Discontinuous Quasi-Variational Relations with Applications

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1 Introduction:

A number of important applications of game theory involve discontinuous payoff functions. Building on previous work of Dasguta and Maskin (1986), Simon (1987), and others, Reny (1999) derived a number of existence results for games with discontinuous payoffs by relaxing upper semicontinuity of payoffs (such as Simon's (1987) reciprocal upper semicontinuity or Dasgupta and Maskin's (1986) upper semicontinuity of the sum of payoffs) and lower semicontinuity of payoffs (such as the notion of payoff security). If strategy sets are convex and payoffs are quasiconcave in own actions, then these relaxations of upper and lower semicontinuity can be applied to derive pure-strategy existence results.[1] Several recent papers have investigated the extent to which these results for games with discontinuous payoff functions can be extended to the case in which a player's preference order need not be representable by a utility function. Reny (1999) introduces the notions of point security and correspondence security for games in which players' preference relations are complete, reflexive, and transitive, and generalizes existence results for strategic-form games with payoff functions found in Reny (2016a), Barelli and Meneghel(2013), and McLennan et al.(2011) . Carmona and Podczeck (2016) introduce the notions of point target security and correspondence target security and provide several further generalizations of these results. For related results in games and models of abstract economies in which agents' preferences need not be representable by utility functions, see Reny (2016b), Nessah, and Tian (2016) and He and Yannelis (2017).

Contemporaneous with this research program in game theory, a number of authors have studied various related equilibrium problems derived from the seminal work of Ky Fan. The Ky Fan equilibrium problem is a fundamental result of non-linear analysis with myriad applications in optmization theory,

[1] For an excellent survey of this literature including extensive references, see Carmona [**?**]. For more recent results, see Reny [**?**].

fixed point theory, mathematical economics and game theory to name just a few. The problem has the following statement:

Ky Fan Equilibrium Problem: Given a set X and a function $f : X \times X \to \mathbb{R}$, find $\overline{x} \in X$ such that $f(\overline{x}, y) \leq 0$ for each $y \in X$.

The seminal result concerning the existence of a solution to the Ky Fan equilibrium problem is the following.

Theorem 1 (Fan, 1972) : Suppose that X is a compact, convex, nonempty subset of a Hausdorff topological vector space Y and that $f : X \times X \to \mathbb{R}$ satisfies

1. $x \mapsto f(x, y)$ is lower semi-continuous for each $y \in X$.
2. $y \mapsto f(x, y)$ is quasi-concave for each $x \in X$.
3. $f(x, x) \leq 0$ for each $x \in X$.

Then there exists $\overline{x} \in X$ such that $f(\overline{x}, y) \leq 0$ for each $y \in X$.

As an application, Theorem 1 provides an existence proof for Nash equilibrium using the well known Nikaido-Isoda mapping. Given a game with players $N = \{1, .., n\}$, strategy sets X_i and payoff functions $u_i : X \to \mathbb{R}$, where $X = X_1 \times \cdots \times X_n$, define $f : X \times X \to \mathbb{R}$ as

$$f(x, y) = \sum_{i=1}^{n} u_i(x_{-i}, y_i) - \sum_{i=1}^{n} u_i(x).$$

A strategy profile $\overline{x} = (\overline{x}_1, .., \overline{x}_n) \in X$ is a Nash equilibrium if and only if $\overline{x}$ solves the Ky Fan equilibrium problem for f and X. For example, if each X_i is nonempty, convex and compact, each $x_{-i} \mapsto u_i(x_{-i}, y_i)$ is lower semi-continuous for each y_i, $x \mapsto \sum_{i=1}^{n} u_i(x)$ is upper semi-continuous and $y \mapsto \sum_{i=1}^{n} u_i(x_{-i}, y_i)$ is quasi-concave for each x_{-i}, then the game has an equilibrium.

Fan's theorem has been generalized in many directions where the focus has been on the ways in which lower semi-continuity and quasi-concavity can be relaxed. For recent contributions, see Nessah and Tian(2013) and the references cited there and for a comprehensive survey, see Tarafdar and Chowdhury (20080.. Extensions to the case of general relations may be found in Luc D T (2008), Lin L J and Wang S Y (2009), Balaj M and Luc D T (2010) , Balaj M and Lin L J (2011) , Luc D T, Sarabi E, and Soubeyran A (2010), Balaj M and Lin L J (2010), and Yang (2016).

It is the goal of this paper to provide further generalizations in terms of relations by applying recent weakened notions of continuity that have proved to be very useful in establishing the existence of Nash equilibrium in games with discontinuous payoffs.

2 Preliminary Notation and Definitions.

Let X and Y be topological spaces and let $\varphi : X \to Y$ be a correspondence. We say that $\varphi : X \to Y$

1. is compact (non-empty) valued if $\varphi(x)$ is a compact (non-empty) subset of Y

2. is upper hemi-continuous at $x \in X$ if for every open set $V \subseteq Y$ with $\varphi(x) \subseteq V$, there exists an open set U with $x \in U$ such that $\varphi(x') \subseteq V$ for all $x' \in U$.

3. is lower hemi-continuous at $x \in X$ if for for every open set $V \subseteq Y$ with $\varphi(x) \cap V \neq \varnothing$, there exists an open set U with $x \in U$ such that $\varphi(x') \cap V \neq \varnothing$ for all $x' \in U$.

4. is upper hemi-continuous (upper hemi-continuous) if φ is upper hemi-continuous (lower hemi-continuous) at each $x \in X$.

5. has open lower sections if the set $\varphi^{-1}(y) = \{x \in X | y \in \varphi(x)\}$ is open in X for each $y \in Y$

6. has the local intersection property if for each $x \in X$, there exists an open set $U(x)$ containing x such that $\cap_{x' \in U(x)} \varphi(x') \neq \varnothing$.

3 Quasi-Variational Relation Problems without Continuity

The *quasi-variational equilibrium problem* is more general than the Ky Fan equilibrium problem and has been the subject of much research in the last two decades.

Quasi-Variational Equilibrium Problem: Given a set X, a function $f : X \times X \rightarrow \mathbb{R}$ and a correspondence $K : X \rightarrow X$, find $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and $f(\overline{x}, y) \leq 0$ for each $y \in K(\overline{x})$.

For a basic existence theorem , we have the following result of Lin and Ansari (2010) (see their Theorem 3.1 and Remark 3.1)

Theorem 2: Suppose that X is a Hausdorff topological vector space. Suppose that $K : X \rightarrow X$ is a corresondence and $f : X \times X \rightarrow \mathbb{R}$ is a function. Suppose that

(i) For every $y \in X$, the set $\{x \in X | f(x, y) \leq 0\}$ is closed and for every $x \in X$, the set $\{y \in X | f(x, y) > 0\}$ is convex.

(ii) The set F of fixed points of K is closed in X.

(iii) The correspondence K is non-empty valued and convex valued with open lower sections

Then there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and

$$f(\overline{x}, y) \leq 0 \text{ for all } y \in K(\overline{x}).$$

Remark: Assumption (ii) in Theorem 2 cannot be dropped. For example, suppose that $X = [0, 1]$,

$$\begin{aligned} K(x) &= [0, \frac{1}{2}] \text{ if } x \neq \frac{1}{2} \\ &= [0, \frac{1}{2}[\text{ if } x = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} f(x, y) &= 0 \text{ if } y \leq x \\ &= 1 \text{ if } y > x \end{aligned}$$

Then for each x$\in X$ with $x \in K(x)$, there exists a $y \in K(x)$ such that $f(x, y) > 0$. In this example, the set of fixed points of K is not closed.

The equilibrium problem for generalized games introduced in Debreu (1952) is a special case of the QVEP. Again consider a game with players $N = \{1, .., n\}$, strategy sets X_i and payoff functions $u_i : X_1 \times \cdots \times X_n \to \mathbb{R}$. To each player. we associate a feasible action corrrespondence $K_i : X \to X_i$ where $K_i(x) \subseteq X_i$ is the set of actions available to i given the strategy profile x. A profile $\overline{x} = (\overline{x}_1, .., \overline{x}_n) \in X$ is an equilibrium of the generalized game if for each player i,

$$\overline{x}_i \in \arg \max_{y_i \in K_i(\overline{x})} u_i(\overline{x}_{-i}, y_i).$$

Defining $f(x, y) = \sum_{i=1}^{n} u_i(x_{-i}, y_i) - \sum_{i=1}^{n} u_i(x)$ and $K(x) = \times_{i \in N} K_i(x)$, it is clear that $\overline{x} \in X$ is an equilibrium in the generalized game if and only if $\overline{x}$ solves the QVEP problem for f and K.

