \Phi + T_{P''}(A \cap P'') - T_{P''}(A \cap P'' + d'')) \; - \; (T(A) - \Phi) \\
&= f(m) - f(m-1)
\end{aligned}$$

This concludes the proof. □

B.5 Proofs of Theorems 2, 3, and 4

We prove a known result for completeness (Murota, 2019, Proposition 4.14).

Lemma 10. *An additively separable vectorial function preserves the vectorial substitutes condition.*

Proof. Let a vectorial function U satisfy the vectorial substitutes condition and thus M$^\natural$-concavity (Lemma 2), and a vectorial function W be additively separable with respect to a family of concave-extensible functions $\{f_P\}_{P \in \mathcal{P}}$. Take any $\mathbf{z}, \mathbf{z}' \in \tau(2^D)$ and $\hat{P} \in \mathcal{P}$ such that $\mathbf{z}(\hat{P}) > \mathbf{z}'(\hat{P})$. Since U is M$^\natural$-concave, either $U(\mathbf{z}-\mathbf{i}^{\hat{P}})+U(\mathbf{z}'+\mathbf{i}^{\hat{P}}) \geq U(\mathbf{z})+U(\mathbf{z}')$, or there exists $\hat{P}' \in \mathcal{P}$ such that $\mathbf{z}'(\hat{P}') > \mathbf{z}(\hat{P}')$ and $U(\mathbf{z} - \mathbf{i}^{\hat{P}} + \mathbf{i}^{\hat{P}'}) + U(\mathbf{z}' + \mathbf{i}^{\hat{P}} - \mathbf{i}^{\hat{P}'}) \geq U(\mathbf{z}) + U(\mathbf{z}')$, or both.

In the first case, $U(\mathbf{z}-\mathbf{i}^{\hat{P}})+U(\mathbf{z}'+\mathbf{i}^{\hat{P}})-U(\mathbf{z})-U(\mathbf{z}')$ is nonnegative. Moreover, by the additive separability of W and $\mathbf{z}(\hat{P}) > \mathbf{z}'(\hat{P})$,

$$\begin{aligned}
&U(\mathbf{z} - \mathbf{i}^{\hat{P}}) + U(\mathbf{z}' + \mathbf{i}^{\hat{P}}) - U(\mathbf{z}) - U(\mathbf{z}') \\
&= f_{\hat{P}}(\mathbf{z}(\hat{P}) - 1) + f_{\hat{P}}(\mathbf{z}'(\hat{P}) + 1) - f_{\hat{P}}(\mathbf{z}(\hat{P})) - f_{\hat{P}}(\mathbf{z}'(\hat{P})) \geq \; 0.
\end{aligned}$$

So we have $(U+W)(\mathbf{z} - \mathbf{i}^{\hat{P}}) + (U+W)(\mathbf{z}' + \mathbf{i}^{\hat{P}}) - (U+W)(\mathbf{z}) - (U+W)(\mathbf{z}') \geq 0$.

In the second case, we can similarly show

$$(U+W)(\mathbf{z} - \mathbf{i}^{\hat{P}} + \mathbf{i}^{\hat{P}'}) + (U+W)(\mathbf{z}' + \mathbf{i}^{\hat{P}} - \mathbf{i}^{\hat{P}'}) \geq (U+W)(\mathbf{z}) + (U+W)(\mathbf{z}').$$

Combining these two cases proves that $U+W$ is M$^\natural$-concave. □

Analogous to Lemma 5, it is easy to see the following result.

Lemma 11. *The class of vectorial functions that preserve the vectorial substitutes condition is closed under addition.*

Proof of Theorem 4. The necessity part follows from Proposition 6. For sufficiency, in view of Proposition 10 and Lemma 11, we only need to show that given a vectorial function U associated with a revenue function R that satisfies the substitutes condition, and a vectorial function W associated with a cardinally concave transfer function T, $U+W$ satisfies the vectorial substitutes condition. But Theorem 1 says that $R+T$ satisfies the substitutes condition. Because $U+W$ is associated with $R+T$, it satisfies the vectorial substitutes condition. $\square$

Adding group separable transfer functions to group concave revenue functions gives us a new subclass of functions that satisfy the substitutes condition.

Lemma 12. *The sum of a group concave revenue function and a group separable transfer function satisfies the substitutes condition.*

Proof. We show that $R+T$, where R is a group separable revenue function and T is a group concave transfer function, satisfies the single-improvement property, that is, for any salary schedule $\mathbf{s}$ and $A \subset D$ such that $A \notin X(\mathbf{s}; R+T)$, we need to find $A' \subset D$ such that $V(A; \mathbf{s}, R+T) < V(A'; \mathbf{s}, R+T)$, $|A \setminus A'| \leq 1$, and $|A' \setminus A| \leq 1$.