In this paper, we are concerned with a generalization of the QVEP in which the function f is replaced with a relation. This extension is stated ass follows:

Quasi-Variational Relation Problem: Given set X and a relation $R \subseteq X \times X$ and a correspondence $K : X \to X$, find $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and

$$(\overline{x}, y) \in R \text{ for all } y \in K(\overline{x}).$$

Remark: The QVEP is the special case in which $R = \{(x, y) \in X \times X | f(x, y) \leq 0)$.

The QVRP is more flexible than the QVEP. For example the QVRP includes as a special case, the

Quasi-Variational Inclusion Problem: Given a sets X and S and a correspondence $\varphi : X \times X \to S$, a point $s \in S$ and a correspondence $K : X \to X$, find $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and $s \in \varphi(\overline{x}, y)$ for each $y \in K(\overline{x})$.

Our main goal is to investigate the extent to which the existence of equilibrium can be established when both the correspondence K and the relation R

may exhibit discontinuities. In particular, we are interested in relaxing continuity of R by employing recent ideas from the theory of discontinuous games.

Definition: Let X be a topological space and $R \subseteq X \times Y$ be a relation.

(i) R has closed upper sections if $\{x \in X : (x,y) \in R\}$ is closed in X for each $y \in Y$.

(ii) R is transfer semi-continuous if $x \in X$ and $(x,y) \notin R$ imply that there exists an open set $U(x)$ containing x and $y^* \in X$ such that $(x', y^*) \notin R$ for all $x' \in U(x)$.

Remark: If R is a closed set in $X \times Y$, then R has closed upper sections and if R has closed upper sections, then the relation R is transfer semi-continuous.

Transfer semi-continuity of a relation R generalizes the notion of 0-transfer lower semi-continuity of a function. That is, $f : X \times Y \to \mathbb{R}$ is 0-transfer lower semi-continuous if and only if the relation $R = \{(x,y) \in X \times Y | f(x,y) \leq 0\}$ is transfer semi-continuous.

Definition: Suppose that X is a topological space, $K : X \to X$ is a corresondence and $R \subseteq X \times X$ a relation. Then R is transfer semi-continuous with respect to K if $x \in X$ and $y \in K(x)$ and $(x,y) \notin R$ imply that there exists an open set $U(x)$ containing x and $y^* \in K(x)$ such that $(x', y^*) \notin R$ for all $x' \in U(x)$.

We can now generalize Theorem 2.

Theorem 3: Suppose that X is a non-empty, compact, convex subset of a Hausdorff topological vector space. Suppose that $K : X \to X$ is a correspondence and $R \subseteq X \times X$ is a relation: Suppose that

(i) R transfer semi-continuous with respect to K.

(ii) K is non-empty valued and convex valued with open lower sections.

(iii) The set F of fixed points of K is closed.

(iv) For each x, $\{y \in X : (x,y) \notin R\}$ is convex.

(v) For each $x, (x,x) \in R$

Then there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and

$$(\overline{x}, y) \in R \text{ for all } y \in K(\overline{x}).$$

Proof: We argue by contradiction.

Step 1: Suppose that the conclusion of the theorem is false. Suppose that the conclusion of the theorem is a false. Then for every $x \in X$ with $x \in K(x)$, there exists a $y \in K(x)$ such that $(x,y) \notin R$. Defining $F = \{x \in X | x \in K(x)\}$ and $P(x) = \{y \in X : (x,y) \notin R\}$ it follows that $P(x) \cap K(x) \neq \varnothing$ and convex for each $x \in F$. Furthermore, F is non-empty by Browder's fixed point theorem. Next, define a corresondence $\psi : X \to X$ as

$$\begin{aligned} \psi(x) &= K(x) \cap P(x) \text{ if } x \in F \\ &= \varnothing \text{ if } x \notin F \end{aligned}$$

We will show that ψ has the local intersection property. Since dom$\psi = F$, we must show that for each $x \in F$, there exists an open set $U(x)$ in X containing x such that $\cap_{x' \in U(x)}[K(x') \cap P(x')] \neq \varnothing$. If $x \in F$, then there exists $y \in X$ such that $(x, y) \notin R$. Since $R \subseteq X \times X$ is transfer semi-continuous with respect to K, there exists an open set $U_1(x)$ containing x and $y^* \in K(x)$ such that $(x', y^*) \notin R$ for all $x' \in U_1(x)$. Since $y^* \in K(x)$ and K has open lower sections, there exists an open set $U_2(x)$ containing x such that $y^* \in K(x')$ for all $x' \in U_2(x)$. Therefore, $U_1(x) \cap U_2(x)$ is an open set containing x and $y^* \in K(x') \cap P(x')$ for all $x' \in U_1(x) \cap U_2(x)$.

Step 2: Combining Lemma 3 in Prokopovych (2011) and Lemma 5.1 in Yannelis and Prabhakar(1983), there exists a convex valued correspondence $\varphi : X \to X$ with open lower sections such that $dom\varphi = dom\psi$ and $\varphi(x) \subseteq \psi(x)$ for all x. Next, define a correspondence $T : X \to X$ as follows:

$$\begin{aligned} T(x) &= \varphi(x) \text{ if } x \in F \\ &= K(x) \text{ if } x \notin F \end{aligned}$$

Then φ is nonempty valued since $dom\varphi = dom\psi = F$. Furthemore, ψ is convex valued. Next, we claim that T has open lower sections. Since φ and K have open lower sections, it suffices to show that

$$T^{-1}(y) = \varphi^{-1}(y) \cup [K^{-1}(y) \cap (X \backslash F)]$$

for each $y \in X$. It is clear that $T^{-1}(y) \subseteq \varphi^{-1}(y) \cup [K^{-1}(y) \cap (X \backslash F)]$ so suppose that $x \in \varphi^{-1}(y) \cup [K^{-1}(y) \cap (X \backslash F)]$. If $x \in K^{-1}(y) \cap (X \backslash F)$, then $y \in K(x)$ and $T(x) = K(x)$ implying that $y \in T(x)$. If $x \in \varphi^{-1}(y)$ and $x \in F$, then $y \in \varphi(x)$ and $T(x) = \varphi(x)$ implying that $y \in T(x)$. If $x \in \varphi^{-1}(y)$ and $x \notin F$, then $y \in \varphi(x)$ and $\varphi(x) \subseteq K(x) \cap P(x) \subseteq K(x)$ implying that $y \in T(x)$. Applying Browder's fixed point theorem, we conclude that there exists $x \in X$ such that $x \in T(x)$. Consequently, $x \in \varphi(x)$ implying that $x \in P(x)$ which contradicts assumption (v).

4 A generalization with local convexity

In Theorem 3, we assume that the correspondence K has open lower sections so that Browder's fixed point theorem can be brought to bear. Note, however, that the Hausdorff topological vector space X in Theorem 3 need not be locally convex. If we strengthen this assumption and assume that X is locally convex, then we can weaken both the asumptions that the relation R is transfer continuous and K has open lower sections. This alternative result is possible as an application of a recent fixed point theorem of He and Yannelis (2017).

We begin with a generalization of transfer continuity inspired by another recent idea from the theory of discontinuous games due to Reny (2016). If X and Z are subsets of topological vector spaces with Z convex, we will follow Reny (2016) and call a correspondence $\varphi : X \to Z$ *co-closed* if the corresondence $x \in X \mapsto$con$\varphi(x) \subseteq Z$ has closed graph in the relative topology on $X \times Z$.[2]

[2] Here, con$\varphi(x)$ denotes the convex hull of $\varphi(x)$.

Definition: Suppose that X is a topological space, $K : X \to X$ is a correspondence and $R \subseteq X \times X$ is a relation. Let $P(x) = \{y \in X | (x, y) \notin R\}$. Then R is point secure in X with respect to K if whenever $y \in K(x)$ and $(x, y) \notin R$, there exists an open set $U(x)$ containing x and a continuous function $d : U(x) \to X$ such that $d(x') \in K(x') \cap P(x')$ for all $x' \in U(x)$.

Definition: Suppose that X is a topological spaces, $K : X \to X$ is a correspondence and $R \subseteq X \times X$ is a relation. Let $P(x) = \{y \in X | (x, y) \notin R\}$. Then R is correspondence secure in X with respect to K if whenever $y \in K(x)$ and $(x, y) \notin R$, there exists an open set $U(x)$ containing x and a co-closed correspondence $d : U(x) \to X$ such that $d(x') \subseteq K(x') \cap P(x')$ for all $x' \in U(x)$.

Remark: Point security of R with respect to K implies correspondence security of R with respect to K. From Step 1 of the proof of Theorem 3, it follows that, if K has open lower sections and if R is transfer semi-continuous with respect to K, then R is point secure with respect to K. However, transfer semi-continuity of R with respect to K need not imply that R is point secure with respect to K. Let $X = [0, 1]$. Let $K_1(x) = [0, 1]$ and let

$$\begin{aligned} K_2(x) &= \{\frac{1}{2}\} \text{ if } x \neq \frac{1}{2} \\ &= \{0\} \text{ if } x = \frac{1}{2} \end{aligned}$$

Suppose that f is defined as follows:

$$\begin{aligned} f(x, y) &= 1 \text{ if } x = 1 \\ &= 1 \text{ if } y = 0 \\ &= 0 \text{ otherwise.} \end{aligned}$$

Defining $R = \{(x, y | f(x, y) \leq 0\}$, it follows that R is transfer semi-continuous with respect to K but R is not point secure with respect to K.

Next, we record a fixed point theorem of He and Yannelis (2017) that is important for our result.

Definition: Let X and Y be topological spaces. A correspondence $\varphi : X \to Y$ has the continuous inclusion property if for each $x \in X$, there exists an open set $U(x)$ containing x and a co-closed correspondence $d : U(x) \to X$ such that $d(x') \subseteq \varphi(x')$.

Remark: If $\varphi : X \to Y$ has open lower sections, then φ has the continuous inclusion property

Theorem 4: Suppose that X is a non-empty, convex, compact subset of a Hausdorff locally convex topological vector space. Suppose that $\varphi : X \to X$ is a non-empty valued, convex valued corresondence with the continuous inclusion property. Then φ has a fixed point.

Theorem 5: Suppose that X is a non-empty, convex, compact subset of a Hausdorff locally convex topological vector space. Suppose that $K : X \to X$ is a corresondence and $R \subseteq X \times X$ is a relation. Suppose that

(i) R is correspondence secure with respect to K.

(ii) K is non-empty valued and convex valued with the continuous inclusion property and the set of fixed points of K is closed.

(iii) For each x, $\{y \in X : (x, y) \notin R\}$ is convex.

(iv) For each x, $(x, x) \in R$

Then there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and

$$(\overline{x}, y) \in R \text{ for all } y \in K(\overline{x}).$$

Proof: We argue by contradiction. Suppose that the conclusion of the theorem is false. Then for every $x \in X$ with $x \in K(x)$, there exists a $y \in K(x)$ such that $(x, y) \notin R$. Defining $F = \{x \in X | x \in K(x)\}$ and $P(x) = \{y \in X : (x, y) \notin R\}$ it follows that $P(x) \cap K(x) \neq \varnothing$ and convex for each $x \in F$. Furthermore, F is non-empty by Theorem 2 in He-Yannelis. Define

$$\begin{aligned} T(x) &= K(x) \cap P(x) \text{ if } x \in F \\ &= K(x) \text{ if } x \notin F \end{aligned}$$

Consequently, there exists exists an open set $U(x)$ containing x and a co-closed correspondence $d : U(x) \to X$ such that $d(x') \subseteq K(x') \cap P(x')$ for all $x' \in U(x)$. Define

$$\begin{aligned} T(x) &= K(x) \cap P(x) \text{ if } x \in F \\ &= K(x) \text{ if } x \notin F \end{aligned}$$

Since K has the continuous inclusion property and X/F is open, it follows that T has the continuous inclusion property so applying Theorem 2 in He-Yannelis, we conclude that there exists $\overline{x} \in X$ with $\overline{x} \in T(\overline{x})$. Therefore, $\overline{x} \in K(\overline{x}) \cap P(\overline{x})$ contradicting (iv).

Example: Let $X_1 = [0, 1] = X_2$. Let $K_1(x) = [0, 1]$ and let

$$\begin{aligned} K_2(x) &= [0, \frac{1}{2}] \text{ if } 0 \leq x_1 < 1 \\ &=]0, 1] \text{ if } x_1 = 1 \end{aligned}$$

Suppose that $u_1 \equiv 0$ and u_2 is defined as follows:

$$\begin{aligned} u_2(x_1, x_2) &= 1 \text{ if } x_1 = 1 \\ &= 1 \text{ if } x_2 = 0 \\ &= 0 \text{ otherwise.} \end{aligned}$$

Defining

$$R = \{(x,y) \in X \times X | \sum_{i=1}^{2} [u_i(x_{-i}, y_i) - u_i(x)] \leq 0$$

it follows that R is point secure with respect to K. In this example, K is neither upper hemi-continuous nor lower hemi-continuous (and therefore does not have open lower sections). However, K does have the continuous inclusion property so applying Theorem 5**,** we conclude the game admits a Nash equilibrium.

We next provide an application of Theorem 5 to generalized games. when the players choose the profile x.

Application: Consider a game with players $N = \{1, .., n\}$, strategy sets X_i and payoff functions $u_i : X_1 \times \cdots \times X_n \to \mathbb{R}$. Suppose that each X_i is a compact, convex, non-empty subset of a locally convex Hausdorff TVS. To each player, we associate a feasible action corrrespondence $K_i : X \to X_i$ where $K_i(x) \subseteq X_i$ is the set of actions available to i. Suppose that

(a) For every real λ, each u_i is λ-transfer lower semi-continuous in x_{-i} with respect to K_i: if $y_i \in K_i(x)$ and $u_i(x_{-i}, y_i) > \lambda$ then there exists an open set $U_i(x)$ containing x and $y_i^* \in K_i(x)$ such that $u_i(x'_{-i}, y_i) > \lambda$ for all $x' \in U_i(x)$.

(b) The aggregate payoff $x \mapsto \sum_{i=1}^{n} u_i(x)$ is upper semi-continuous.

(c) Each K_i is non-empty valued and convex valued with open lower sections and the set of fixed points of K is closed.

(d) For each x_{-i}, $\{y_i \in X_i : u_i(x_{-i}, y_i) > 0\}$ is convex.

Then there exists a profile $\overline{x} = (\overline{x}_1, .., \overline{x}_n) \in X_1 \times \cdots \times X_n$ such that

$$\overline{x}_i \in \arg \max_{y_i \in K_i(\overline{x})} u_i(\overline{x}_{-i}, y_i) \text{ for each i.}$$

Defining

$$R = \{(x,y) \in X \times X | \sum_{i=1}^{n} u_i(x_{-i}, y_i) - \sum_{i=1}^{n} u_i(x) \leq 0$$

and $K(x) = \times_{i \in N} K_i(x_{-i})$, it is clear that $\overline{x} \in X$ is an equilibrium in the generalized game if and only if $\overline{x}$ solves the QVRP problem for R and K. Conditions (ii) and (iii) of Theorem 5 are implied by assumptions (c) and (d) and condition (iv) is obviously satisfied so to apply Theorem 3 we need to use assumptions (a) and (b) to establish that R is correspondence secure with respect to K. The argument adapts the proof of Lemma 1 in Prokopovych and Yannelis(2014). Suppose that $(x,y) \in X \times X$ and that $\sum_{i=1}^{n} u_i(x_{-i}, y_i) - \sum_{i=1}^{n} u_i(x) > 0$. Then

$$\sum_{i=1}^{n} u_i(x_{-i}, y_i) > \lambda > \sum_{i=1}^{n} u_i(x)$$

for some λ. So there exist numbers λ_i such that $\sum_i \lambda_i = \lambda$ and $u_i(x_{-i}, y_i) > \lambda_i$ for each i. Applying Assumption (a), there exists for each i an open set $U_i(x)$ containing x and $y_i^* \in K_i(x)$ such that $u_i(x'_{-i}, y_i^*) > \lambda_i$ for all $x' \in U_i(x)$. Aplying assumption b, there exists an open set $V(x)$ containing x such that $\lambda > \sum_{i=1}^n u_i(x')$ for all $x' \in V(x)$. Therefore, $\sum_{i=1}^n u_i(x'_{-i}, y_i^*) > \sum_i \lambda_i = \lambda > \sum_{i=1}^n u_i(x')$ for all $x' \in [\cap_i U_i(x)] \cap V(x)$ implying that R is transfer semi-continuous with respect to K.