Within each group P, denote the profit function associated with its revenue function R_P by $V_P(\,\cdot\,; \mathbf{s}|_P, R_P)$.[26] Suppose that there exists $P^* \in \mathcal{P}$ and $d, d' \in P^*$ such that $d \in A$, $d' \notin A$, and $V_{P^*}((A \cap P^*) - d + d'; \mathbf{s}|_{P^*}, R_P) > V_{P^*}(A \cap P^*; \mathbf{s}|_{P^*}, R_P)$. In such a case, because $V(A - d + d'; \mathbf{s}, R) > V(A; \mathbf{s}, R)$ and $T(A - d + d') = T(A)$, we have $V(A - d + d'; \mathbf{s}, R+T) > V(A; \mathbf{s}, R+T)$. We can set $A' = A - d + d'$ for this simple case. Thus, without loss of generality, we assume nonexistence of such P^*.

Hence, within every group $P \in \mathcal{P}$, $A \cap P$ offers the maximal profit under the exact constraint of $|A \cap P|$ (given the salary schedule $\mathbf{s}|_P$ and revenue function R_P). For every $P \in \mathcal{P}$, define $f_P : [0, |P|]_{\mathbb{Z}} \to \mathbb{R}$ such that

$$f_P(m) = \max\{V_P(B; \mathbf{s}|_P, R_P) : B \subset P \text{ and } |B| = m\},$$

which is the maximal profit under the exact constraint of m within group P. So $V_P(A \cap P; \mathbf{s}|_P, R_P) = f_P(|A \cap P|)$ for each $P \in \mathcal{P}$. Define $W : \tau(2^D) \to \mathbb{R}$ such that $W(\mathbf{z}) = \sum_{P \in \mathcal{P}} f_P(\mathbf{z}(P))$ for each $\mathbf{z} \in \tau(2^D)$.

By Lemma 20 of KSY, each f_P is concave-extensible, and thus W is additively separable. Define $U : \tau(2^D) \to \mathbb{R}$ to be associated with T. According to Proposition 10, $W + U$ is M$^\natural$-concave; it satisfies a multi-unit single-improvement

[26] Here, $\mathbf{s}|_P$ is the restriction of salary schedule $\mathbf{s}$ to P.

property (Murota, 2003; Milgrom and Strulovici, 2009). In other words, at least one of the following 3 statements must be true: there exists $\tilde{P} \in \mathcal{P}$ such that $(W+U)(\tau(A)+\mathbf{i}^{\tilde{P}}) > (W+U)(\tau(A))$; there exists $\tilde{P} \in \mathcal{P}$ such that $(W+U)(\tau(A)-\mathbf{i}^{\tilde{P}}) > (W+U)(\tau(A))$; there exists $\tilde{P}, \tilde{P}' \in \mathcal{P}$ such that $(W+U)(\tau(A)-\mathbf{i}^{\tilde{P}}+\mathbf{i}^{\tilde{P}'}) > (W+U)(\tau(A))$.

In the first case, we can use Lemma 19 of KSY to find $d^* \in \tilde{P} \setminus A$ such that $V_{\tilde{P}}((A \cap \tilde{P}) + d^*; \mathbf{s}|_{\tilde{P}}, R_{\tilde{P}}) = f_{\tilde{P}}(|A \cap \tilde{P}| + 1)$. Consequently,

$$V(A+d^*; \mathbf{s}, R+T) = (W+U)(\tau(A)+\mathbf{i}^{\tilde{P}}) > (W+U)(\tau(A)) = V(A; \mathbf{s}, R+T),$$

so $A + d^* \subset D$ is a single-improvement opportunity and we can set $A' = A + d^*$. Similarly, in the second case, we can find $A - d^* \subset D$ with $d^* \in \tilde{P} \cap A$ as a single-improvement opportunity; in the third case, we can find $A - d^* + d^\dagger \subset D$ with $d^* \in \tilde{P} \cap A$ and $d^\dagger \in \tilde{P}' \setminus A$ as a single-improvement opportunity. $\square$

Proof of Theorem 3. The necessity part follows from Proposition 4 because all involved spline concave revenue functions are group concave. For sufficiency, consider a group concave revenue function R, a group separable transfer function T^1, and a cardinally concave transfer function T^2. Lemma 12 says that $R + T^1$ satisfies the substitutes condition; and so Theorem 1 implies that $(R + T^1) + T^2$ satisfies the substitutes condition. $\square$

Proof of Theorem 2. The necessity part follows from Proposition 3 because all within-group binary unit-demand revenue functions are group separable. For sufficiency, consider a group separable revenue function R, an additively separable transfer function T^1, and a group concave transfer function T^2. Lemma 12 says that $R + T^2$ satisfies the substitutes condition; and so Theorem 1 implies that $(R + T^2) + T^1$ satisfies the substitutes condition. $\square$

B.6 Proofs of Proposition 5 and Proposition 7

When $|\mathcal{P}| = 2$, we have the following result.