5 Skew Symmetric Games

To introduce this idea, let S be a nonempty set. A function $\varphi : S \times S \to \mathbb{R}$ is *skew-symmetric* if $\varphi(x,y) = -\varphi(y,x)$ for all $(x,y) \in S \times S$. Obviously, skew symmetry implies that $\varphi(x,x) = 0$ for all $x \in S$. A relation $\succsim$ in $S \times S$ has a skew-symmetric representation if there exists a skew-symmetric function $\varphi : S \times S \to \mathbb{R}$ satisfying

$$y \succsim x \Leftrightarrow \varphi(x,y) \leq 0.$$

From the definition, it follows that every relation $\succsim$ admitting a skew-symmetric representation is reflexive and complete, and if $\succsim$ admits a utility representation $u : S \to \mathbb{R}$, then $\succsim$ admits the skew-symmetric representation $\varphi(x,y) = u(x) - u(y)$.

Definition: A qualitative game is a collection $G = (X_i, \succsim_i)_{i=1}^n$, where n is a finite number of players, X_i is a nonempty set of actions for player i, and $\succsim_i$ is a preference relation for player i defined on the set $X := \times_{i=1}^n X_i$ of action profiles, i.e., $\succsim_i$ is a binary relation in $X \times X$.

We say that a qualitative game $G = (X_i, \succsim_i)_{i=1}^n$ is a skew symmetric game (SSYM) if for each i there exists a skew symmetric map $\varphi_i : X \times X \to \mathbb{R}$ satisfying

$$y \succsim_i x \Leftrightarrow \varphi_i(x,y) \leq 0, \quad (x,y) \in X \times X.$$

A Nash equilibrium of an SSYM game $(X_i, \varphi_i)_{i=1}^n$ is a strategy profile $(x_1, ..., x_n) \in \times_{i=1}^n X_i$ such that for each i,

$$\varphi_i((y_i, x_{-i}), x) \leq 0, \quad \text{for all } y_i \in X_i.$$

For example, suppose that each X_i is a compact, nonempty, convex subset of $\mathbb{R}^{m_i}$ for some $m_i \geq 1$. In addition, suppose that each φ_i is continuous on $X \times X$ and $y_i \mapsto \varphi_i((y_i, x_{-i}), x)$ is quasiconcave for each $x \in X$. Now define

$$\mu_i(z) := \arg\max_{x_i \in X_i} \varphi_i((x_i, z_{-i}), z), \qquad \text{for each } z \in X.$$

Then, combining Berge's Maximum Theorem and the Kakutani Fixed Point Theorem, it follows that there exists $x^* \in X$ such that

$$x^* \in \mu_1(x^*) \times \cdots \times \mu_n(x^*),$$

i.e.,

$$\varphi_i((x_i, x^*_{-i}), x^*) \leq \varphi_i(x^*, x^*) = 0.$$

Discontinuous SSYM games are studied in Carbonell-Nicolau and McLean (2018).

There is also an obvious approach to existence using some version of the Ky-Fan inequality. Let

$$f(x, y) = \sum_{i=1}^{n} \varphi_i((y_i, x_{-i}), x).$$

Then $x^* \in X$ is an equilibrium if and only if

$$F(x, x^*) \leq 0, \text{ for each } x \in X.$$

Theorem 6: Suppose that the SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ where each X_i is a non-empty, compact, convex subset of a Hausdorff locally convex TVS .To each player, we associate a feasible action corrrespondence $K_i : X \to X_i$. Define a relation $R \subseteq X \times X$ as

$$R = \{(x, y) \in X \times X | \sum_{i=1}^{n} \varphi_i((y_i, x_{-i}), x) \leq 0\}$$

and a correspondence $K : X \to X$ with $K(x) = K_1(x) \times \cdots \times K_n(x)$ for each $x \in X$. Suppose that

(i) R is correspondence secure in X with respect to K.

(ii) K is non-empty valued and convex valued with the continuous inclusion property.

(iii) For each x, $\{y \in X : (x, y) \notin R\}$ is convex.

Then there exists a profile $\overline{x} = (\overline{x}_1, .., \overline{x}_n) \in X_1 \times \cdots \times X_n$ such that

$$\overline{x}_i \in \arg \max_{y_i \in K_i(\overline{x})} u_i(\overline{x}_{-i}, y_i) \text{ for each i.}$$

6 Generalized Quasi-Variational Relation Problems

Two other significant problems in non-linear analysis are the generalized equilibrium problem and the generalized quasi-variational equilibrium problem.

Generalized Equilibrium Problem (GEP): Given sets X and Z, a function $g : X \times X \times Z \to \mathbb{R}$ and a correspondence $\varphi : X \to Z$, Find $\overline{x} \in X$ and $\overline{z} \in \varphi(\overline{x})$ such that

$$g(\overline{x}, y, \overline{z}) \leq 0 \text{ for all } y \in X.$$

Application: Suppose that $X \subseteq \mathbb{R}^n$ is convex, $Z = \mathbb{R}^n$ and $f : X \to \mathbb{R}$ is a concave function. Let $\partial^+ f(x)$ denote the superdifferential of f at x. If $\overline{x}$ and $\overline{z}$

solve the GEP with $\varphi(x) = \partial^+ f(x)$ and $g(x, y, z) = z \cdot (y - x)$, then $\overline{z} \in \partial^+ f(\overline{x})$ and $\overline{z} \cdot (y - \overline{x}) \leq 0$ for all $y \in X$. Therefore

$$f(y) - f(\overline{x}) \leq \overline{z} \cdot (y - \overline{x}) \text{ for all } y \in X$$

implying that $\overline{x}$ solves the problem maximize $f(x)$ subject to $x \in X$

Application: Suppose that $X = \Delta$ is the unit simplex in $\mathbb{R}^n$ and $\varphi : \Delta \to \mathbb{R}^n$ is an excess demand correspondence satisfying $x \cdot z \leq 0$ for all $z \in \varphi(x)$. If $\overline{x}$ and $\overline{z}$ solve the GEP with $g(x, y, z) = z \cdot (y - x)$, then $\overline{z} \in \varphi(\overline{x})$ and $\overline{z} \cdot (y - \overline{x}) \leq 0$ for all $y \in \Delta$ implying that $\overline{z} \in -\mathbb{R}^n$.

A problem that includes the generalized equilibrium problem and the quasi-equilibrium problem as special cases is the eneralized quasi-variational equilibrium problem:

Generalized Quasi-Variational Equilibrium Problem (GQVE): Given a sets X and Z, a function $g : X \times X \times Z \to \mathbb{R}$ and two correspondednces $K : X \to X$ and $\varphi : X \to Z$, Find $\overline{x} \in X$ and $\overline{z} \in \varphi(\overline{x})$ such that $\overline{x} \in K(\overline{x})$ and

$$g(\overline{x}, y, \overline{z}) \leq 0 \text{ for all } y \in K(\overline{x}).$$

If $g(x, y, z) = f(x, y)$ for some function f, then the problem specializes to the quasi-equilibrium problem and if in additon $K(x) = X$ for all x, we recover the Ky Fan equilibrium problem as a special case. If $g(x, y, z) = z \cdot (x - y)$, then the GVEP specializes to the generalized quasi-variational inequality problem. The GQVE problem was introduced in Chan and Pang (1982) as part of their study of the generalized quasi-variational inequality problem in finite dimensions.

The GQVE problem can be further generalized to relations.