Lemma 13. *Suppose there are two groups. The sum of two vectorial functions that both satisfy the vectorial substitutes condition satisfies it.*

Proof. Given two vectorial functions U and U' that both satisfy the vectorial substitutes condition, we show that $U+U'$ is M$^\natural$-concave. Consider any $\mathbf{z}, \mathbf{z}' \in \tau(2^D)$ and $P \in \mathcal{P}$ such that $\mathbf{z}(P) > \mathbf{z}'(P)$. There is only one other $P' \in \mathcal{P}$.

If $|\mathbf{z}| \leq |\mathbf{z}'|$, we can apply the second condition in Theorem 5 to both U and U' to obtain $U(\mathbf{z}) + U(\mathbf{z}') \leq U(\mathbf{z} - \mathbf{i}^P + \mathbf{i}^{P'}) + U(\mathbf{z}' + \mathbf{i}^P - \mathbf{i}^{P'})$ and $U'(\mathbf{z}) + U'(\mathbf{z}') \leq U'(\mathbf{z} - \mathbf{i}^P + \mathbf{i}^{P'}) + U'(\mathbf{z}' + \mathbf{i}^P - \mathbf{i}^{P'})$. Combining these two inequalities gives us

$$(U + U')(\mathbf{z}) + (U + U')(\mathbf{z}') \leq (U + U')(\mathbf{z} - \mathbf{i}^P + \mathbf{i}^{P'}) + (U + U')(\mathbf{z}' + \mathbf{i}^P - \mathbf{i}^{P'}).$$

If $|\mathbf{z}| > |\mathbf{z}'|$ and $\mathbf{z}'(P') \geq \mathbf{z}(P')$, we can apply the first condition in Theorem 5 to both U and U' to obtain $U(\mathbf{z}) + U(\mathbf{z}') \leq U(\mathbf{z} - \mathbf{i}^P) + U(\mathbf{z}' + \mathbf{i}^P)$ and $U'(\mathbf{z}) + U'(\mathbf{z}') \leq U'(\mathbf{z} - \mathbf{i}^P) + U'(\mathbf{z}' + \mathbf{i}^P)$. (Note that, due to $\mathbf{z}'(P') \geq \mathbf{z}(P')$, we cannot replace $\mathbf{i}^P$ with $\mathbf{i}^{P'}$ in these inequalities.) Combining these two inequalities gives us

$$(U + U')(\mathbf{z}) + (U + U')(\mathbf{z}') \leq (U + U')(\mathbf{z} - \mathbf{i}^P) + (U + U')(\mathbf{z}' + \mathbf{i}^P).$$

If $|\mathbf{z}| > |\mathbf{z}'|$ and $\mathbf{z}'(P') < \mathbf{z}(P')$, we can apply the definition of M$^\natural$-concavity to U and U'. Because the second case of the definition is never possible due to $\mathbf{z}'(P') \leq \mathbf{z}(P')$, the first case must be true. So $U(\mathbf{z}) + U(\mathbf{z}') \leq U(\mathbf{z} - \mathbf{i}^P) + U(\mathbf{z}' + \mathbf{i}^P)$ and $U'(\mathbf{z}) + U'(\mathbf{z}') \leq U'(\mathbf{z} - \mathbf{i}^P) + U'(\mathbf{z}' + \mathbf{i}^P)$. They are the same as the last case. □

Proof of Proposition 7. For necessity, since the zero vectorial function satisfies the vectorial substitutes condition, preserving it requires a vectorial function to satisfy it itself. The sufficiency part follows from Lemma 13. □

Proof of Proposition 5. We start with a group concave revenue function R, a group separable transfer function T, and a group concave transfer function T'. To show that $R + (T + T')$ satisfies the substitutes condition, we first note that $R + T'$ satisfies it by Lemma 13. Then, we know $(R + T') + T$ satisfies it by Theorem 2. □