Generalized quasi-variational relation problem (GQVR): Given sets X and Z, a relation $R \subseteq X \times X \times Z$ and two correspondednces $K : X \to X$ and $\varphi : X \to Z$, find $\overline{x} \in X$ and $\overline{z} \in \varphi(\overline{x})$ such that $\overline{x} \in K(\overline{x})$ and

$$(\overline{x}, y, \overline{z}) \in R \text{ for all } y \in K(\overline{x}).$$

Our goal is to prove an existence result for the GQVR problem with minimal continuity assumptions. The assumptions that we do impose are again motivated by weakened contnuity ideas that have been developed for Nash equilibrium existence results for discontinuouus games.

We begin with a result that does not assume local convexity and generalizes Theorem 3 above by extending the notion of transfer semi-continuity.

Definition: Suppose that X and Z topological spaces. Suppose that $K : X \to X$ is a corresondence, $\varphi : X \to Z$ is a correspondence and $R \subseteq X \times X \times Z$ is a relation. Then R is transfer semi-continuous with respect to K and φ if for all $(x, y, z) \in X \times X \times Z$ such that $x \in X, x \in K(x), z \in \varphi(x), y \in K(x)$ and

$(x,y,z) \notin R$, there exists an open set $V(x,z)$ containing (x,z) and $y^* \in K(x)$ such that $(x',y^*,z') \notin R$ for all $(x',z') \in V(x,z)$.

Theorem 7: Suppose that X is a non-empty, convex, compact subset of a Hausdorff topological vector space and that Z is a Hausdorff TVS. Suppose that $K : X \to X$ is a correspondence, $\varphi : X \to Z$ is a correspondence and $R \subseteq X \times X \times Z$ is a relation. Suppose that

(i) R is transfer semi-continuous with respect to K and φ

(ii) K and φ are non-empty valued and convex valued with open lower sections

(iii) The set $\{(x,z) \in X \times Z | (x,z) \in K(x) \times \varphi(x)\}$ is closed.

(iv) For each $(x,z) \in X \times Z$, $\{y \in X | g(x,y,z) > 0\}$ is convex.

(v) For every $z \in Z$, $(x,x,z) \in R$.

Then there exists $\overline{x} \in X$ and $\overline{z} \in Z$ such that $\overline{x} \in K(\overline{x}), \overline{z} \in \varphi(\overline{x})$ and

$$(\overline{x},y,\overline{z}) \in R \text{ for all } y \in K(\overline{x}).$$

Proof: We again argue by contradiction.

Step 1: Suppose that the conclusion of the theorem is a false. Then for every $(x,z) \in X \times Z$ with $x \in K(x)$ and $z \in \varphi(x)$, there exists a $y \in K(x)$ such that $(x,y,z) \notin R$. Therefrore, $P(x,z) \cap K(x) \neq \varnothing$ where

$$P(x,z) = \{y \in X | (x,y,z) \notin R\}.$$

Defining $F = \{(x,z) \in X \times Z | x \in K(x) \text{ and } z \in \varphi(x)\}$, it follows from (ii) and Browder's Theorem that $F \neq \varnothing$ and that, for every $(x,z) \in F$, the set $P(x,z) \cap K(x) \neq \varnothing$ and convex. Define a corresondence $\psi : X \times Z \to X$ as

$$\begin{aligned} \psi(x,z) &= P(x,z) \cap K(x) \text{ if } (x,z) \in F \\ &= \varnothing \text{ if } (x,z) \notin F. \end{aligned}$$

We show that ψ has the local intersection property**.** Since $\mathrm{dom}\psi = F$, we must show that for each $(x,z) \in F$, there exists an open set $U(x,z)$ in $X \times Z$ containing (x,z) such that

$$\cap_{(x',z') \in U(x,z)} P(x',z') \cap K(x') \neq \varnothing.$$

If $(x,z) \in K(x) \times \varphi(x)$, then $P(x,z) \cap K(x) \neq \varnothing$. Since $R \subseteq X \times X \times Z$ is transfer semi-continuous in (x,z) with respect to $K \times \varphi$, there exists an open set $V_1(x,z)$ containing (x,z) and $y^* \in K(x)$ such that $(x',y^*,z') \notin R$ for all $(x',z') \in V_1(x,z)$. Since $y^* \in K(x)$ and K has open lower sections, there exists an open $V_2(x)$ containing x such that $y^* \in K(z)$ for all $x' \in V_2(x)$. Therefore, $U(x,z) = V_1(x,z) \cap [V_2(x) \times Z]$ is an open set containng (x,z) and $y^* \in P(x',z') \cap K(z')$ for all $(x',z') \in U(x,z)$.

Step 2: Again combining Lemma 3 in Prokopovych (2011) and Lemma 5.1 in Yannelis and Prabhakar(1983), there exists a convex valued correspondence $\rho : X \times Z \to X$ with open lower sections such that $dom\varphi = dom\psi$ and $\rho(x,z) \subseteq$

$\psi(x,z)$ for all (x,z). Next, define a correspondence $T : X \times Z \to X \times Z$ as follows:

$$\begin{aligned} T(x,z) &= \rho(x,z) \times \varphi(x) \text{ if } (x,z) \in F \\ &= K(x) \times \varphi(x) \text{ if } (x,z) \notin F. \end{aligned}$$

Then T is nonempty valued since $dom\rho = dom\psi$. Furthemore, T is convex valued. Next, we claim that T has open lower sections. To see this, choose $(\xi,\eta) \in X \times Z$ and $(x,z) \in \{(x,z)|(\xi,\eta) \in T(x,z)\}$.

Suppose that $(x,z) \notin K(x) \times \varphi(x)$, Then there exists an open set $U_1(x,z)$ containing (x,z) such that $(x',z') \notin K(x') \times \varphi(x')$ for all $(x',z') \in U_1(x,z)$. Since $K(x) \times \varphi(x)$ has open lower sections, there exists an open set $U_2(x,z)$ containing (x,z) such that $(\xi,\eta) \in K(x') \times \varphi(x')$ for all $(x',z') \in U_2(x,z)$. If $(x',z') \in U_1(x,z) \cap U_2(x,z)$, then $T(x',z') = K(x') \times \varphi(x')$ (since $(x',z') \notin K(x') \times \varphi(x')$) and $(\xi,\eta) \in K(x') \times \varphi(x')$. Therefore, $(\xi,\eta) \in T(x',z')$ for all $(x',z') \in U_1(x,z) \cap U_2(x,z)$.

Suppose that $(x,z) \in K(x) \times \varphi(x)$. Then there exists an open set $U(x,z)$ containing (x,z) such that $(\xi,\eta) \in \rho(x',z') \times \varphi(x')$ for all $(x',z') \in U(x,z)$ since ρ has open lower sections. If $(x',z') \in K(x') \times \varphi(x')$, then $T(x',z') = \rho(x',z') \times \varphi(x')$ implying that $(\xi,\eta) \in T(x',z')$. If $(x',z') \notin K(x') \times \varphi(x')$, then $T(x',z') = K(x') \times \varphi(x')$. From the construction of $\rho(x',z')$, it follows that $\rho(x',z') \times \varphi(x') \subseteq K(x') \times \varphi(x')$ implying that $(\xi,\eta) \in T(x',z')$. Applying Browder's fixed point theorem, we conclude that there exists $(x,z) \in X \times Z$ such that $(x,z) \in T(x,z)$. Consequently, $(x,z) \in \rho(x,z) \times \varphi(x)$ implying that $x \in \rho(x,z)$ which contradicts assumption (v).

To generalize Theorem 5, we need extensions of point and correspondence security.

Definition: Suppose that X is a non-empty, convex, compact subset of a Hausdorff topological vector space and that Z is a Hausdorff TVS. Suppose that $K : X \to X$ is a corresondence, $\varphi : X \to Z$ is a correspondence and and $R \subseteq X \times X \times Z$ is a relation. Let $P(x,z) = \{y \in X|(x,y,z) \notin R\}$. Then R is point secure in $X \times Z$ with respect to K and φ if whenever $x \in K(x), y \in K(x), z \in \varphi(x)$ and $(x,y,z) \notin R$, there exists an open set $U(x,z)$ containing (x,z) and a continuous function $d : U(x,z) \to X$ such that $d(x',z') \in K(x') \cap P(x',z')$ for all $(x',z') \in U(x,z)$.

Definition: Suppose that X is a non-empty, convex, compact subset of a Hausdorff topological vector space and that Z is a Hausdorff TVS. Suppose that $K : X \to X$ is a corresondence, $\varphi : X \to Z$ is a correspondence and and $R \subseteq X \times X \times Z$ is a relation. Let $P(x,z) = \{y \in X|(x,y,z) \notin R\}$. Then R is correspondence secure in $X \times Z$ with respect to K and φ if whenever $x \in K(x), y \in K(x), z \in \varphi(x)$ and $(x,y,z) \notin R$, there exists an open set $U(x,z)$ containing (x,z) and a co-closed correspondence $d : U(x,z) \to X$ such that $d(x',z') \subseteq K(x') \cap P(x',z')$ for all $(x',z') \in U(x,z)$.

Remark: Correspondence security of $R \subseteq X \times X \times Z$ with respect to K and φ generalizes point security of $R \subseteq X \times X$ with respect to K. From Step 1 of the proof of Theorem 7, it follows that, if K and φ are non-empty valued and convex valued with open lower sections and if R is transfer semi-continuous with respect to K and φ, then R is point secure in $X \times Z$ with respect to K and φ.

Theorem 8: Suppose that X is a non-empty, convex, compact subset of a locally convex Hausdorff topological vector space and that Z is a locally convex Hausdorff TVS. Suppose that $K : X \to X$ is a correspondence, $\varphi : X \to Z$ is a correspondence and $R \subseteq X \times X \times Z$ is a relation. Suppose that

(i) R is correspondence secure with respect to K and φ

(ii) K and φ are non-empty valued and convex valued with the continuous inclusion property and the set $\{(x, z) \in X \times Z | (x, z) \in K(x) \times \varphi(x)\}$ is closed.

(iii) For each (x, z), $\{y \in X : (x, y, z) \notin R\}$ is convex.

(iv) $(x, x, z) \in R$ for every $z \in Z$.

Then there exists $\overline{x} \in X$ and $\overline{z} \in Z$ such that $\overline{x} \in K(\overline{x}), \overline{z} \in \varphi(\overline{x})$ and

$$(\overline{x}, y, \overline{z}) \in R \text{ for all } y \in K(\overline{x}).$$

Proof: We argue by contradiction. Suppose that the conclusion of the theorem is false. Let $F = \{(x, z) \in X \times Z | (x, z) \in K(x) \times \varphi(x)\}$ and define a correspondence

$$\begin{aligned} T(x, z) &= [K(x) \cap P(x, z)] \times \varphi(x) \text{ if } (x, z) \in F \\ &= K(x) \times \varphi(x) \text{ if } (x, z) \notin F \end{aligned}$$

It follows from Theorem 2 in He-Yannelis that $F \neq \varnothing$. In addition, for every $(x, z) \in F$, the set $P(x, z) = \{y \in X | (x, y, z) \notin R\} \neq \varnothing$ and convex. Consequently, there exists exists an open set $U(x, z)$ containing (x, z) and a co-closed correspondence $d : U(x, z) \to X$ such that $d(x', z') \subseteq K(x') \cap P(x', z')$ for all $(x', z') \in U(x, z)$. Consequently, $K \times \varphi$ has the continuous inclusion property and F is closed, it follows that T has the continuous inclusion property so applying Theorem 2 in He-Yannelis, we conclude that there exists $(\overline{x}, \overline{z}) \in X \times Z$ with $(\overline{x}, \overline{z}) \in T((\overline{x}, \overline{z}))$ implying that $\overline{x} \in K(\overline{x}) \cap P(\overline{x}, \overline{z})$ This contradicts (v).

Application: In this application, we consider an optimization-based approach to generalized games. Let X_i be a non-empty, compact, convex subset of a Banach space V_i with dual V_i^* and let $u_i : X \to \mathbb{R}$ denote the payoff function of player i. Let $K_i : X \to X_i$ denote the feasible set correspondence for player i where $K_i(x)$. For $x \in X$, let $N_{K_i(x)}(x_i) \subseteq V_i^*$ denote the normal cone to $K_i(x)$ at x_i, i.e.,

$$N_{K_i(x)}(x_i) = \{z \in V_i^* | \langle z, y - x_i \rangle \leq 0 \text{ for all } y \in K_i(x).$$

Then a strategy profile is a Nash equilibrium if for each i we have $\overline{x}_i \in K_i(\overline{x})$ and

$$\partial_i^+ u_i(\overline{x}_{-i}, \overline{x}_i) \cap N_{K_i(\overline{x})}(\overline{x}_i) \neq \varnothing.$$

where $\partial_i^+ u_i(\overline{x}_{-i}, \overline{x}_i)$ denotes the superdifferential of $\partial_i^+ u_i(\overline{x}_{-i}, \cdot)$ evaluated at $\overline{x}_i$. To formulate the problem as a GQVE, let

$$\begin{aligned} g(x,y,z) &= \sum_i \langle z_i, y_i - x_i \rangle \\ \varphi(x) &= \prod_i \varphi_i(x_{-i}, x_i) = \prod_i \partial_i^+ u_i(x) \\ K(x) &= \prod_i K_i(x). \end{aligned}$$

Then a strategy profile $\overline{x} = (\overline{x}_1, .., \overline{x}_n)$ is a Nash equilibrium if there exists an $x \in X$ and $\overline{z} \in \varphi(x)$ such that

$$\sum_i \langle \overline{z}_i, y_i - \overline{x}_i \rangle \leq 0 \text{ for all } y \in K(\overline{x}).$$

Note that $\partial_i^+ u_i(x_{-i}, y_i)$ is a convex, w^* closed set in X^* for each x_{-i} and y_i. Furthermore, $g : X \times X \times X^* \to \mathbb{R}$ is continuous. Now suppose that for each i, $x \mapsto \partial_i^+ u_i(x)$ is non-empty valued with the continuous inclusion property and that K is non-empty valued and convex valued with the continuous inclusion property. Then the generalized game has an equilibrium. To see this note that, since g is continuous, it follows that $R = (x,y,z) | \sum_i \langle z_i, y_i - x_i \rangle > 0\}$ is correspondence secure with respect to K and φ. Now the conclusion follows from Theorem 8.

7 Alternative approaches to the GQVE Problem without continuity

A different approach to the existence of solutions to the GQVE problem is possible as an application of Kneser's Minimax Theorem.

Theorem 9: (Kneser, 1952): Let A be a non-empty, convex subset of a locally convex TVS and let B be a non-empty compact convex subset of a locally convex TVS. Furthermore, suppose that $f : A \times B \to \mathbb{R}$ is function satisfying $b \mapsto f(a,b)$ is lower-semicontinuous and convex for each $a \in A$ and $a \mapsto f(a,b)$ is concave for each $b \in B$. Then

$$\sup_{a \in A} \min_{b \in B} f(a,b) = \min_{b \in B} \sup_{a \in A} f(a,b).$$

Theorem 10: Suppose that X is a non-empty, convex, compact subset of a locally convex Hausdorff topological vector space and that Z is a locally convex Hausdorff TVS. Suppose that $K : X \to X$ is a corresondence, $\varphi : X \to Z$ is a correspondence and $g : X \times X \times Z \to R$ is a function. Suppose that

(i) For every $(x,y) \in X \times X$, the function

$$z \in Z \mapsto g(x,y,z)$$

is lower semi-continuous and convex. For every $y \in X$, the function

$$x \in X \mapsto \inf_{z \in \varphi(x)} g(x, y, z)$$

is correspondence secure with respect to K.

(ii) K is non-empty valued and convex valued with the continuous inclusion property and φ is non-empty valued, convex valued and compact valued.

(iii) For each (x, z), the function $y \to g(x, y, z)$ is concave.

(iv) For every $x \in X$, $\inf_{z \in F(x)} g(x, x, z) \leq 0$

Then there exists $\overline{x} \in X$ and $\overline{z} \in Z$ such that $\overline{x} \in K(\overline{x}), \overline{z} \in \varphi(\overline{x})$ and

$$g(\overline{x}, y, \overline{z}) \leq 0 \text{ for all } y \in K(\overline{x}).$$

Proof: Define a function $f : X \times X \to \mathbb{R}$ as $f(x, y) = \inf_{z \in \varphi(x)} g(x, y, z)$. Then f is correspondence secure with respect to K and K has the continuous inclusion property. It follows from **Theorem 5** that there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and $f(\overline{x}, y) \leq 0$ for all $y \in K(\overline{x})$. That is, there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and

$$\sup_{y \in K(\overline{x})} \inf_{z \in \varphi(\overline{x})} g(\overline{x}, y, z) \leq 0.$$

Noting that $y \mapsto g(\overline{x}, y, z)$ is concave on $K(\overline{x})$ (since $y \mapsto g(x, y, z)$ is concave and $K(\overline{x})$ is a convex set) and that $z \mapsto g(\overline{x}, y, z)$ is convex on $\varphi(\overline{x})$ and lsc, we conclude from Kneser's Theorem that

$$\sup_{y \in K(\overline{x})} \inf_{z \in \varphi(\overline{x})} g(\overline{x}, y, z) = \inf_{z \in \varphi(\overline{x})} \sup_{y \in K(\overline{x})} g(\overline{x}, y, z)$$

Since $z \in \varphi(\overline{x}) \mapsto \sup_{y \in K(\overline{x})} g(\overline{x}, y, z)$ is lower semi-continuous (as the supremum of a collection of lower semi-continuous functions) and $\varphi(\overline{x})$ is compact, it follows that there exists a $\overline{z} \in \varphi(\overline{x})$ such that $g(\overline{x}, y, \overline{z}) \leq 0$ for all $y \in K(\overline{x})$.

Another approach to the GQVE problem is possible following an idea of Aubin and Ekeland. For the simpler case of the QVE problem, the proof of the next result is essentially identical to the proof of Theorem 21 in Aubin and Ekeland (1984)

Theorem 11: Suppose that

(i) For every $y \in X$, the function $x \in X \mapsto f(x, y)$ is lower semi-continuous and $y \in X \mapsto g(x, y)$ is concave.

(ii) The correspondence K is convex valued and for each $p \in Y$, the set $\{x \in X | p \cdot x \leq \sup_{y \in K(x)} p \cdot y\}$ is closed.

(iii) The set $\{x \in X | \sup_{y \in K(x)} f(x, y) \leq 0\}$ is closed.

Then there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and

$$f(\overline{x}, y) \leq 0 \text{ for all } y \in K(\overline{x}).$$

Our generalization of Theorem 11 to GQVE problems uses a technique appearing in Shih and Tan and Yao based on Kneser's Minimax Theorem.

Theorem 12: Let X be a compact, convex, non-empty subset of a locally convex TVS. Let $K : X \to X$ be non-empty valued, compact valued and convex valued and let $F : X \to Y$ be non-empty valued, convex valued and closed valued. Suppose that

(i) For every $y \in X$, the function $x \in X \mapsto \inf_{z \in F(x)} g(x, y, z)$ is lsc.

(ii) For each $p \in Y$, the set $\{x \in X | p \cdot x \leq \sup_{y \in K(x)} p \cdot y\}$ is closed.

(iii) The set

$$\{x \in X | \sup_{y \in K(x)} \inf_{z \in F(x)} g(x, y, z) \leq 0\}$$

is closed.

(iv) For every $x \in X$, $\inf_{z \in F(x)} g(x, x, z) \leq 0$

Then there exists $\overline{x} \in X$ and $\overline{z} \in F(\overline{x})$ such that $\overline{x} \in K(\overline{x})$ and

$$g(\overline{x}, y, \overline{z}) \leq 0 \text{ for all } y \in K(\overline{x}).$$

Proof: Step 1: First we show that there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and

$$\sup_{y \in K(\overline{x})} \inf_{z \in F(\overline{x})} g(\overline{x}, y, z) \leq 0$$

Suppose instead that for every $x \in X$, we have $x \notin K(x)$ or

$$\sup_{y \in K(x)} \inf_{z \in F(x)} g(x, y, z) > 0.$$

If $x \notin K(x)$, then (Hahn-Banach) there exists a $p \in Y$ such that

$$p \cdot x > \sup_{y \in K(x)} p \cdot y.$$

Let

$$U(p) = \{x \in X | p \cdot x > \sup_{y \in K(x)} p \cdot y\}$$

and

$$U_0 = \{x \in X | \sup_{y \in K(x)} \inf_{z \in F(x)} g(x, y, z) > 0\}$$

These sets are open and cover X. Consequently, there exists a finite collection $U_0, U(p_1), .., U(p_m)$ and an associated continuous partition of unity $\beta_0, \beta_1, .., \beta_m$. Let

$$\psi(x, y) = \beta_0(x) \left[\inf_{z \in F(x)} g(x, y, z) \right] + \sum_{i=1}^{m} \beta_i(x) \left[p_i \cdot (x - y) \right]$$

and note that $\psi(x, x) \leq 0$ and that $x \mapsto \psi(x, y)$ is lsc and $y \mapsto g(x, y)$ is concave. Applying the Ky Fan theorem, it follows that there exists $\overline{x} \in X$ such that

$$\beta_0(\overline{x}) \left[\inf_{z \in F(\overline{x})} \left[g(\overline{x}, y, z) \right] \right] + \sum_{i=1}^{m} \beta_i(\overline{x}) \left[p_i \cdot (\overline{x} - y) \right] \leq 0.$$

If $\beta_0(\overline{x}) > 0$, then $\overline{x} \in U_0$ implying that there exists a $\overline{y} \in K(\overline{x})$ such that $\inf_{z\in F(\overline{x})} g(\overline{x}, \overline{y}, z) > 0$. If $\beta_0(\overline{x}) = 0$, choose $\overline{y} \in K(\overline{x})$ and $i > 0$ with $\beta_i(\overline{x}) > 0$. Then $\overline{x} \in U(p_i)$ implies that

$$p \cdot \overline{x} > \sup_{y\in K(\overline{x})} p \cdot y \geq p \cdot \overline{y}$$

Therefore, there exists $\overline{y} \in K(\overline{x})$ such that

$$\beta_0(\overline{x}) \left[\inf_{z\in F(\overline{x})} [g(\overline{x}, y, z)] \right] + \sum_{i=1}^{m} \beta_i(\overline{x}) \left[p_i \cdot (\overline{x} - y)\right] > 0$$

an obvious impossibility. This completes the proof of Step 1

Step 2: Define $f : K(\overline{x}) \times F(\overline{x}) \to R$ as $f(y, z) = g(\overline{x}, y, z)$. Since $y \mapsto g(\overline{x}, y, z)$ is concave and $z \mapsto g(\overline{x}, y, z)$ is lsc and convex, it follows from Kneser's theorem that

$$\sup_{y\in K(\overline{x})} \inf_{z\in F(\overline{x})} g(\overline{x}, y, z) = \inf_{z\in F(\overline{x})} \sup_{y\in K(\overline{x})} g(\overline{x}, y, z)$$

implying that

$$\inf_{z\in F(\overline{x})} \sup_{y\in K(\overline{x})} g(\overline{x}, y, z) \leq 0$$

Since $z \in X \mapsto g(\overline{x}, y, z)$ is lsc and $F(\overline{x})$ is compact, it follows that there exists a $\overline{z} \in F(\overline{x})$ such that $g(\overline{x}, y, \overline{z}) \leq 0$ for all $y \in K(\overline{x})$.

8 Weakening convexity

The proof of Theorem 12 proceeds in two steps. First, a Ky Fan type inequality is established and then a Kneser type minimax theorem is applied. Consequently, it is natural to look at the possible ways in which convexity and lower semi-continuity might be weakened in each step. However, a careful examination of the proof of Step 1 indicates that, in order to derive a Ky Fan type result, any weakened notion of concavity or lower semi-continuity must be preserved by addition and this limits the possibilities for generalization. However, we can weaken the convexity assumption in step 2 by appealing to Fan's generalization of Kneser's Theorem.

Definition: Let X and Y be sets.

A function $f : X \times Y \to \mathbb{R}$ is called convex-like on Y if for any $y_1, y_2 \in Y$ and nonnegative numbers λ_1, λ_2 summing to 1, there exists y$^* \in Y$ such that

$$f(x, y^*) \leq f(x, y_1)\lambda_1 + f(x, y_2)\lambda_2$$

for all x$\in X$.

A function $f : X \times Y \to \mathbb{R}$ is called concave-like on X if for any $x_1, x_2 \in Y$ and nonnegative numbers λ_1, λ_2 summing to 1, there exists x$^* \in X$ such that

$$f(x^*, y) \geq f(x_1, y)\lambda_1 + f(x_2, y)\lambda_2$$

for all $y \in Y$.

Theorem 13: (Fan, 1953): Let A be a non-empty set, B a non-empty compact topological space and $f : A \times B \to R$ a function concave-like on A and convex-like on B. Furthermore, suppose that $b \mapsto f(a, b)$ is lower-semicontinuous for each $a \in A$. Then

$$\sup_{a \in A} \min_{b \in B} f(a, b) = \min_{b \in B} \sup_{a \in A} f(a, b)$$

Theorem 14: Let X be compact, convex non-empty subset of a locally convex TVS. Let $K : X \to X$ be non-empty valued, compact valued and convex valued and let $F : X \to Y$ be non-empty valued, convex valued and closed valued. Suppose that for each $(x, z) \in X \times X$, the function $y \mapsto g(x, y, z)$ is concave on X and for each $(x, y) \in X \times X$, $z \mapsto g(x, y, z)$ is lower semi-continuous and convexlike on $F(x)$. Furthermore, suppose that condiotions (i), (ii) and (iii) of Theorem ?? are satisfied.

Then there exists $\overline{x} \in X$ and $\overline{z} \in F(\overline{x})$ such that $\overline{x} \in K(\overline{x})$ and

$$g(\overline{x}, y, \overline{z}) \leq 0 \text{ for all } y \in K(\overline{x}).$$

Proof: From Step 1 of Theorem 12, it follows that there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and

$$\sup_{y \in K(\overline{x})} \inf_{z \in F(\overline{x})} g(\overline{x}, y, z) \leq 0$$

Noting that $y \mapsto g(\overline{x}, y, z)$ is concavelike on $K(\overline{x})$ (since $y \mapsto g(x, y, z)$ is concave and $K(\overline{x})$ is a convex set) and that $z \mapsto g(\overline{x}, y, z)$ is convex-like on $F(\overline{x})$ and lsc, the conclusion is an immediate consequence of Fan's minimax theorem.

9 References

Aubin, J-P and I. Ekeland (1984), Applied Non-Linear Analysis. New York: John Wiley, 1984.

Balaj M and Luc D T (2010) , On mixed variational relation problems, Computers and Mathematics with Applications, 60: 2712–2722

Balaj M and Lin L J (2011) , Generalized variational relation problems with applications, J. Optim. Theor Appl.,148: 1–13

Barelli, P., and I. Meneghel (2013), "A note on the equilibrium existence problem in discontinuous games," *Econometrica* 81, 813-824

Browder, F (1968), The Fixed point theory of Multi-valued Mappings in Topological Vector Spaces, Mathematische Annalen, 177, 283-301.

Carbonell-Nicolau, O., and R.P. McLean (2015), “On the existence of Nash equilibrium in Bayesian games,” mimeograph.

Carmona, G. (2013), *Existence and Stability of Nash Equilibrium.* Singapore: World Scientific Publishing.

Carmona, G., and K. Podczeck (2016), “Existence of Nash equilibrium in ordinal games with discontinuous preferences,” *Economic Theory* 61, 457-478.

Chan, D and J-S Pang, "The Generalized Quasi-Variational Equilibrium Problem", Mathematics of Operations Research, 7 211-222.

Dasgupta, P., and E. Maskin (1986), “The existence of equilibrium in discontinuous economic games, I: Theory,” *Review of Economic Studies* 53, 1-26.

Dasgupta, P., and E. Maskin (1986), “The existence of equilibrium in discontinuous economic games, II: Applications,” *Review of Economic Studies* 53, 27-41.

Fan, Ky, !972), "A Minimax Inequality and Applications", in *Inequalities*, Vol.3, O.Shisha, ed, New Yrk: Academic Press, 1972.

Fishburn, P.C., and R.W. Rosenthal (1986), “Noncooperative games and nontransitive preferences,” *Mathematical Social Sciences* 12, 1-7.

Fishburn, P.C. (1988), *Nonlinear Preference and Utility Theory.* Baltimore, MD: The Johns Hopkins University Press.

Fishburn, P.C. (1988), *Nonlinear Preference and Utility Theory.* Englewood Cliffs, NJ: Prentice Hall.

Hai, N., Khanh, P., and Quan, N. (2009), On the existence of solutions to quasivariational inclusion problems, J Glob Optim, 45:565–581

He, W., and N.C. Yannelis (2016), “Existence of equilibria in discontinuous Bayesian games,” *Journal of Economic Theory* 162, 181-194.

He, W., and N.C. Yannelis (2017), "Equilibria with discontinuous preferences,” *Journal of Mathematical Analysis and Applications, 450*, 1421-1433.

Kneser, H. (1952) Sur un th´eor ‘eme fondamental de la th ´eorie des jeux. C. R. Acad. Sci. Paris, S´er A 234, 2418–2420

Lin, L-J and Q. Ansari (2010), "Systems of Quasi-Variational Relations with Applications", Nonlinear Analysis, 72, 1210-1220.

Lin L J and Wang S Y (2009), Simultaneous variational relation problems and related applications, Computers and Mathematics with Applications, 58: 1711–1721.

Luc D T (2008), An abstract problem in variational analysis, J. Optim. Theory Appl., 138: 65–76.

Luc D T, Sarabi E, and Soubeyran A (2010), Existence of solutions in variational relation problems without convexity, J. Math. Anal. Appl., 364: 544–555.

McLennan, A., Monteiro, P.K., and R. Tourky (2011), “Games with discontinuous payoffs: A strengthening of Reny’s existence theorem,” *Econometrica* 79, 1643-1664.

Monteiro, P.K., and F.H. Page (2007), “Uniform payoff security and Nash equilibrium in compact games,” *Journal of Economic Theory* 134, 566-575.

Nessah, R. (2011), “Generalized weak transfer continuity and the Nash equilibrium,” *Journal of Mathematical Economics* 47, 659-662.

Nessah, R., and G. Tian (2013), "Existence of solution of minimax inequalities, equilibria in games and fixed points without convexity and compactness assumptions," *Journal of Optimization Theory and Applications* 157, 75-95.

Nessah, R., and G. Tian (2016), "On the existence of Nash equilibrium in discontinuous games," *Economic Theory* 61, p. 515-540.

Park, S. (2013) , Evolution of the Minimax Inequality of Ky Fan, Journal of Operators

Prokopovych, P. (2011), "On Equilibrium Existence in Payoff Secure Games," *Economic Theory* 483, 5-16.

Prokopovych, P., and N.C. Yannelis (2014), "On the existence of mixed strategy Nash equilibria," *Journal of Mathematical Economics* 52, 87-97.

Reny, P.J. (1999), "On the existence of pure and mixed strategy Nash equilibria in discontinuous games," *Econometrica* 67, 1029-1056.

Reny, P.J. (2016), "Nash equilibrium in discontinuous games," *Economic Theory* 61, 553-569.

Reny, P.J. (2016), "Equilibrium in discontinuous games without complete or transitive preferences," *Economic Theory Bulletin* 4, 1-4.

Simon, L.K. (1987), "Games with discontinuous payoffs," *Review of Economic Studies* 54, 569-597.

Shafer, W.J. (1974), "The non-transitive consumer," *Econometrica* 42, 913-919.

Shih, M-H and K-K Tan (1985), Generalized Quasi-Variational Inequalities in Locally Convex Topological Vector Spaces, *Journal of Mathematical Analysis and Applications, 108*, 333-343.

Tarafdar, E and M. Chowdhury, Topological Methods for Set-Valued Non-Linear Analysis, Singapore: World Scientific.

Tian, G. (1992), "Generalizations of the FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity," *Journal of Mathematical Analysis and Applications* 170, 457-471.

Yang, Z. (2016), Existence of Solutions of Generalized Quasi-Variational Relation Problems by the Inductive Relations and Some Applications, J Syst Sci Complex, 29: 219–227

Yao, J.C., (1995), Generalized Quasi-Variational Inequaliry Problems with Discontinuous Mappings, Mathematics of Operations Research, 20 465-478.

Yannelis, N. and N.D. Prabhakar (1983), Existence of Maximal elements and Equilibria in Linear Topological Spaces, Journal of Mathematical Economics, 12, 233-245.